Braiding Vineyards

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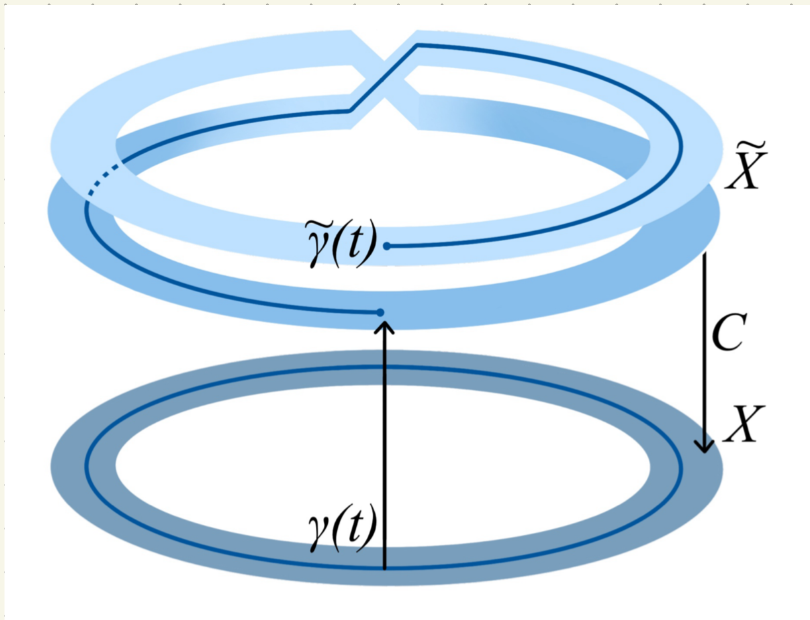
Motivation

Better understand the complexity of vineyards in TDA.

Monodromy: effect where loops in a base space don't lift to loops in a covering space

Consider space X, loop γ, covering $\tilde{X}$, and a lift $\tilde{\gamma}$.

If $\tilde{\gamma}(0) \neq \tilde{\gamma}(2\pi)$, then γ exhibits monodromy.

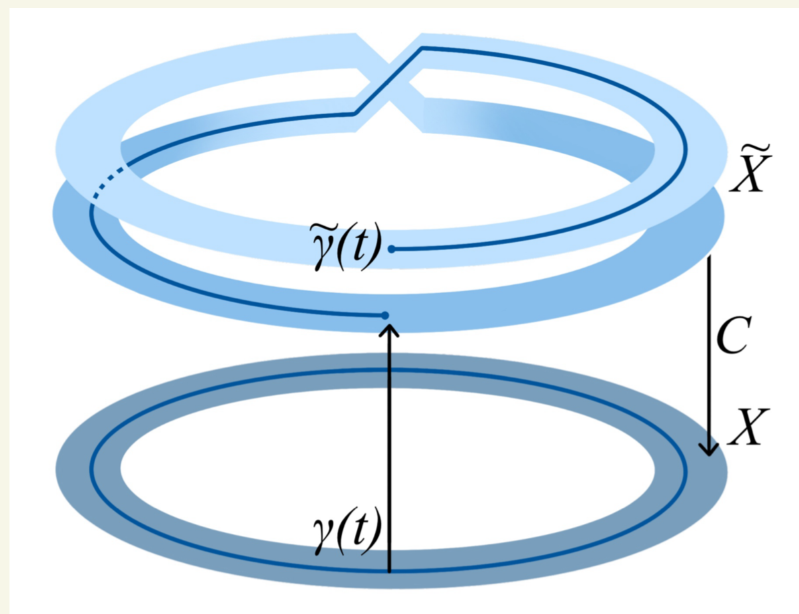


Quantifying monodromy

We say γ has monodromy of order k if k repetitions of the covering loop returns to the starting point, + k is minimal such value!

$$\tilde{\gamma}(0) = \tilde{\gamma}^k(k 2\pi)$$

$$= \underbrace{\tilde{\gamma} \circ \cdots \circ \tilde{\gamma}}_{k \text{ times}}$$



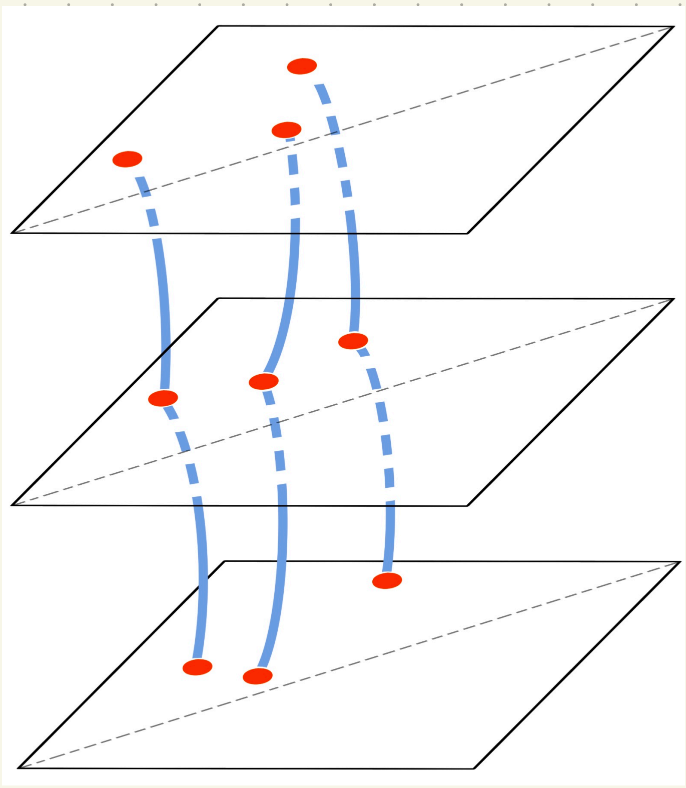
Here: $k=2$

Vineyards

We know persistence is <u>stable</u>.
Cohen-Steiner et al

This means if we take a one parameter family of filter functions, "nearby" functions will have "nearby" diagrams.

So, can match diagrams (via bottleneck or interleaving or whatever) and trace points as f_t changes:



vines & vineyards

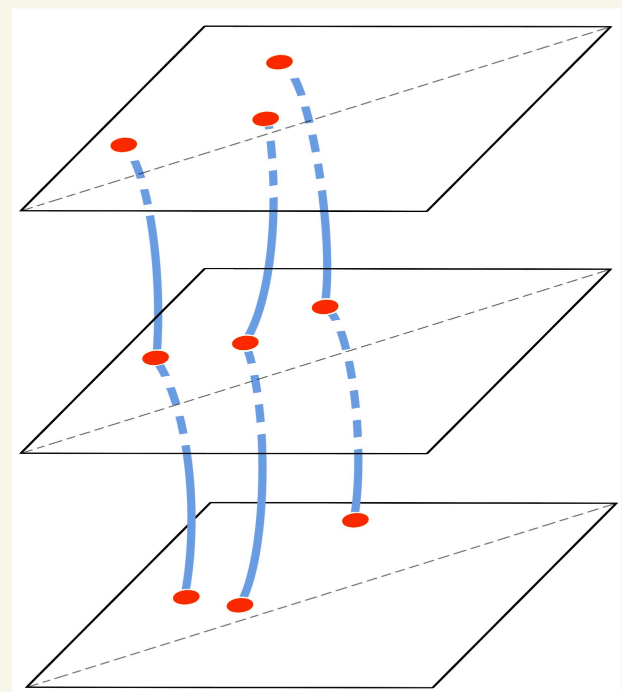
Some necessary assumptions

Genericity + vineyards:

For simplicity, I'll assume no points on our diagrams have multiplicity > 1, and no vines touch the diagonal.

