Distortion, on the Average and in Expectation

Chicago IMSI '23

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work with A. Nikitenko

I. Intuition

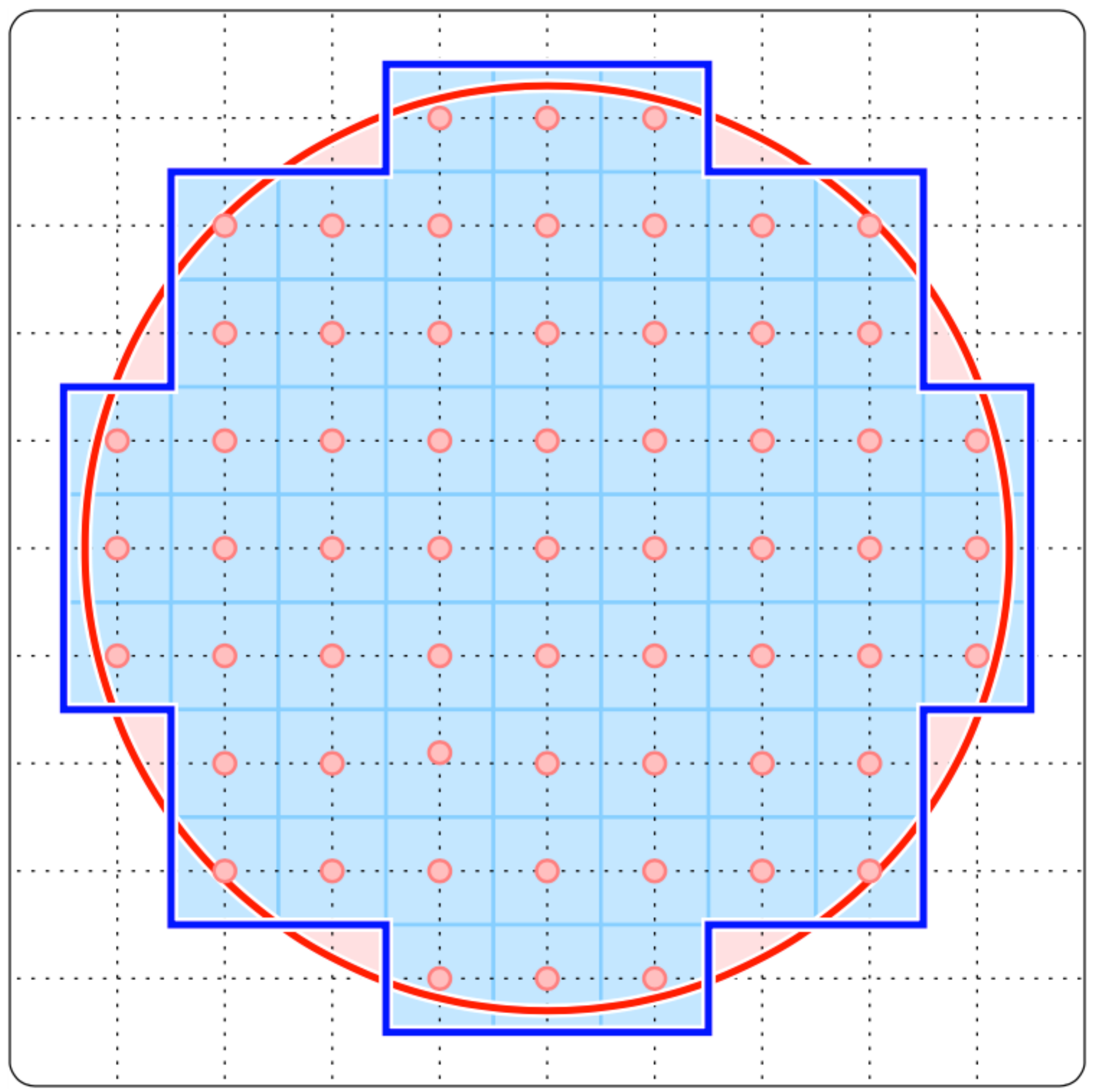
II. Results

III. Methods

IV. Experiments

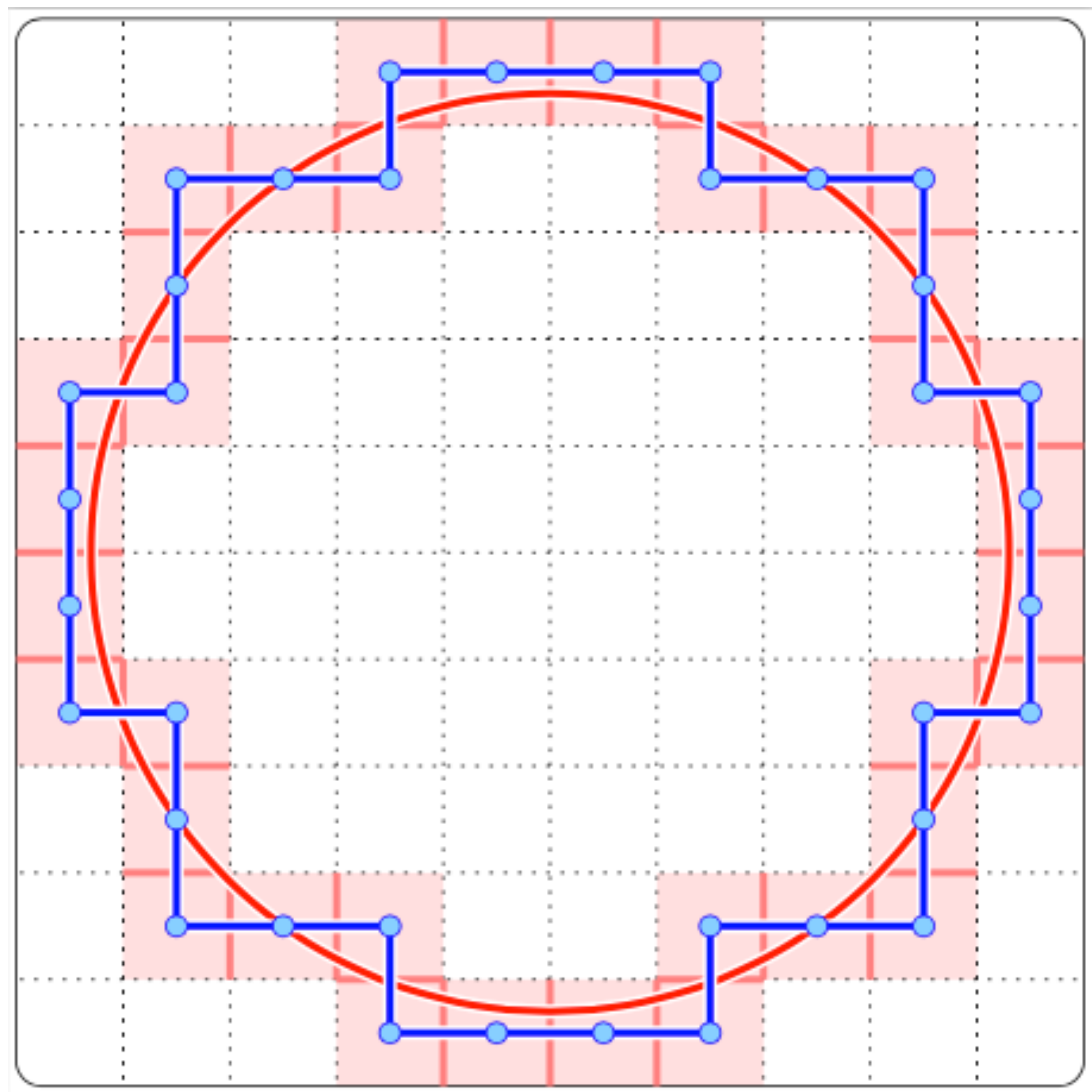
I.1 Square Grid for Disk