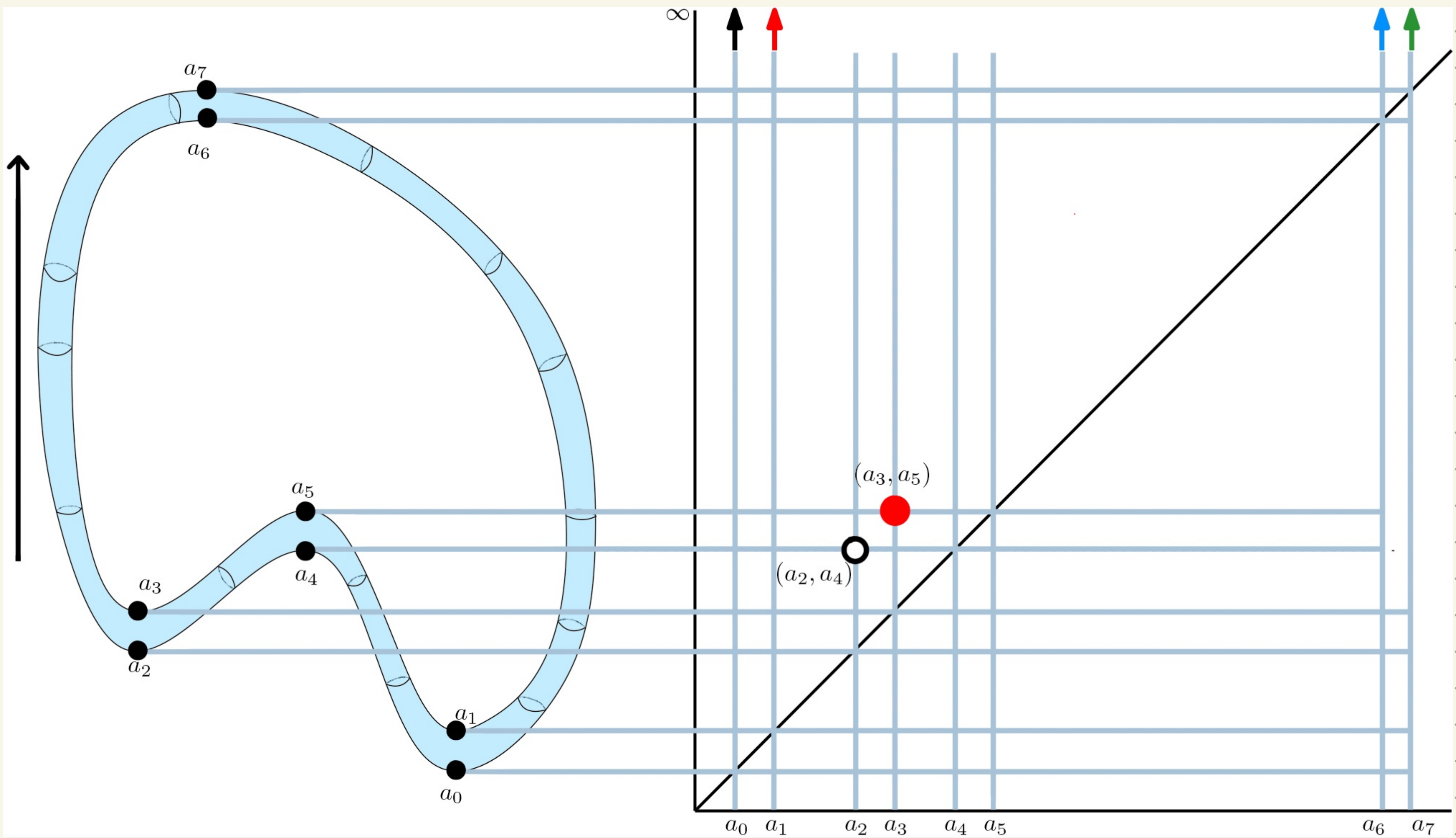
(First is true generically given mild assumptions) Turner 2023

- This suffices for our construction
- Not hard to remove the diagonal assumption.
- For multiplicity > 1, possible but more complex...



Extended persistence

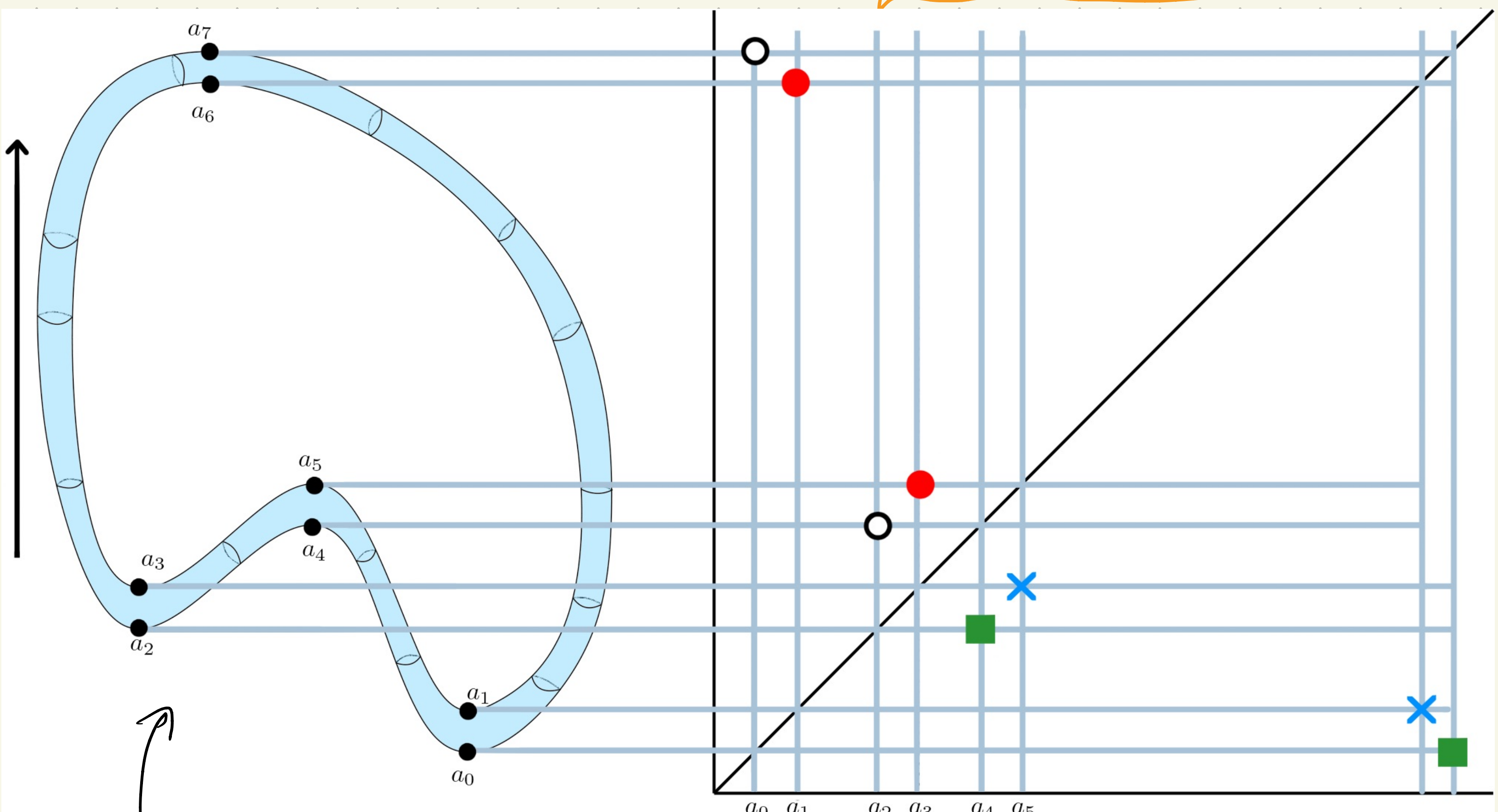
Since we're trying to "braid" vines, we don't want points at ∞:



Solution

In addition to standard sublevel set persistence, we'll also do relative homology of superlevel sets.

Cohen-Steiner et al 2008



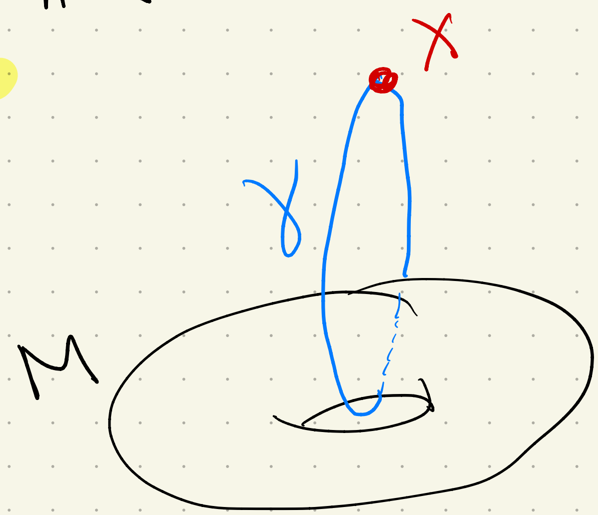
dim $\mathbb{R}^{e+2}$

- ● H_0 (oridary & extended)
- ● ordinary H_1
- ■ relative H_{e+1}
- × relative H_1

Radial Distance function

Let $d(\cdot, x): M \to \mathbb{R}$ be the distance from any $x \in \mathbb{R}^d$ to $M \subset \mathbb{R}^d$

↳ called radial distance fcn



Fix a loop $\gamma : [0, 2\pi] \to \mathbb{R}^d$

↳ observation loop

Set $x = \gamma(t)$ for $t \in [0, 2\pi]$

↳ family of filtrations $d(\cdot, \gamma(t))_M$

$\mathrm{Dgm}(d(\cdot, x)_M)$.

Closed Vineyard map:

$$CV_M : S^1 \to S^1 \times \mathrm{Dgm}$$

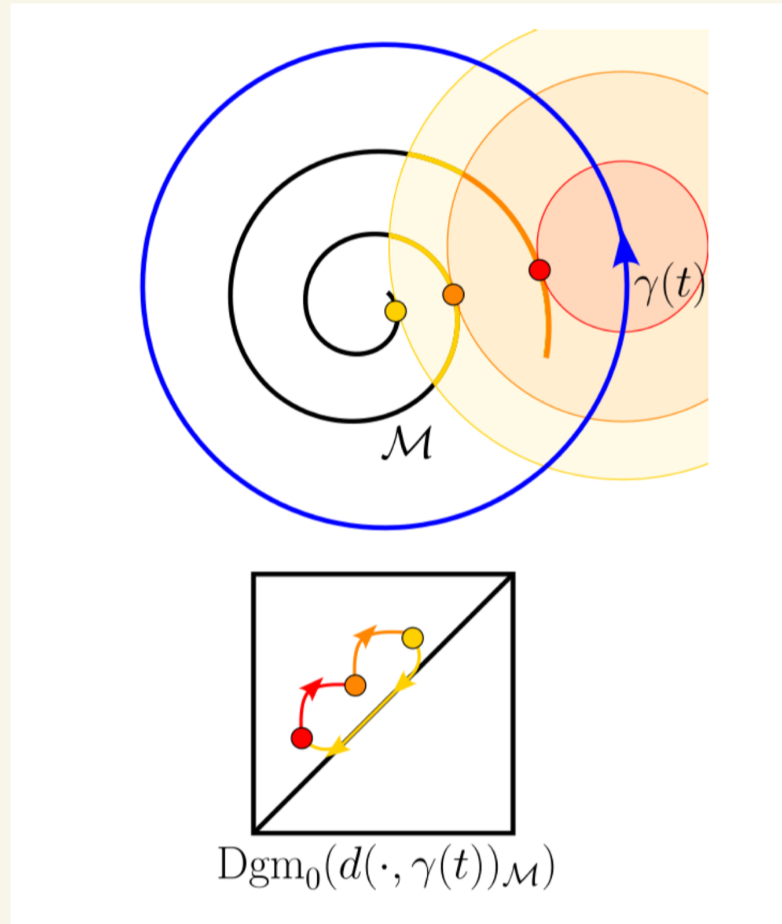
$$t \mapsto (t, \mathrm{Dgm}(d(\cdot, \gamma(t))_M))$$

Monodromy

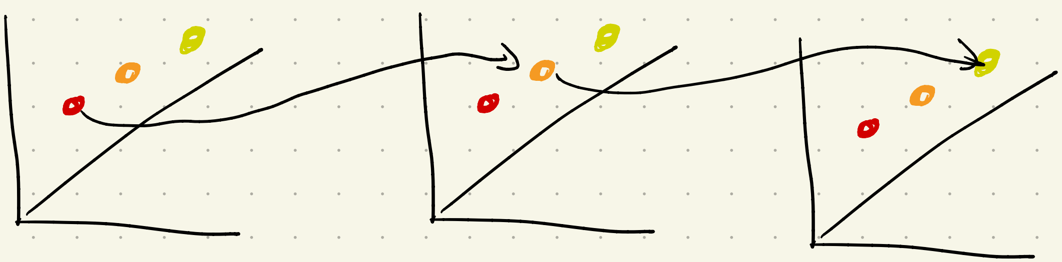
adapted from Arya et al 2024

In this example, the vines induce a map from $\mathrm{Dgm}(d(\cdot, \gamma(0))_{\mathcal{M}})$ to itself

↳ permutes the points!