distortion → 1.0



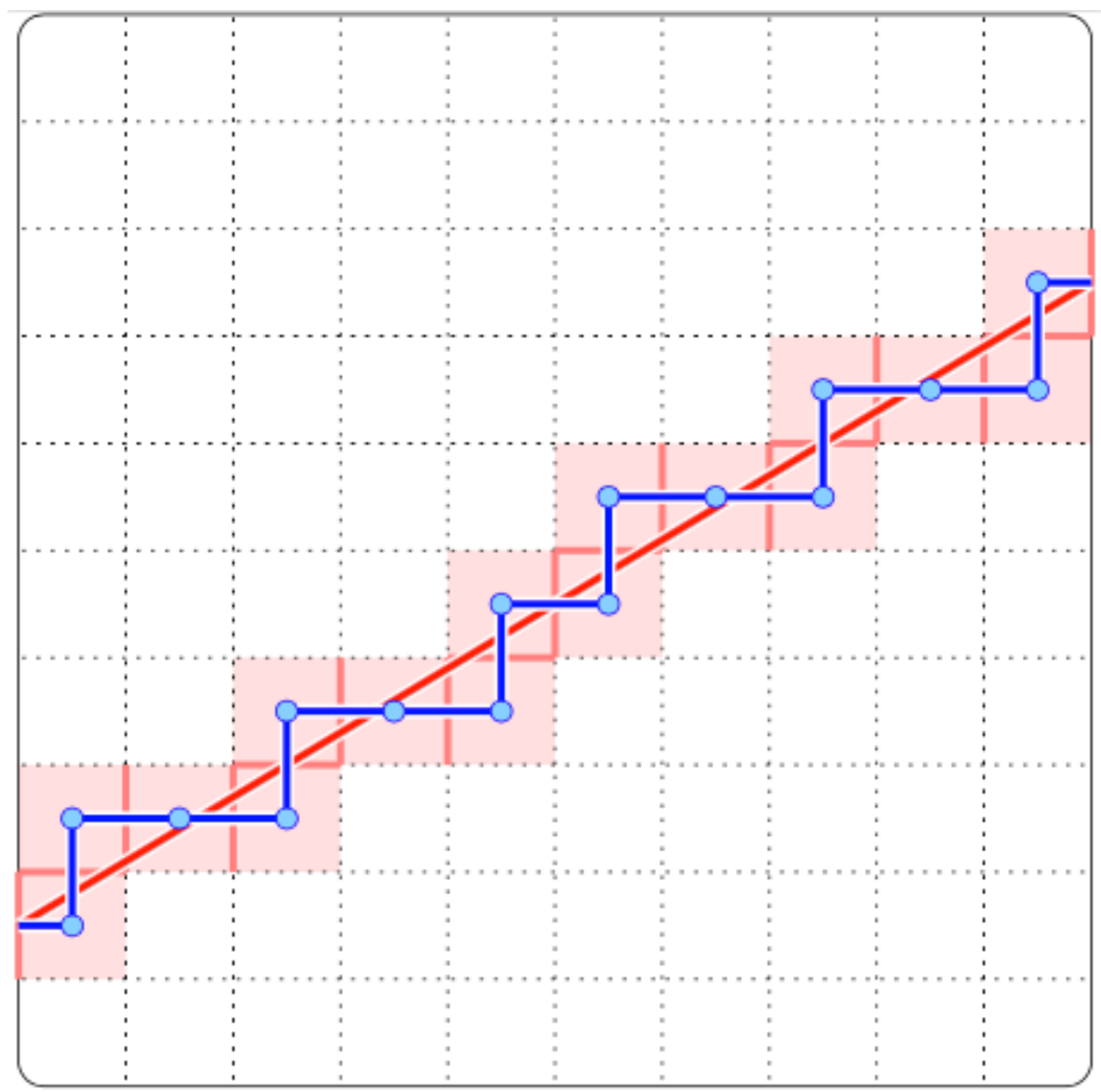
I.2 Staircase for Circle

distortion $\longrightarrow \frac{4}{\pi} = 1.27\ldots$



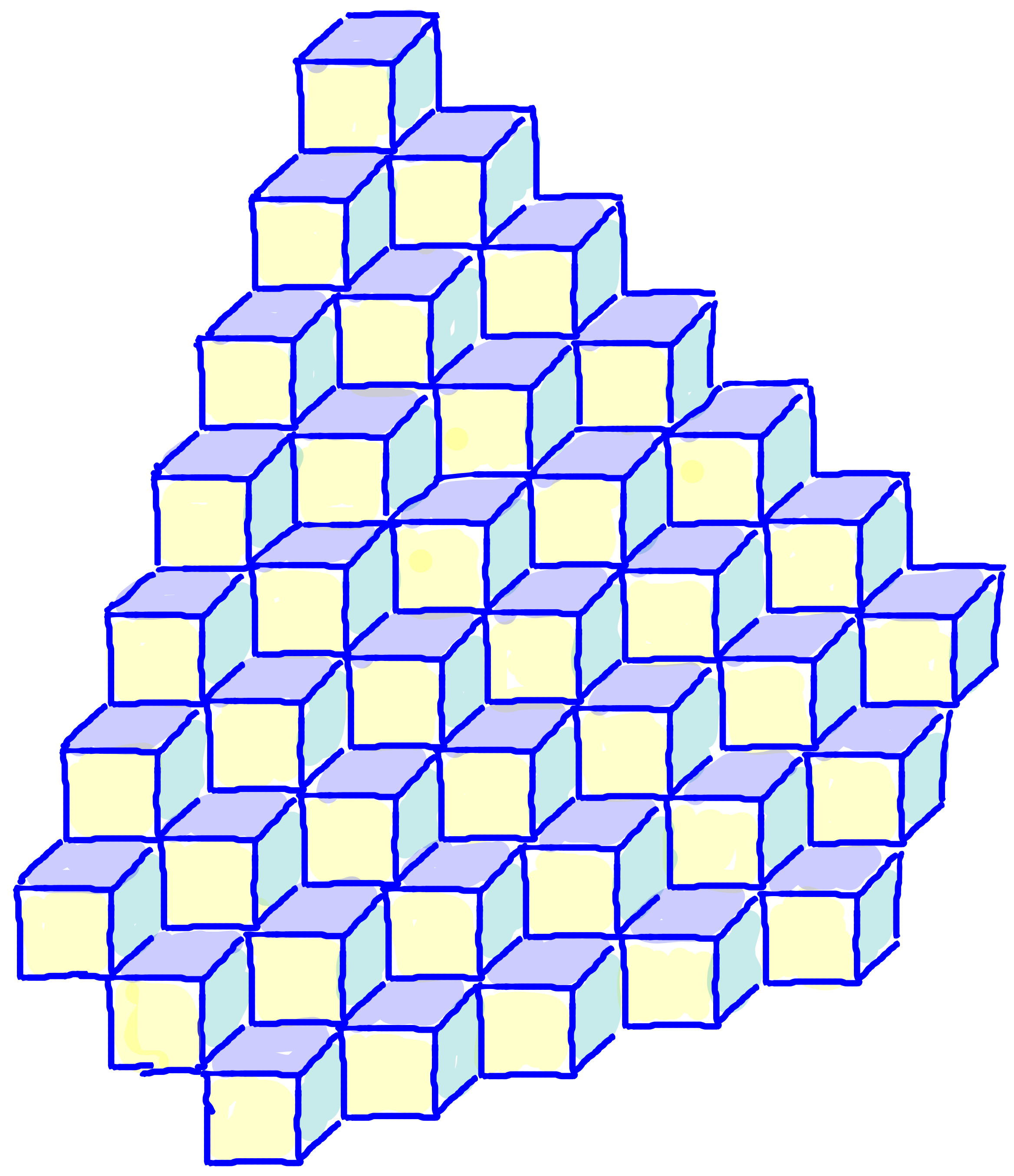
I.3 Staircase for Segment

$$1.0 \leq \text{distortion} \leq \sqrt{2} = 1.41\ldots$$



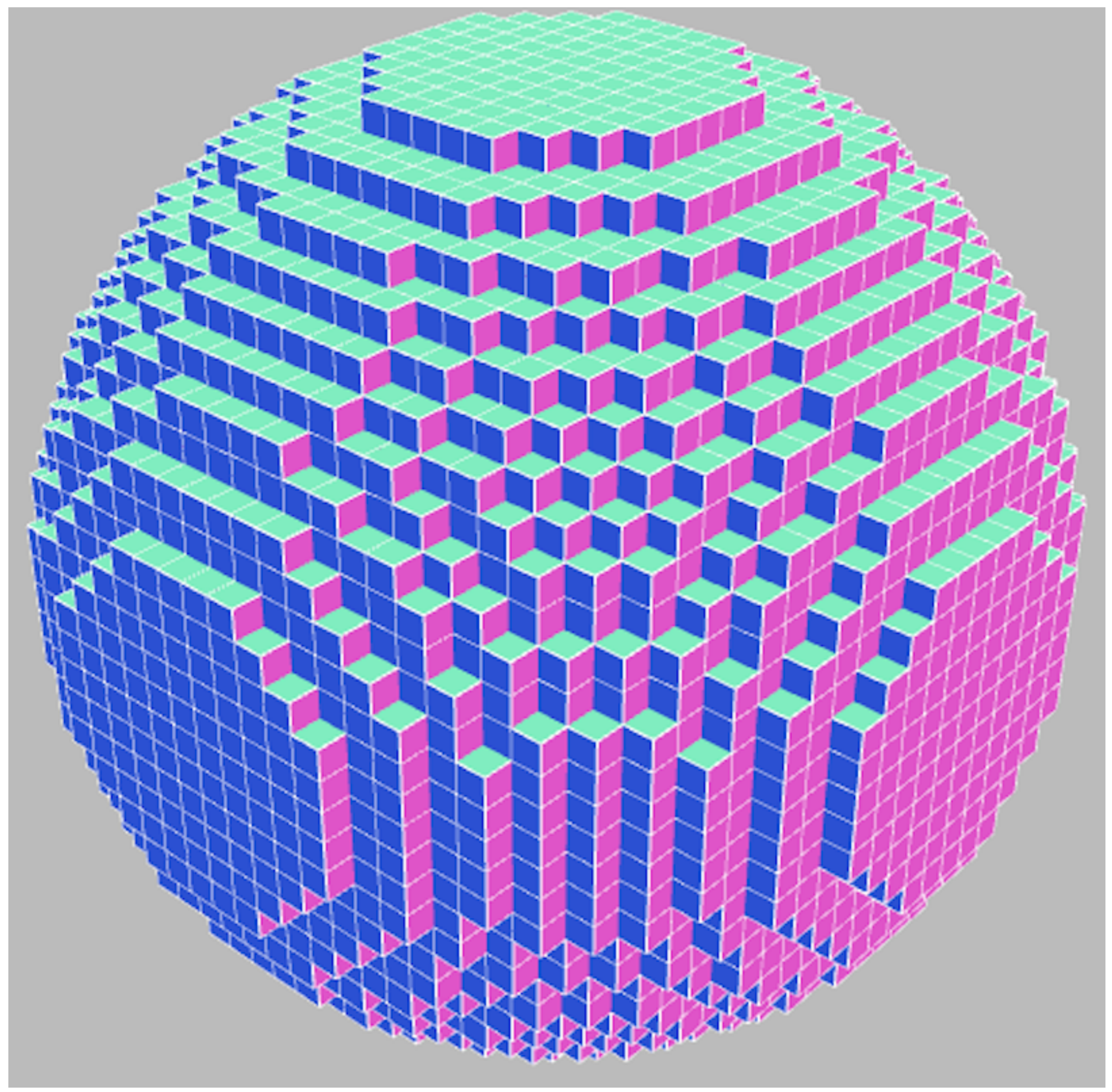