Vineyard:



Here, we have monodromy of order 3.

Previous Work

- Monodromy was first studied by $\boxed{\text{Cerri et al 2013}}$, in the context of multiparameter persistence
  ↳ first example in TDA that we're aware of
- Also by $\boxed{\text{Scaramuccia \& Montain 2025}}$
- $\boxed{\text{Arya et al 2024}}$ show monodromy in H_0 for shapes in $\mathbb{R}^2$ for <u>directional</u> transforms, where filtration is based on height functions, & prove star-shaped objects have <u>no</u> monodromy in $\mathbb{R}^2$

$\boxed{\text{Arya et al 2024}}$

Can we demonstrate monodromy in higher dimensions?

Which objects have it, & how much?

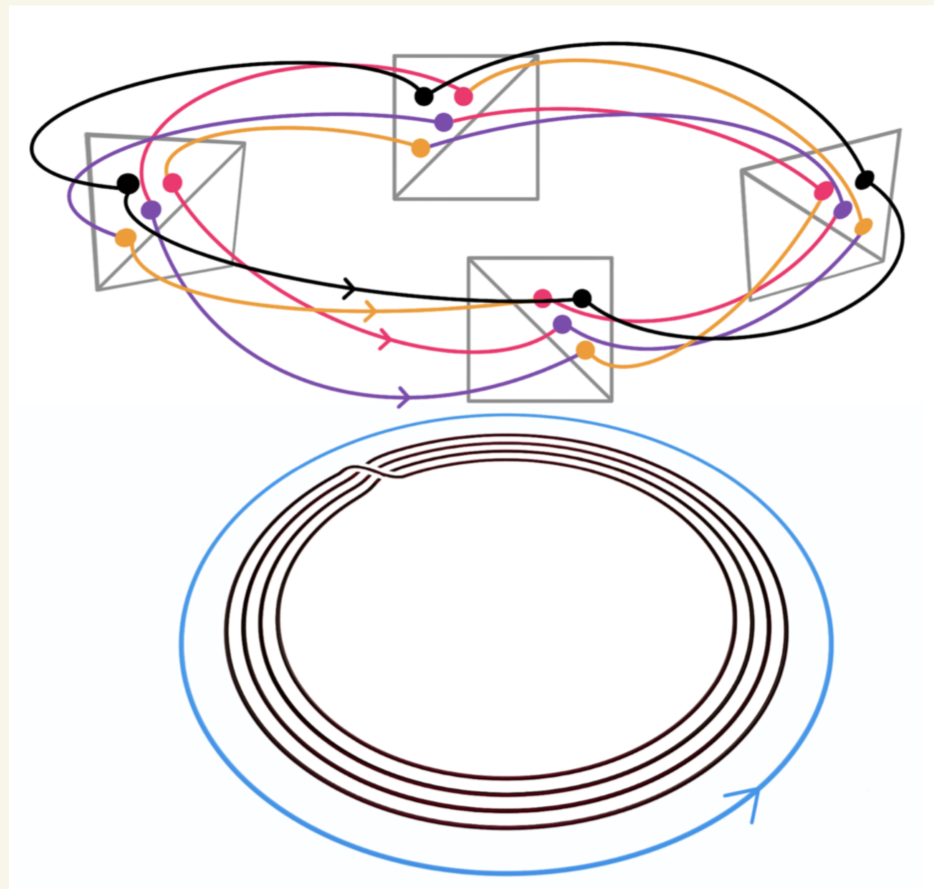
Theorem $\boxed{\text{C., Fillmore, Stephenson, Wintraecken}}$

Monodromy of any order k can be created in the ℓ-vineyard of the radial transform of a manifold M embedded in $\mathbb{R}^{\ell+1}$

Our idea

We know the radial vineyard can reflects parts of the topology of input space M.

If M were a braid, with period $2\pi k$, we could get order k monodromy.

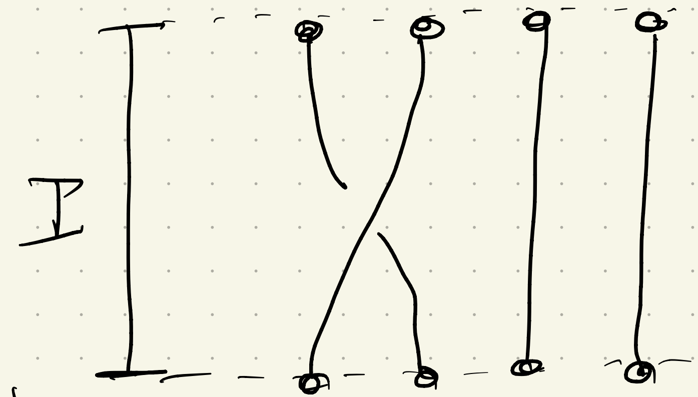


Braids have previously connected to monodromy:

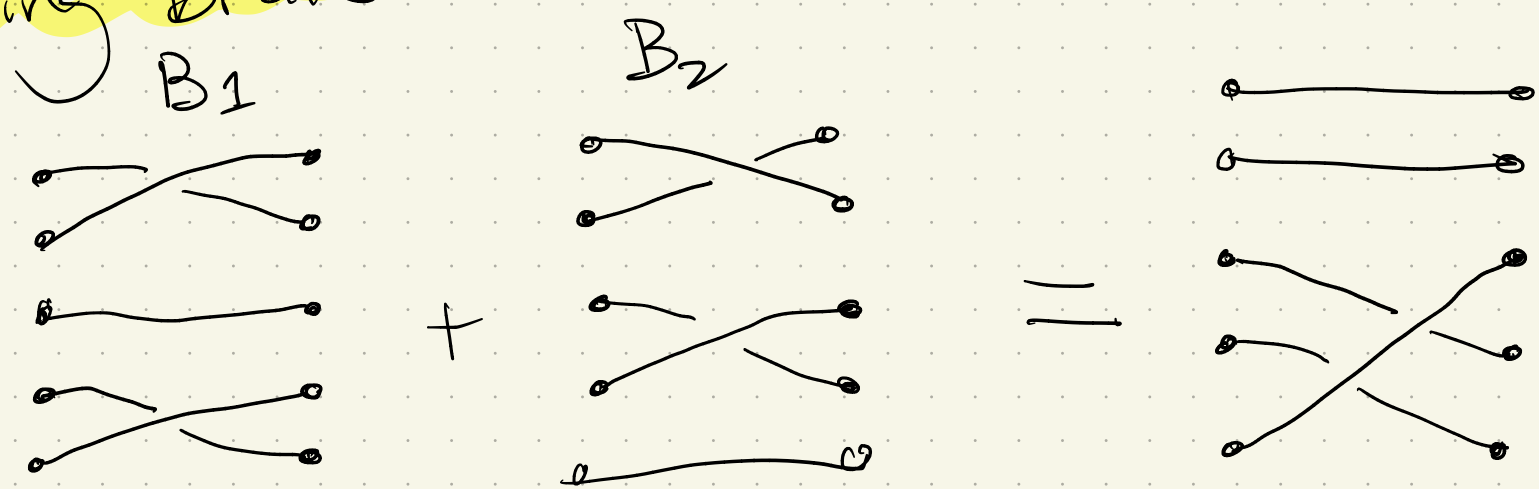
Cohen + Suciu 1997, Cogolludo-Agustín 2011, Salter 2023, Salter 2024

Braids

A braid on m strands is the equivalence class of the disjoint union of m intervals $B_i : I \to D^2 \times I$, monotonically inrecsing wrt I, such that end points are a permutation of start points, under ambient braid isotopy.



Composing braids:

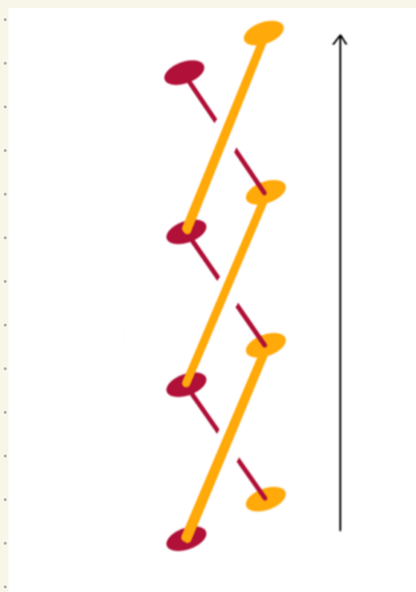


Braids & Knots

If we identify endpoints of a braid & map canonically to torus in $\mathbb{R}^3$, we get a closed braid.

Theorem $\boxed{\text{Alexander 1923}}$

Every knot or link is equivalent to a closed braid.

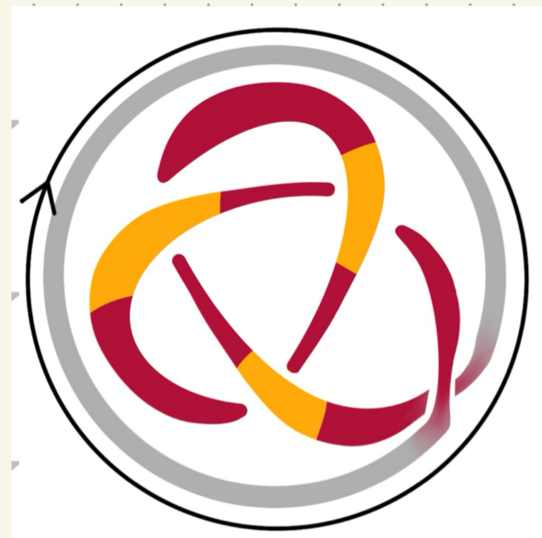


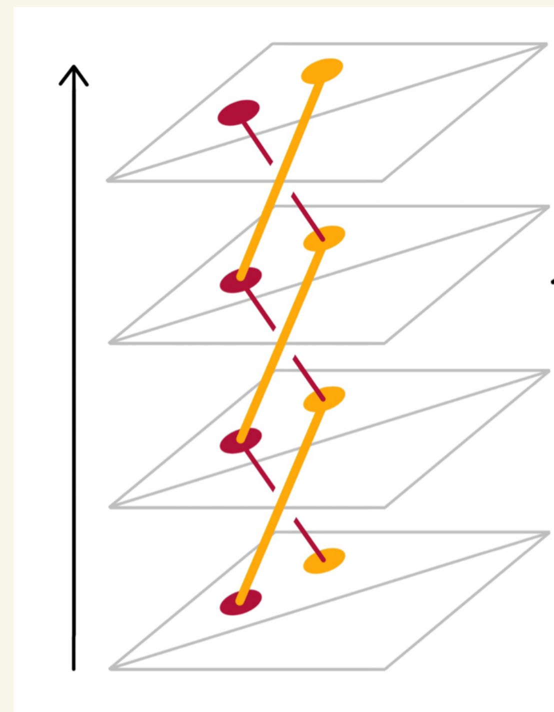
$\Longleftrightarrow$

Question: can vineyards "braid"?

Theorem ($\boxed{\text{C.-Fillmore-Stephenson-Wintraecken}}$)

Given any braid B, there exists a manifold $M \subset \mathbb{R}^d$ & a closed curve $\gamma \subset \mathbb{R}^d$ such that identifying the ends of the vineyard of $d(\cdot, \gamma(t))_M$ will yield a braid B' which is equivalent to B after removing spurious components.





Construction

Note that both M and γ must be carefully constructed to work together!

Overview:

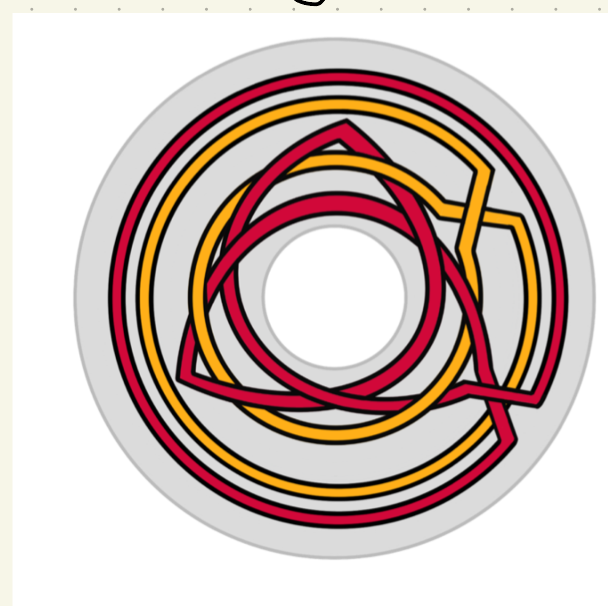
Start with a closed braid $B \subset \mathbb{R}^2$, with k components and s strands.