I.4 Stairscape for Plane

$$1.0 \le \text{distortion} \le \sqrt{3} = 1.73\ldots$$



I.5 Stairscape for Sphere

distortion $\longrightarrow$ $3/2 = 1.5$



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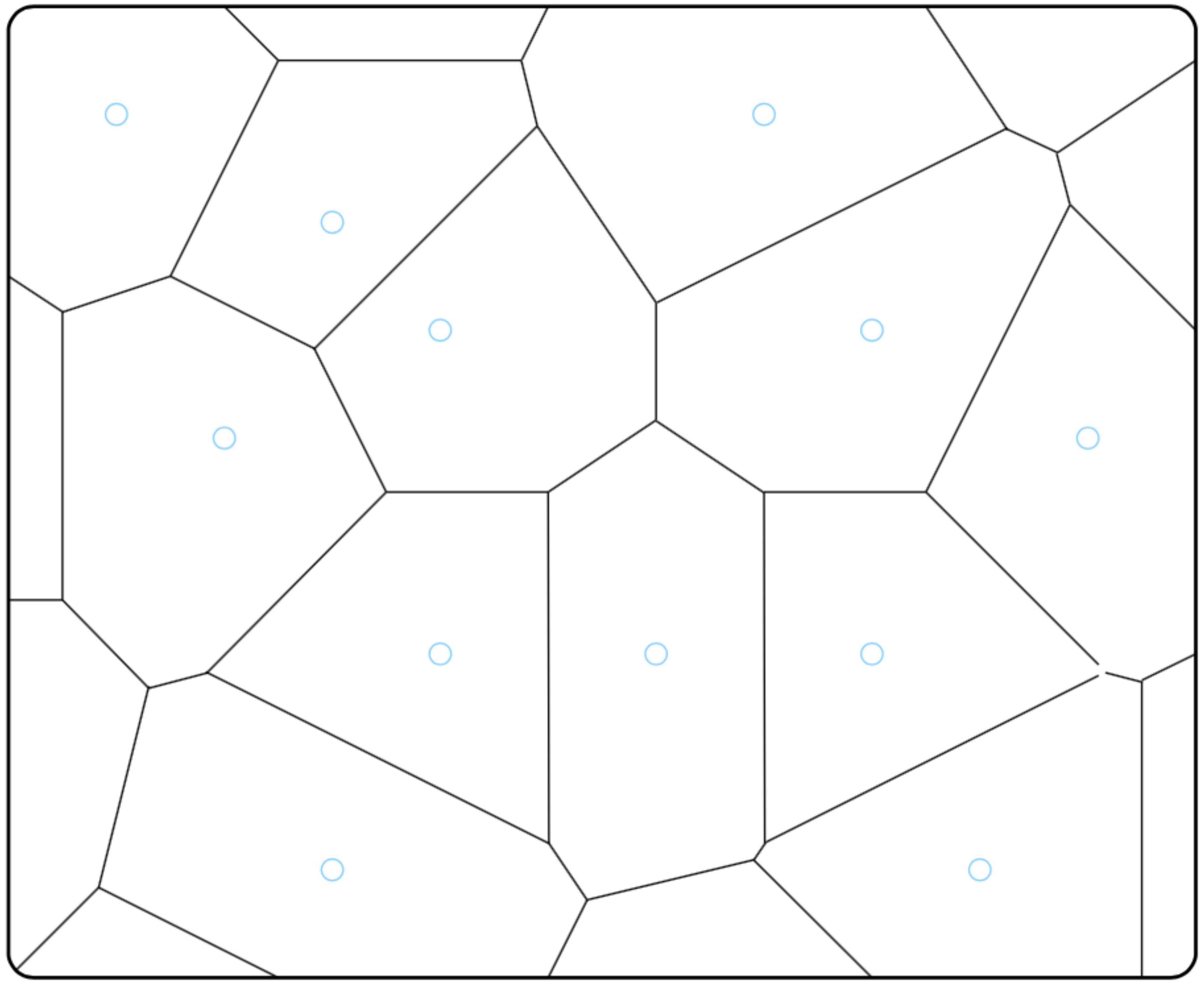
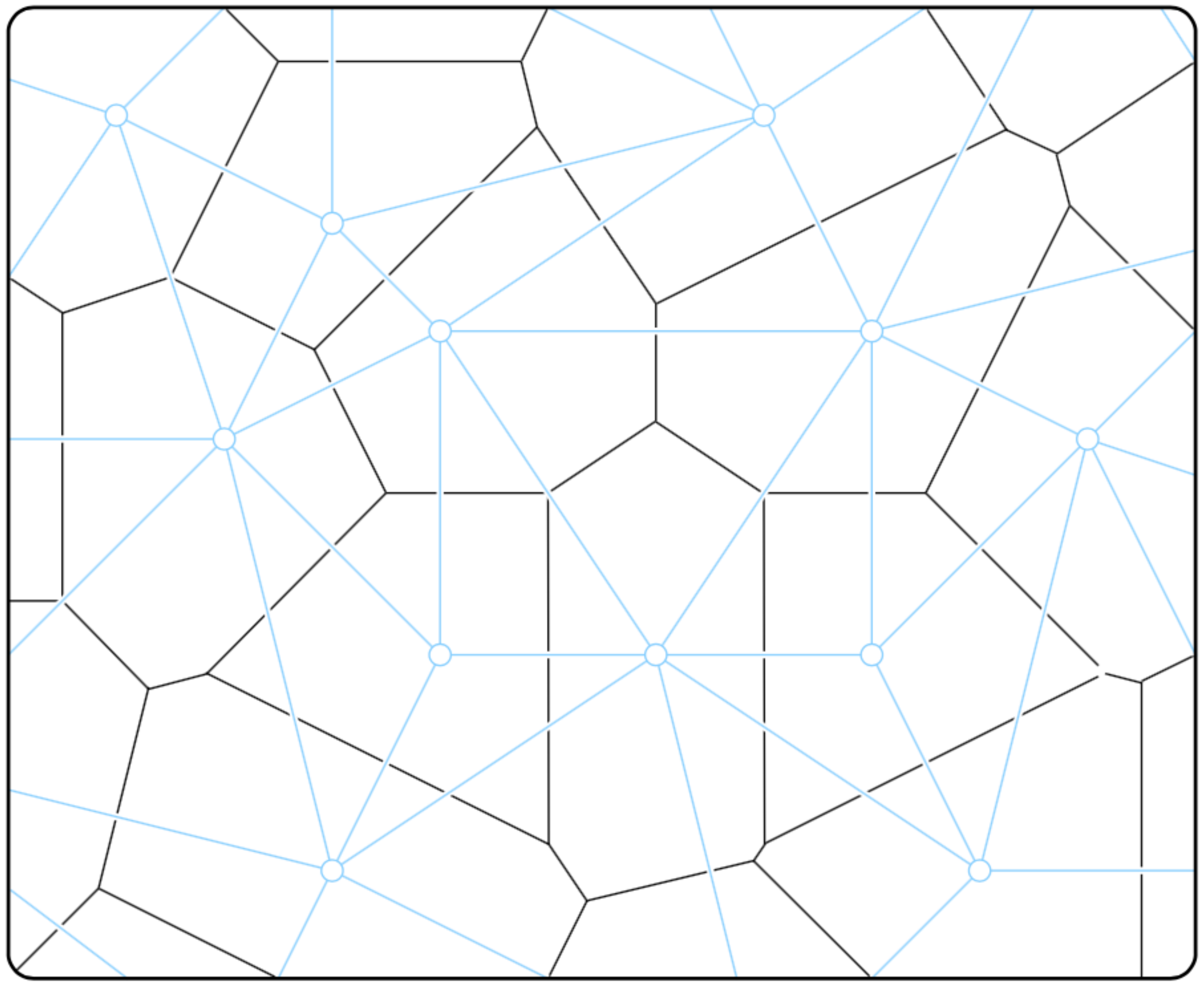
II.1 Voronoi Path/Scape



Vor(A) . . . Voronoi tessellation

II.1 Voronoi Path/Scape

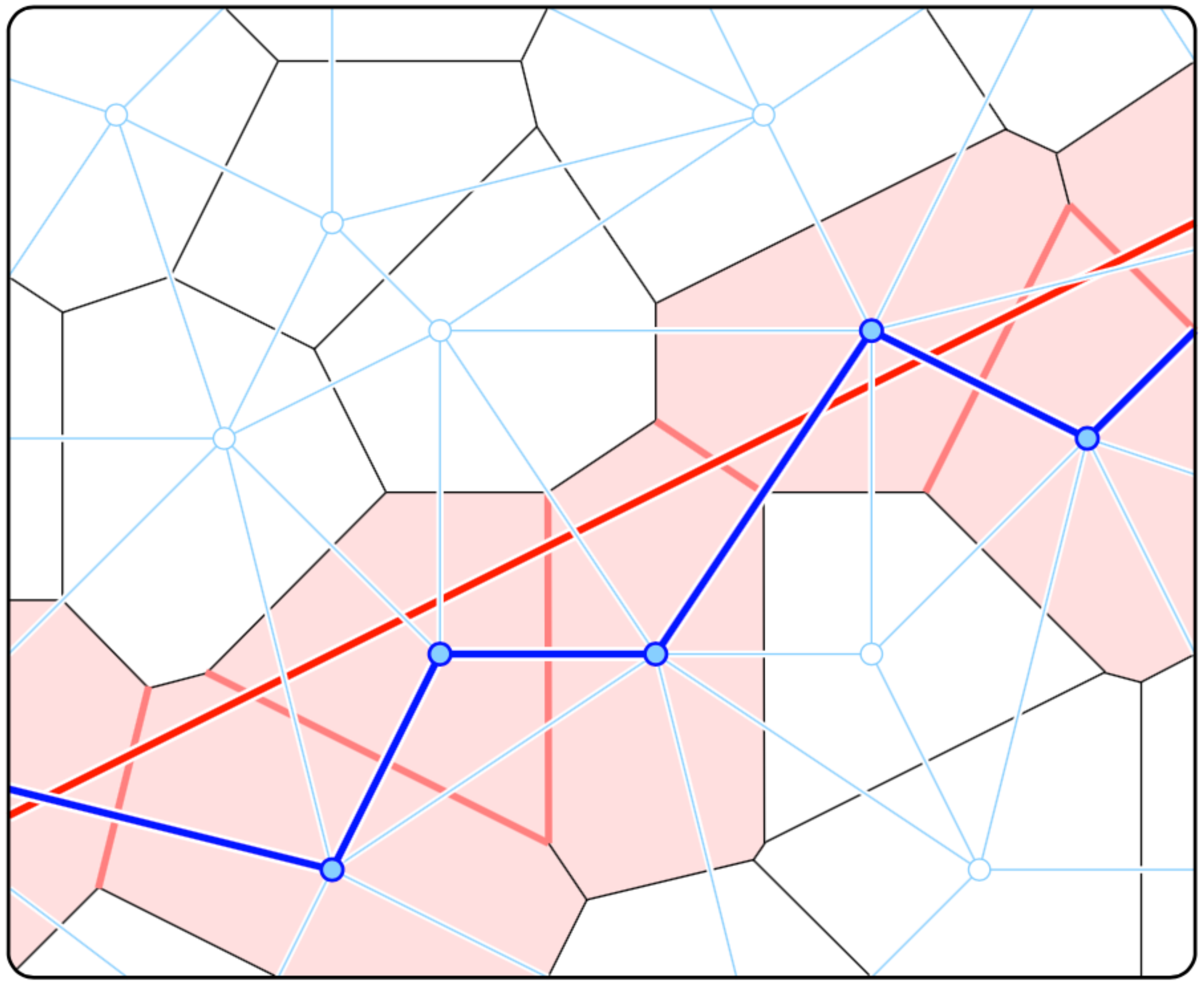


$\mathrm{Vor}(\Delta)$ ... Voronoi tessellation

$\mathrm{Del}(\Delta)$ ... Delaunay mosaic

γ corresponds to γ^*

II.1 Voronoi Path/Scape



$\mathrm{Vor}(A)$ ... Voronoi tessellation

$\mathrm{Del}(A)$ ... Delaunay mosaic

γ corresponds to γ^*

$\mathrm{Scp}_L(A) = \{\gamma \in \mathrm{Del}(A) \mid \gamma^* \cap L \neq \emptyset\}$

is Voronoi path of L and A

II.2 Prior Work

$A \subseteq \mathbb{R}^d$ a Poisson point process, L a line.

2000 : Baccelli, Tchoumatchenko, Zuyev

expected distortion in $\mathbb{R}^2$ is $4/\pi = 1.27\ldots$

2018 : de Castro, Devillers

expected distortion in $\mathbb{R}^d$ is $\sqrt{\frac{2d}{\pi}} + O\left(\frac{1}{\sqrt{d}}\right)$.

II.3 Distortion Constant

$D_{p,d}$	$d=1$	2	3	4	5
$p=1$	1	$4/\pi$	$3/2$	$16/3\pi$	$15/8$
2		1	$3/2$	2	$5/2$
3			1	$16/3\pi$	$5/2$
4				1	$15/8$
5					1

$$D_{p,d} = \binom{d/2}{p/2} = \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{p}{2}+1\right)\Gamma\left(\frac{d-p}{2}+1\right)} = \begin{cases} \dfrac{d!!}{p!!\,(d-p)!!}\,\dfrac{2}{\pi} & \text{if } d \text{ even, } p \text{ odd} \\[2ex] \dfrac{d!!}{p!!\,(d-p)!!} & \text{otherwise} \end{cases}$$

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$1.5 \neq 1.38\ldots = \sqrt{\frac{2\cdot 3}{\pi}}$

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II.4 Average Distortion

Thm.: $\Lambda \subseteq \mathbb{R}^d$ has m.r.p., Ω p-dim. + rectifyable.
Then the average p-volume of the Voronoi scapes of the congruent copies of Ω inside $\mathbb{B}(0,R)$ is

$$\|\Omega\|_p \left[\mathcal{D}_{p,d} + o(1) \right] \text{ as } R \to \infty.$$

II.5 Expected Distortion

Thm.: $A \subseteq \mathbb{R}^d$ is P.P.P. with density $\rho > 0$, $\Omega \subseteq \mathbb{R}^d$ is p-dimensional and rectifyable.

Then

$$\mathbb{E}\left[\| \mathrm{Scp}_{\Omega}(A) \|_p\right] = \mathcal{D}_{p,d} \, \|\Omega\|_p .$$

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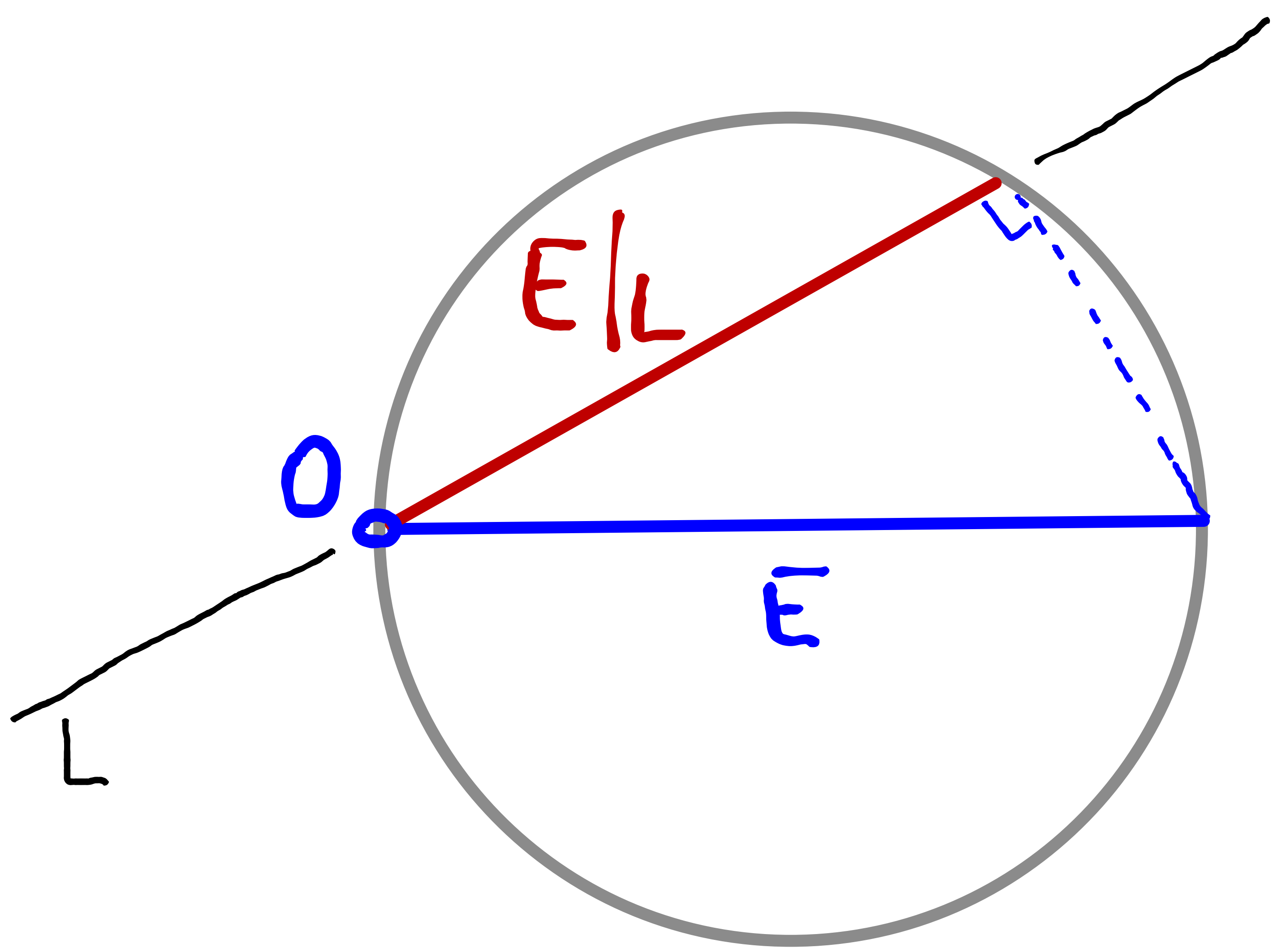
III.1 Grassmannians