[Note: Not showing the braid here, but $k=2$ and $s=3$]

We'll convert this to a manifold in $\mathbb{R}^3$ ⟶

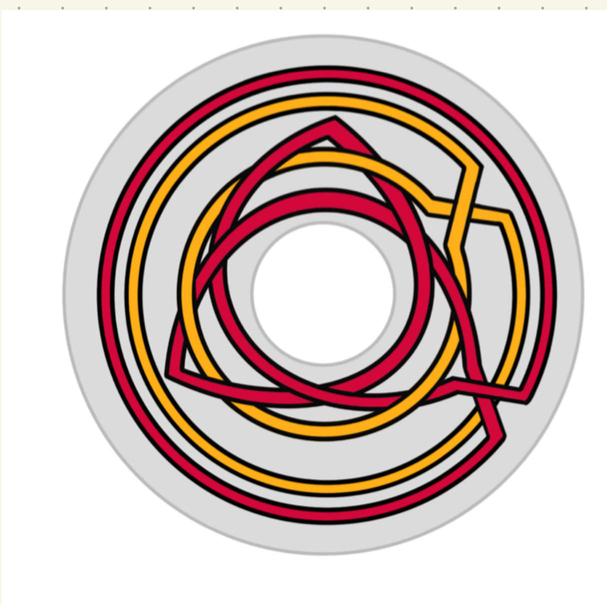
① Redraw in a small neighborhood of an annulus, where strands follow fixed radii

② Then, introduce an extra "twist" per component, & wrap loop around outside of annulus

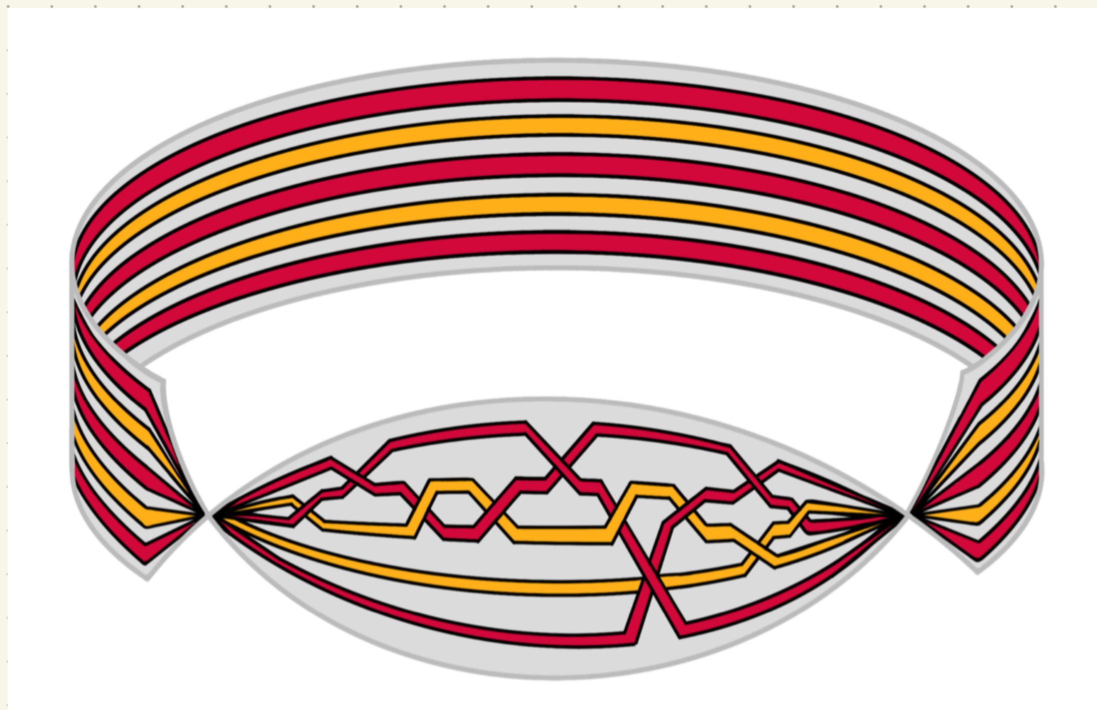


$\Rightarrow$ adds $O(sk)$ crossings and gives $n = s + k$ strands total

3. Deform B so all crossings occur in fraction of annulus, at regular intervals
4. Twist annulus in 3d, so non-crossing part is orthogonal. (Note: no new crossings!)
5. Set observation loop to follow annulus at fixed distance & near "center"



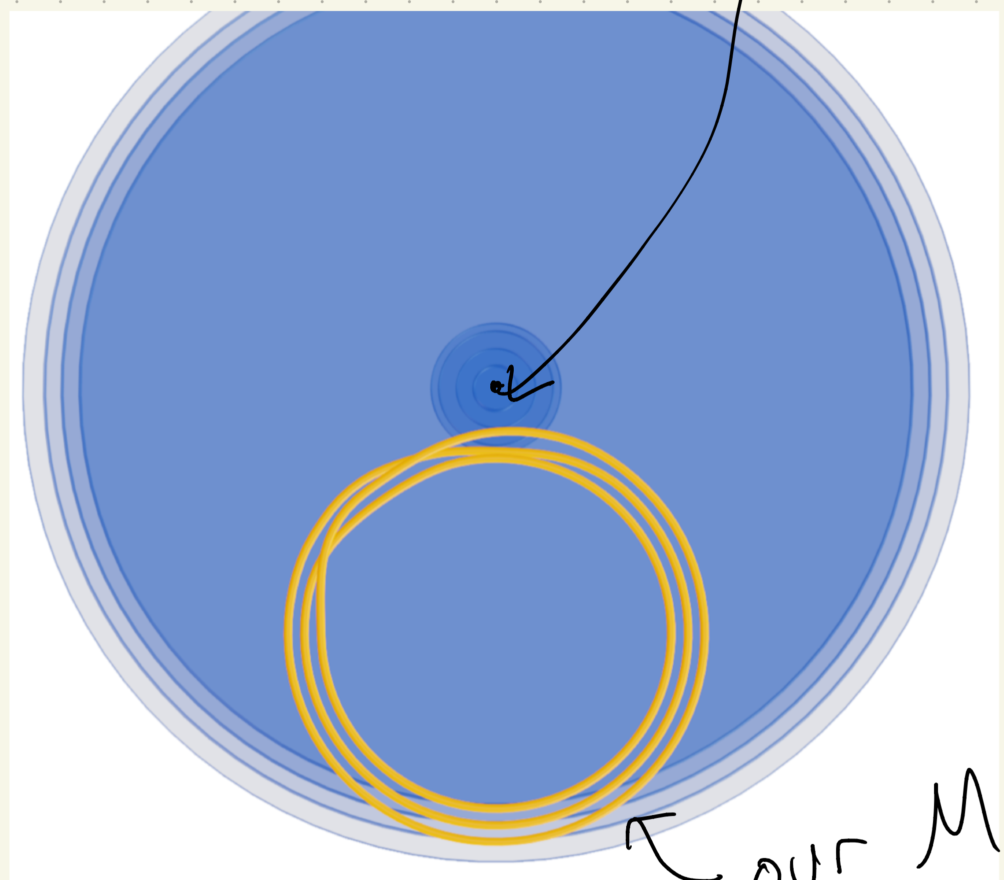
$\Rightarrow$



What is vineyard like?

Several technical lemmas relying on Morse theory + angles show that births in H_0 all occur before deaths!

This means we can use embedding to control vineyard diagrams..

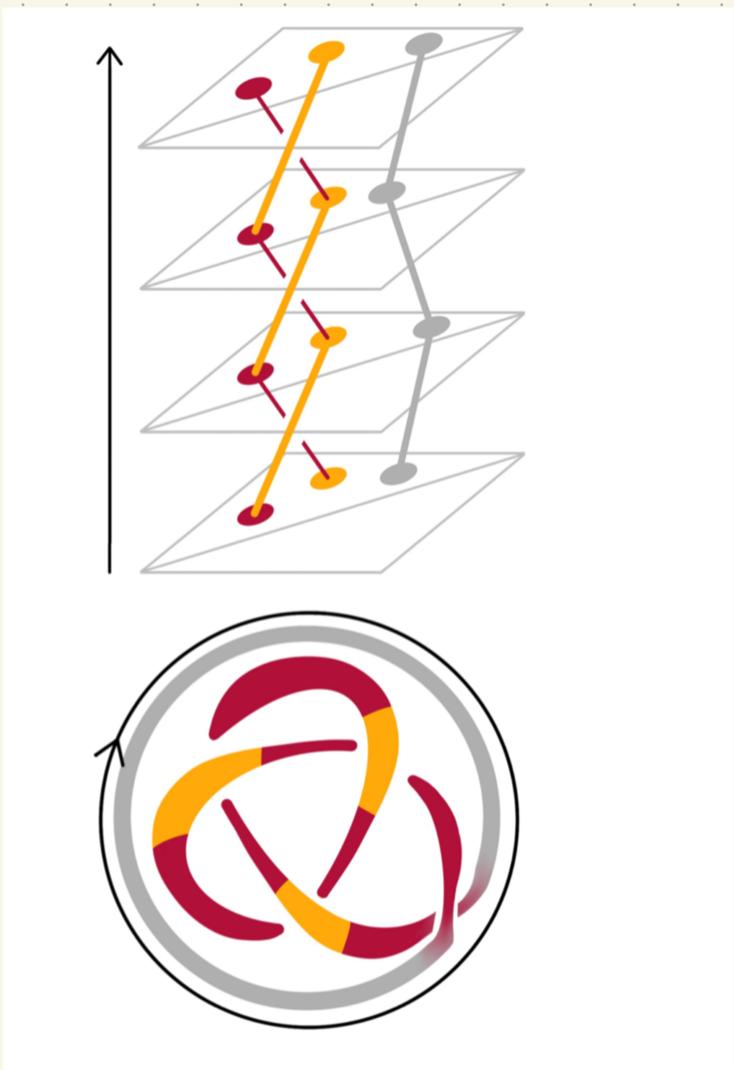


Why the extra strands?

Elder rule in persistence
⇒ first birth + last
death are paired.

We added an outer strand
to account for this.

Result: For each component,
will be an unlinked
strand, which will be
an extra circle in
vineyard.

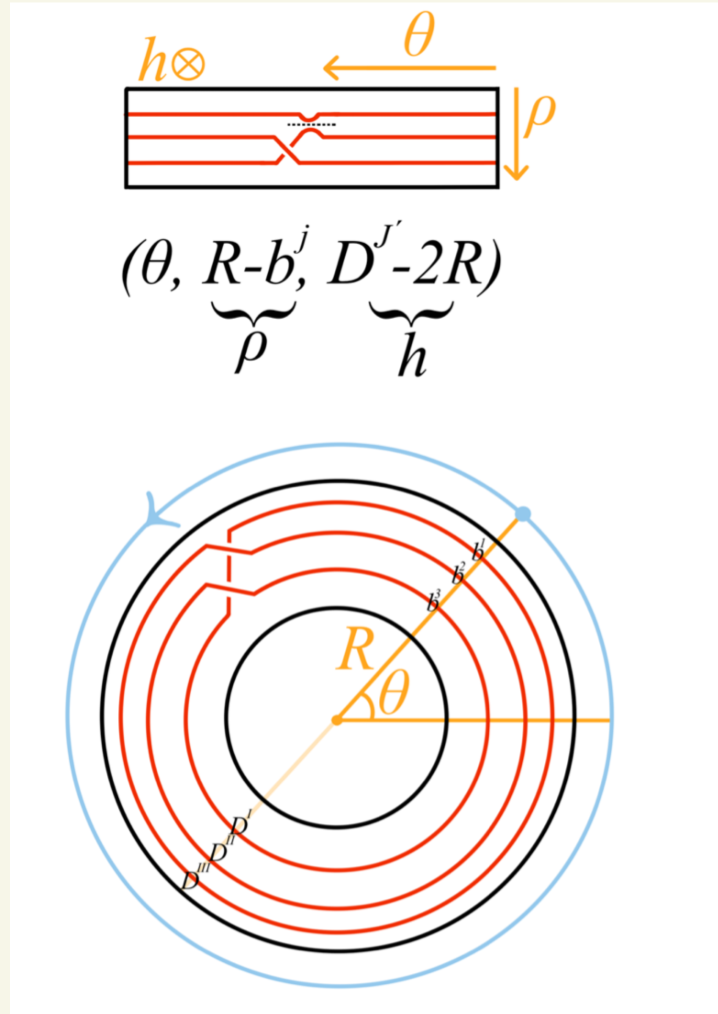


Parameterizing the vineyard

Our vines have annular coordinates:
θ, h & ρ

Away from crossings, vines have distinct ρ so h doesn't matter.

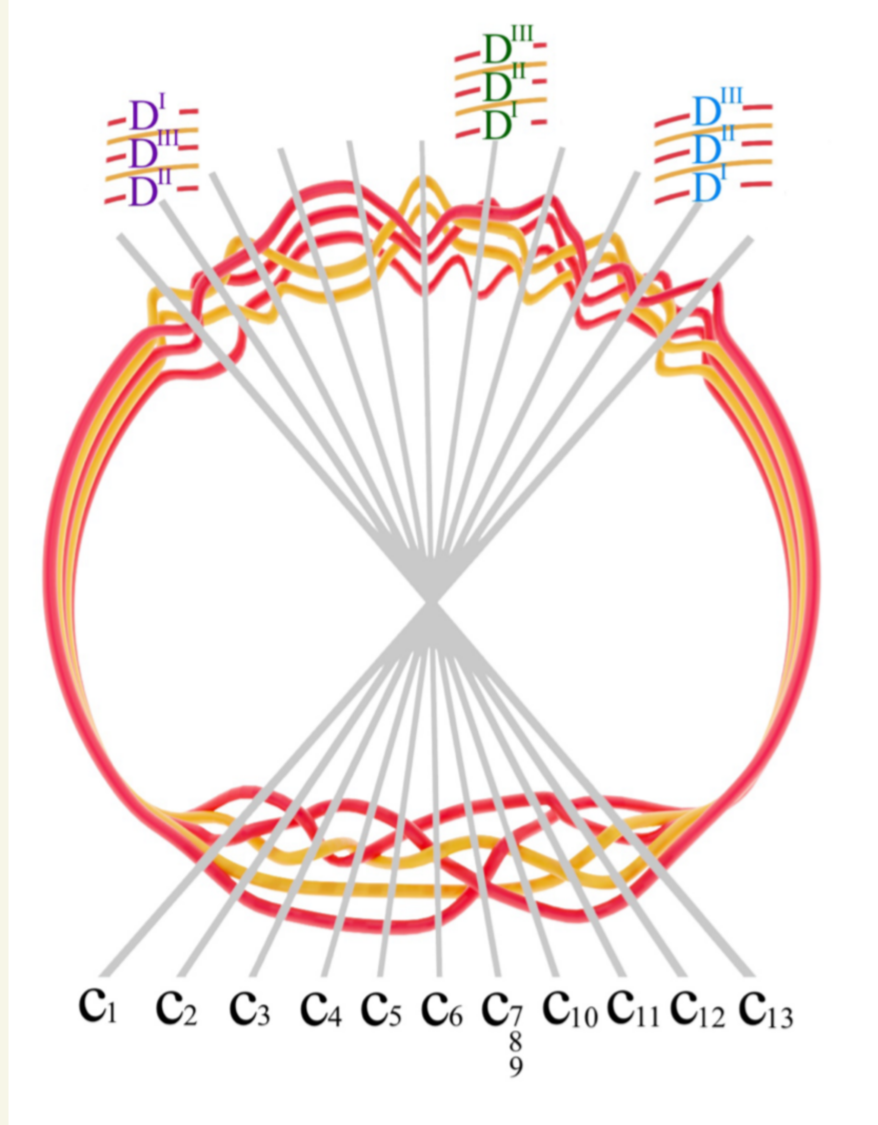
To get correct over/under crossings in vineyard, we need to play with the geometry of our embedding a bit...