$Gr_{p,d}$ ... p-planes in $\mathbb{R}^d$, $\dim Gr_{p,d} = p(d-p)$

$E = [0,1]^p$, $L \in Gr_{p,d}$,

$\|E|_L\|_p$ is p-dim. volume of projection



for $p=1$ and $d=2$

$m^{(1)}_{1,2} = \frac{2}{\pi}$ is avg. length

$m^{(2)}_{1,2} = \frac{1}{2}$ is avg. squared length

III.2 Projection Moments

Def.: The j-th projection moment is the average j-th power of the p-volume:
$$m_{p,d}^{(j)} = \int_{L \in \mathrm{Gr}_{p,d}} \|E|_L\|_p^j \, dL$$

Lemma:
$$m_{p,d}^{(1)} = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{d-p+1}{2}\right)}{\Gamma\left(1/2\right)\Gamma\left(\frac{d+1}{2}\right)},$$
$$m_{p,d}^{(2)} = 1 \Big/ \binom{d}{p}.$$

check that $\mathcal{D}_{p,d} = m_{p,d}^{(1)} \big/ m_{p,d}^{(2)}$.

III.2 Projection Moments

Proof of $m_{p,d}^{(2)} = 1/\binom{d}{p}$.

Cauchy-Binet $\Rightarrow \|E\|_p^2 = \sum_{i=1}^{\binom{d}{p}} \|E_i\|_p^2$.

Expectation on both sides

$$\Rightarrow \quad 1 = \sum_{i=1}^{\binom{d}{p}} \mathbb{E}\left[\|E_i\|_p^2\right]$$

$\square$

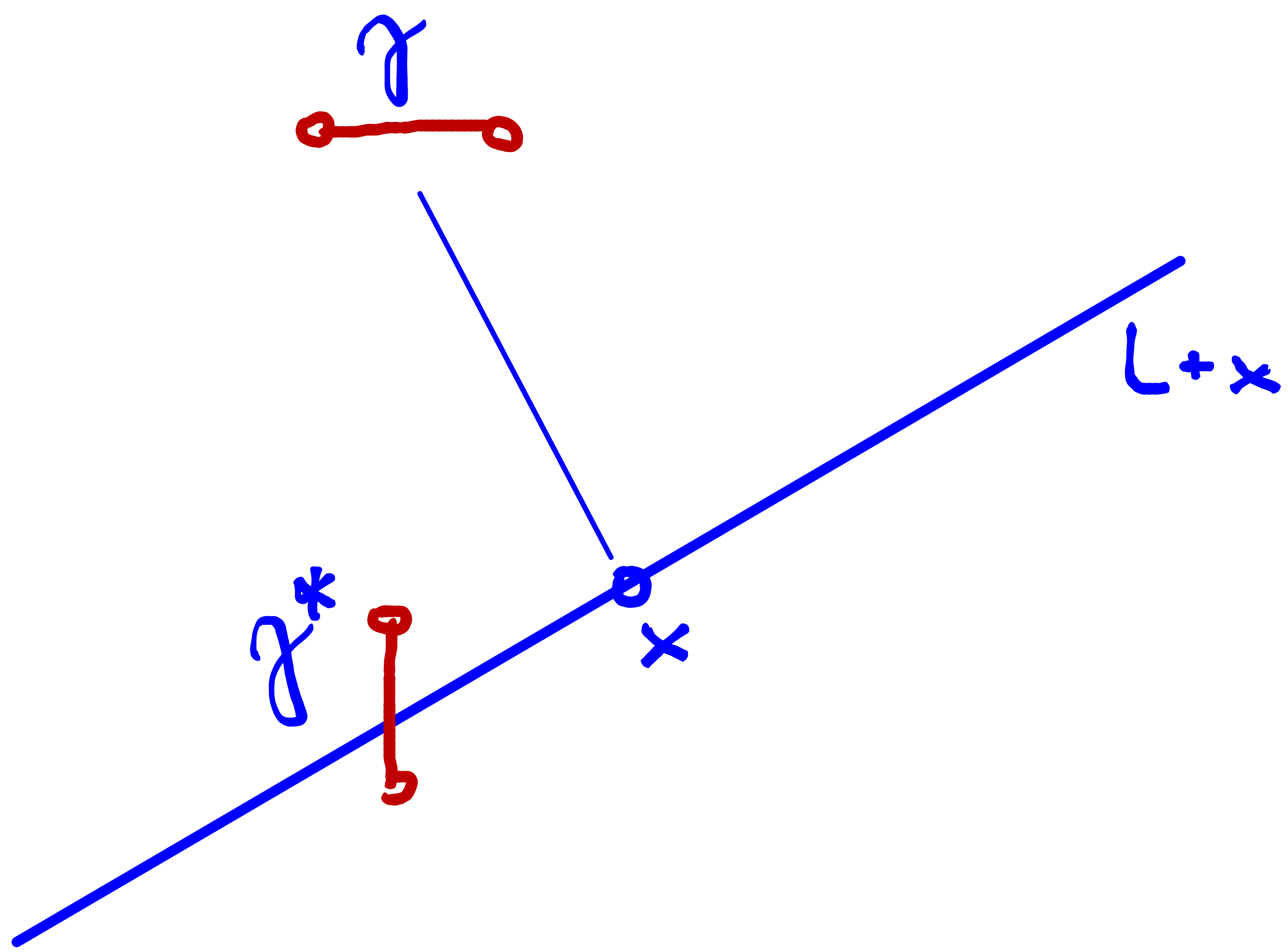
III.3 TILES

$\gamma \in \mathrm{Del}(A)$, $\dim \gamma = p$

$\gamma^* \in \mathrm{Vor}(A)$, $\dim \gamma^* = d - p$

DEF.: The *p-tile* of γ and γ^* is

$$J(\gamma, \gamma^*) = \{ (x, L) \in \mathbb{R}^d \times \mathbb{G}\mathrm{r}_{p,d} \mid x \in \gamma|_{L+x},\ \gamma^* \cap (L+x) \neq \emptyset \}.$$



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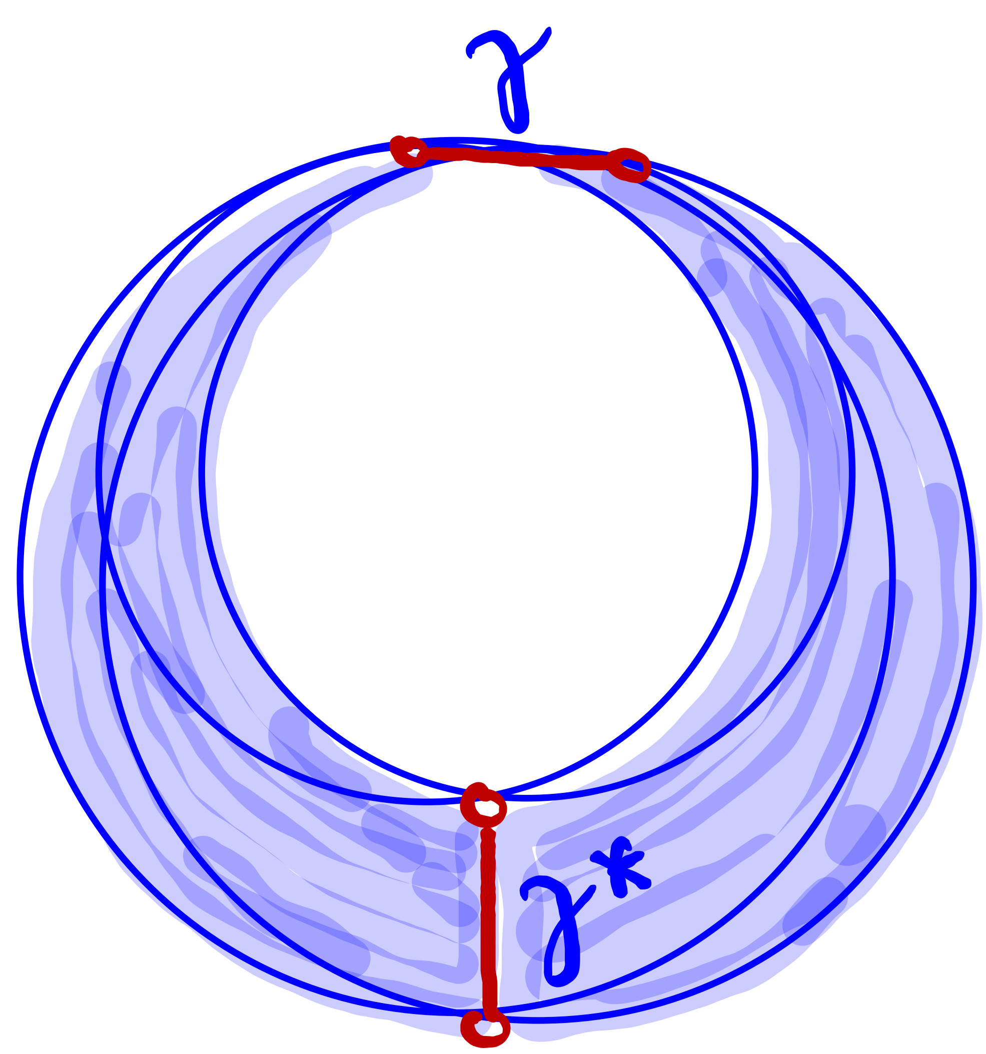


$J(\gamma, \gamma^*)$

$$\dim J(\gamma, \gamma^*) = d + p(d-p)$$

diameter of projection to $\mathbb{R}^d$ is small.

III.4 Volume of Tile

Lemma: $\| J(\gamma, \gamma^*) \| = \|\gamma\|_p \cdot \|\gamma^*\|_{d-p} \Big/ \binom{d}{p}.$

Proof: $x = y + z$

$$\|J\| = \int_{L \in Gr_{p,d}} \int_{y \in L} \mathbb{1}_{y \in \gamma|L} \int_{z \in L^\perp} \mathbb{1}_{(L+z) \cap \gamma^* \neq \emptyset} \, dz \, dy \, dL$$

$$= \|\gamma\|_p \cdot \|\gamma^*\|_{d-p} \cdot \underbrace{\int \cos^2(L, \gamma) \, dL}_{m^{(2)}_{p,d} = 1/\binom{d}{p}}$$

$\square$

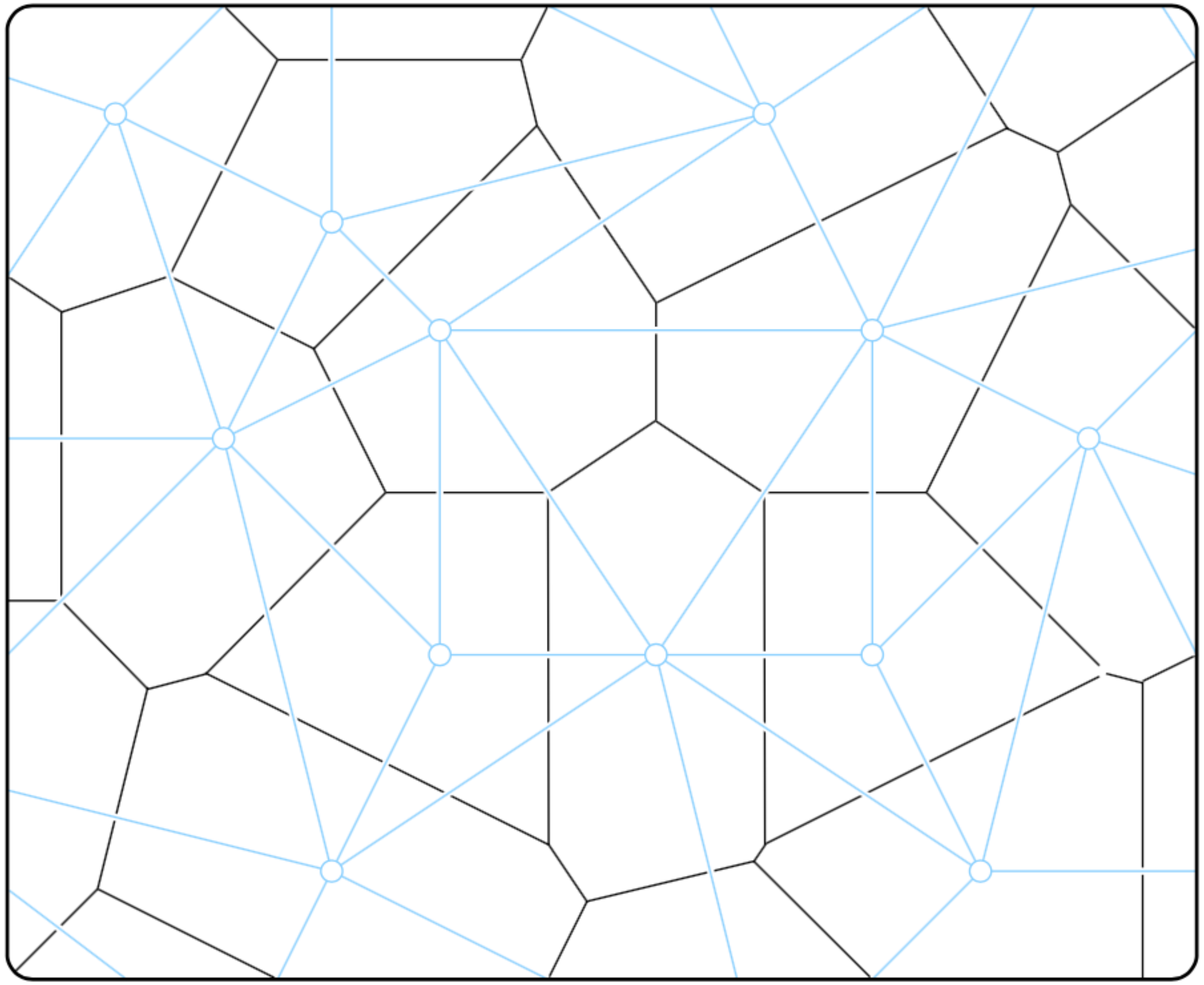
III.5 Mixed Regularity

Def.: $A \subseteq \mathbb{R}^d$ has the mixed regularity property if the boundary tiles of $B(0, R)$ have measure $o(R^d)$.

E.g. if $\exists R_0 \geq 0$ s.t. every ball with radius R_0 contains a point of A.

P.P.P. has mixed regularity property in expectation.

III.6 Mixed Complex



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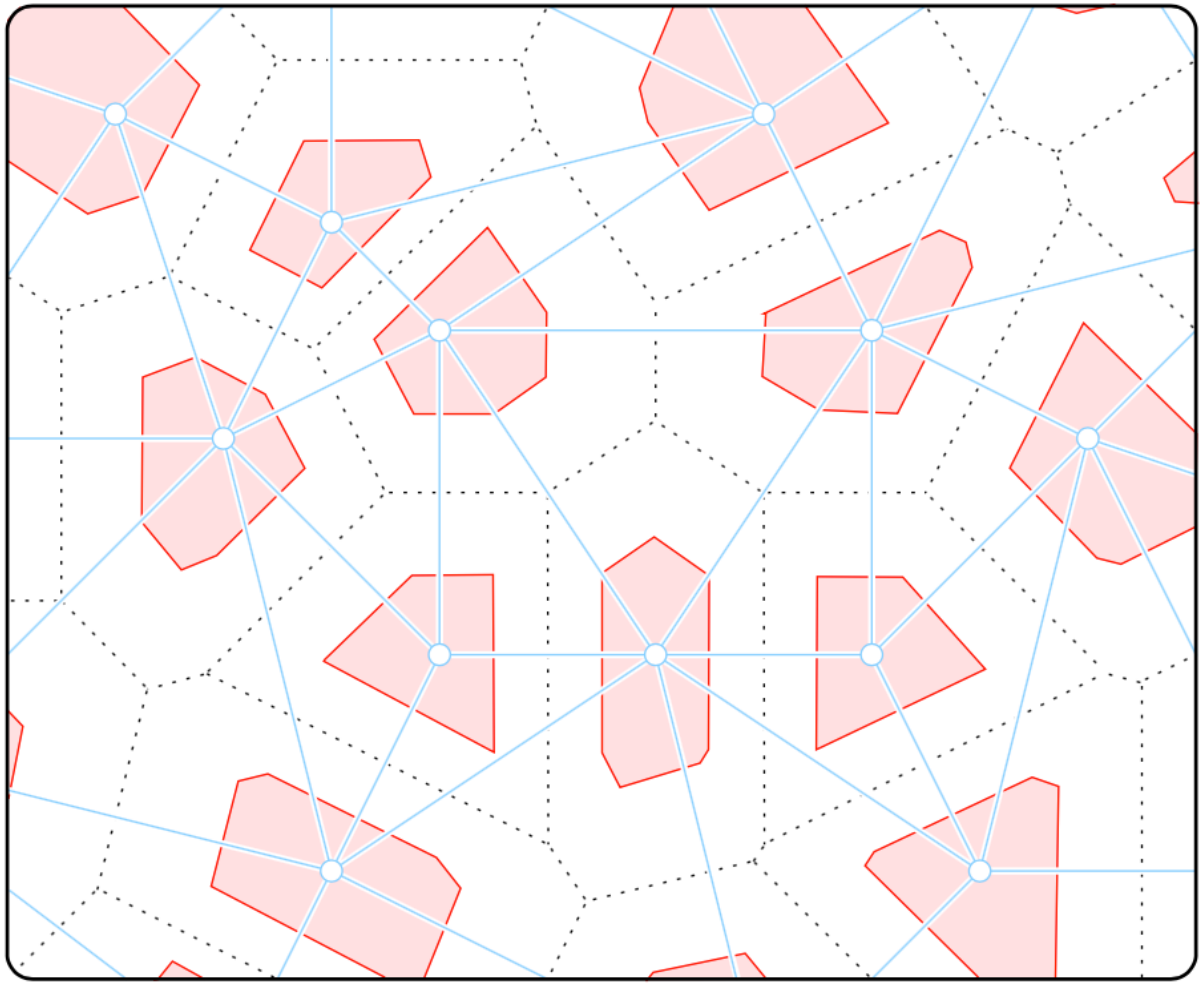
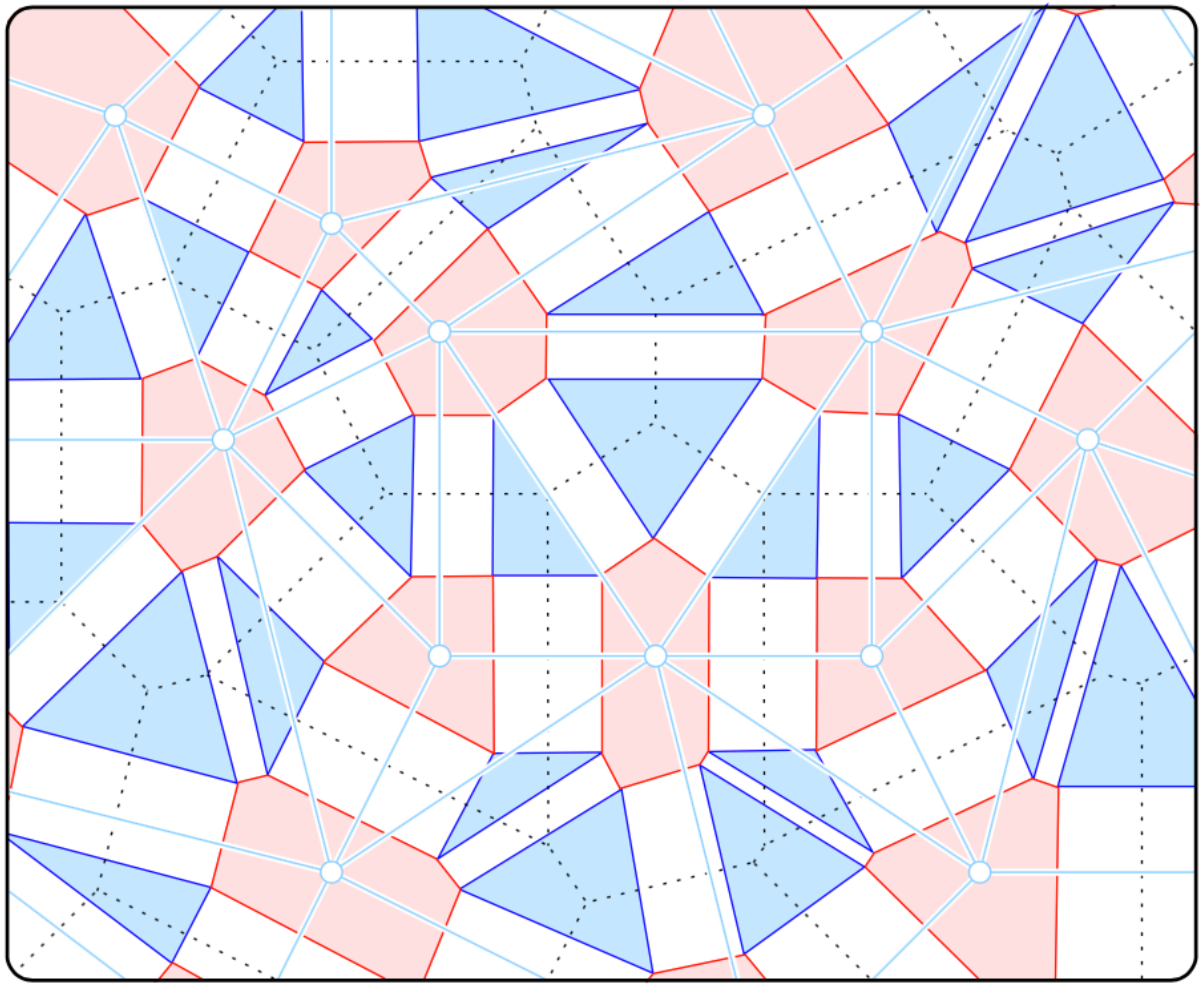


III.6 Mixed Complex



III.7 Mixed Volume

Corollary. $A \subseteq \mathbb{R}^d$ has m.r.p.

$$\sum \|\gamma\|_p \|\gamma^*\|_{d-p} = \nu_d R^d \binom{d}{p} + o(R^d)$$

$\gamma \in \mathrm{DeP}(A)$
inside $\mathbb{B}(0,R)$
$\dim \gamma = p$

as $R \to \infty$.

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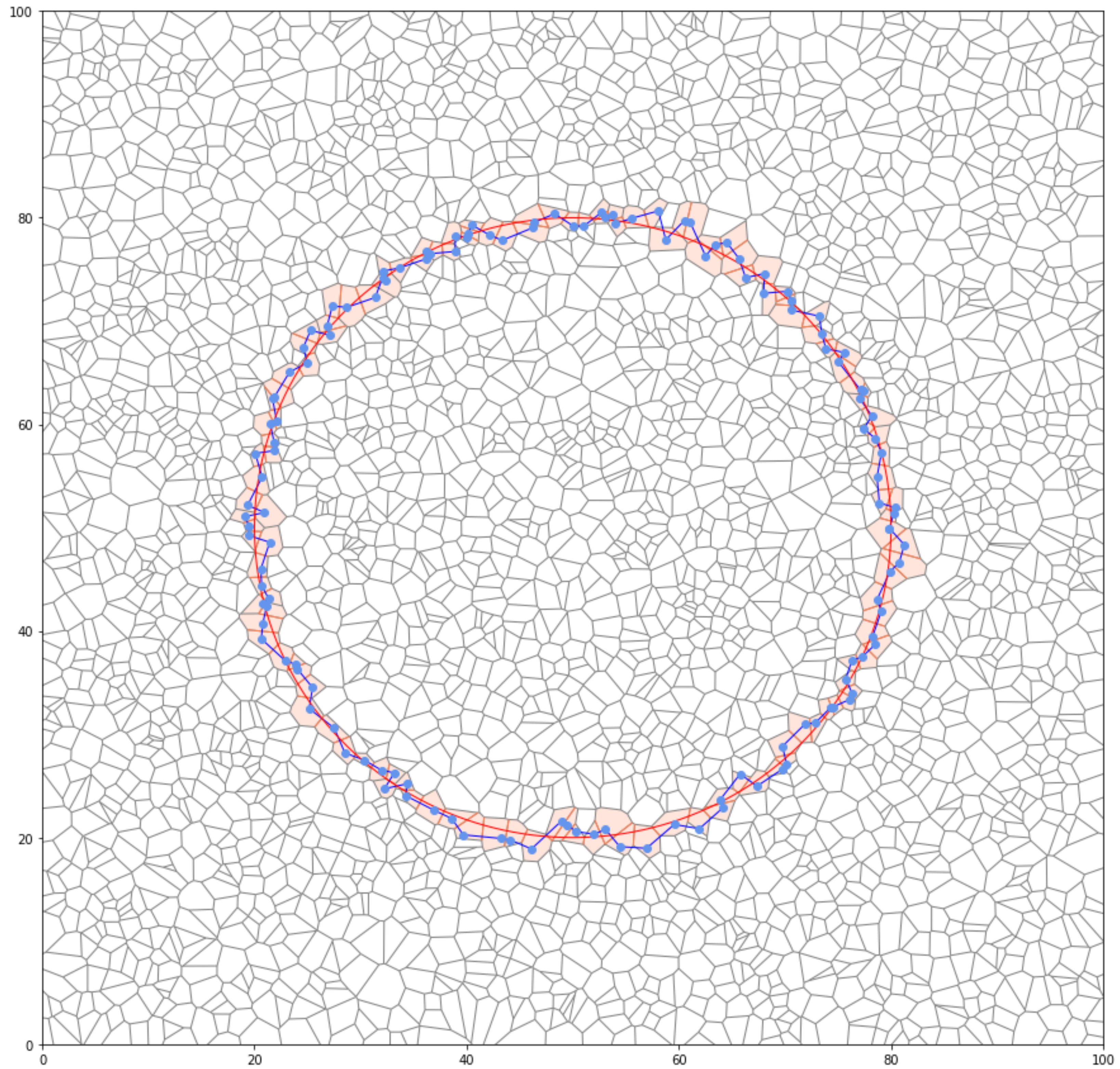
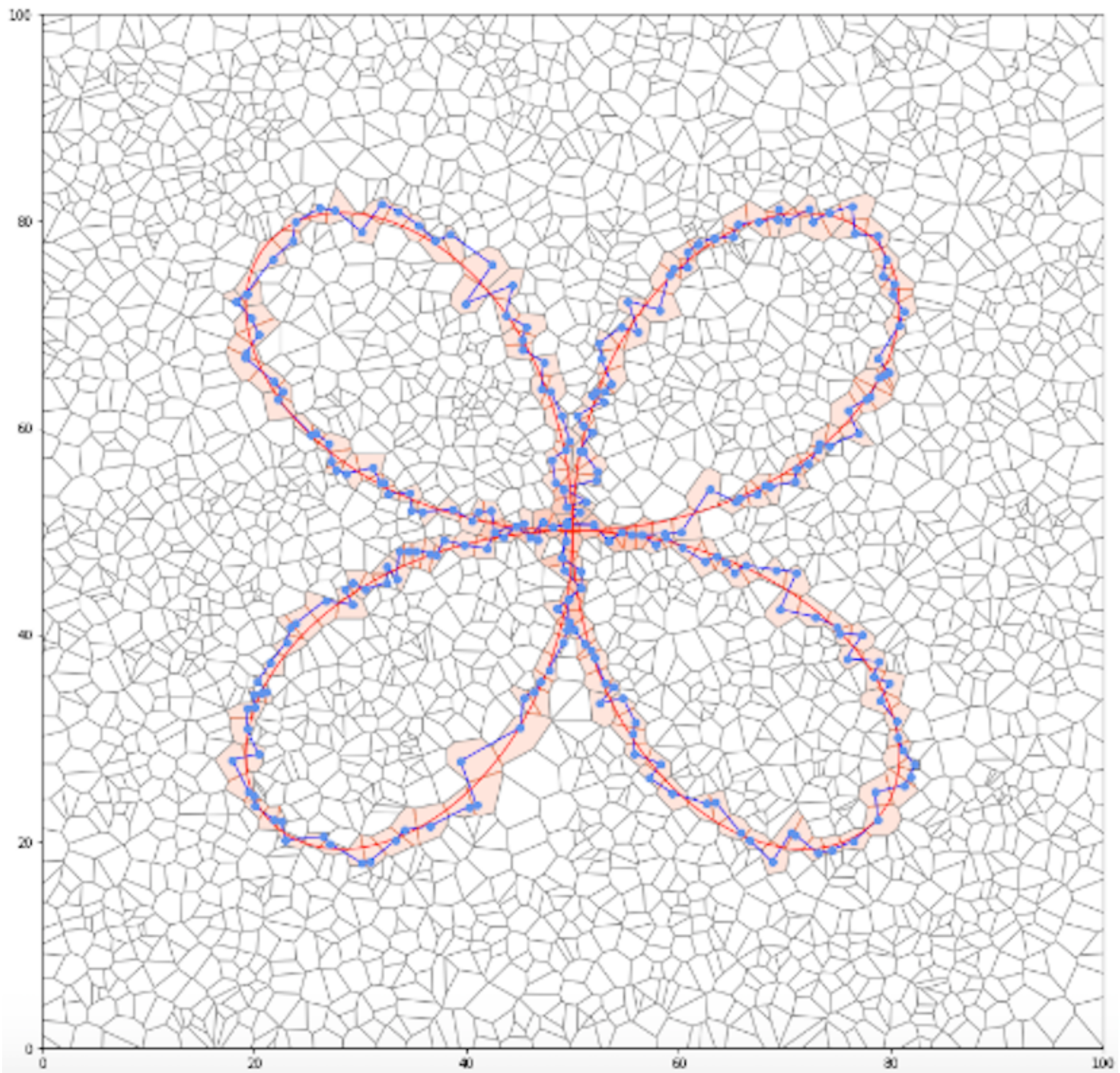
(thanks to R. Biswas, T. Ezubova)

IV.1 Circle in Plane



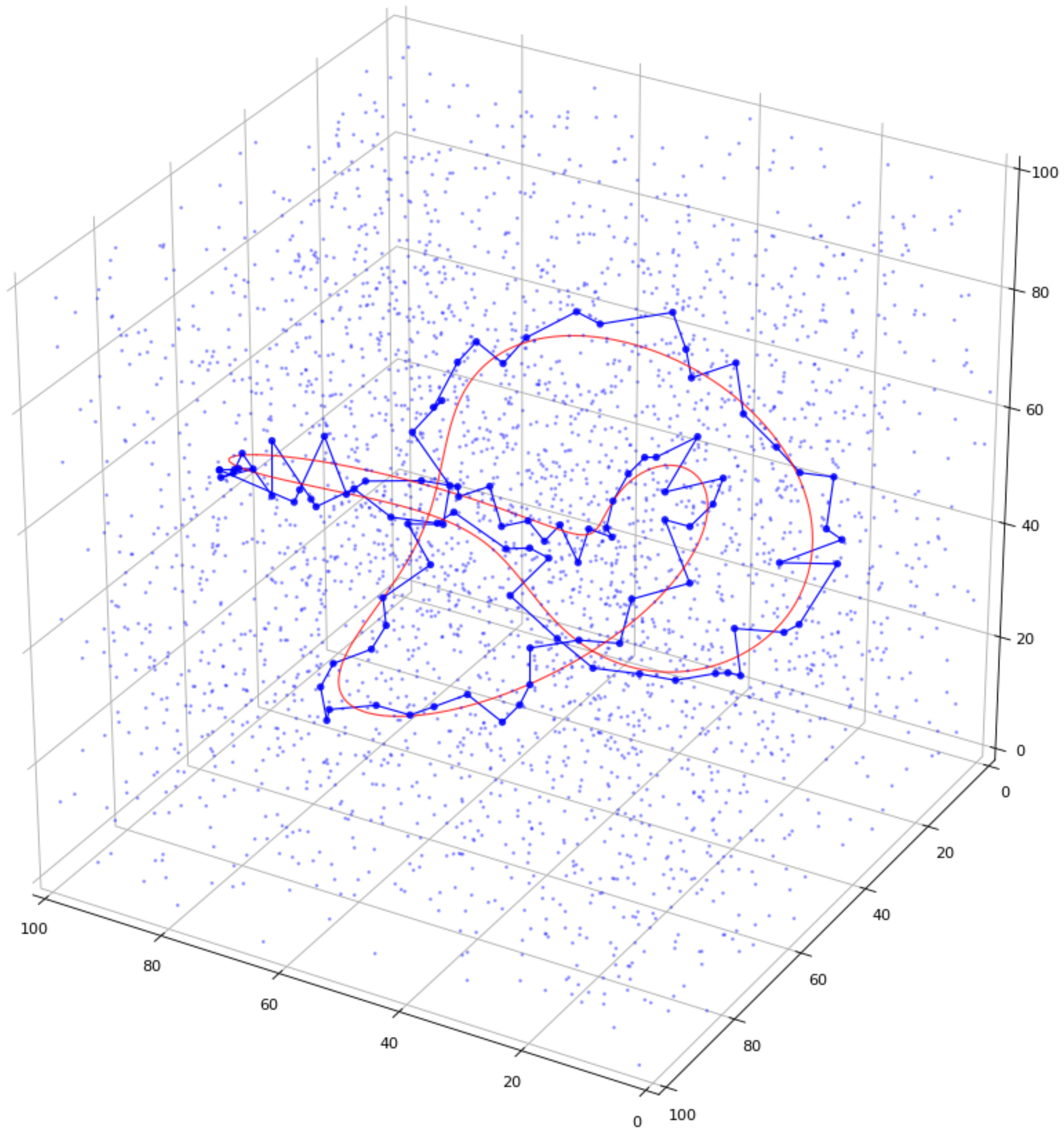
#pts	#lvs	μ	σ
96.18	0.07	1.277	0.117
766.33	0.03	1.268	0.061
6139.71	0.01	1.273	0.036
49144.97	0.01	1.273	0.023

IV.2 Clover in Plane



#pts	#lvs	μ	σ
96.18	1.16	1.265	0.163
766.35	0.34	1.272	0.072
6139.71	0.13	1.273	0.037
49144.97	0.06	1.274	0.022

IV.3 Knot in Space



#pts	#evs	μ	σ
807.80	0.48	1.504	0.179
3 230.77	0.33	1.506	0.129
12 930.16	0.21	1.500	0.096
51 705.84	0.12	1.500	0.071

Thank You