Pushing & Pulling

Our construction ensured that crossings are evenly spaced, & all deaths occur at opposite points.

Therefore, we perturb deaths to make sure vineyard has correct crossings.



(Note: hiding some intense calculations here...)

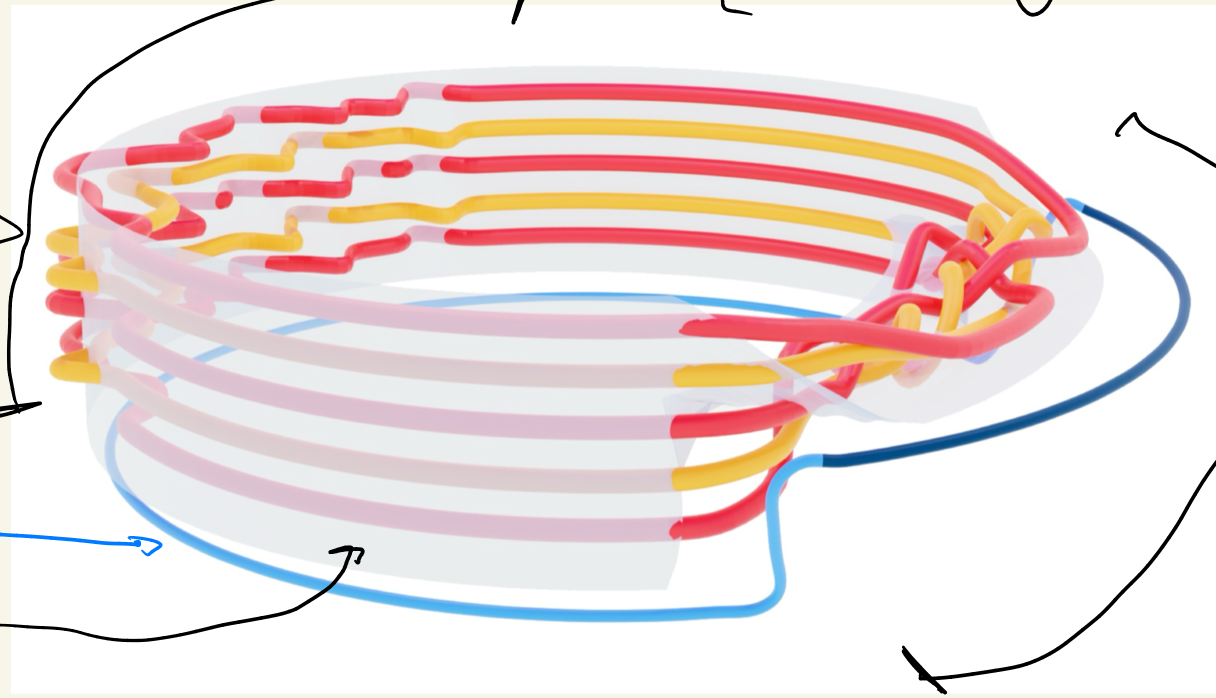
Final result: braids

all {crosings}

Pushed & pulled deaths

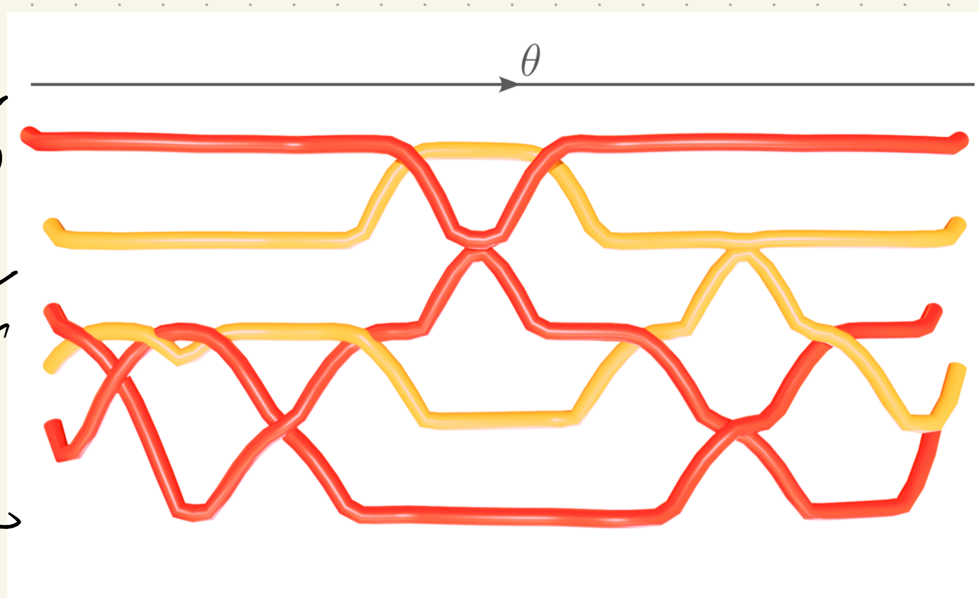
observation loop

base annulus



unlinked outer loops

braid



Vineyard of dark blue portion of the observation loop

Extending to ℓ-persistence in $\mathbb{R}^d$

To have this calculation work for higher dimensions, take $(\ell+1)$-dimensional α-offset of braid $B \times O$ in $\mathbb{R}^3 \oplus \mathbb{R}^{\ell-1} < \mathbb{R}^d$.

↳ Given this manifold construction (& same base loop), vines in our ℓ-vineyard will be α-close to the 0-vineyard.

Some takeaways

- Vineyards can be messy:
  ↳ in particular, comparing them is at least as hard as knot recognition
- The radical persistent transform may be more promising and richer to study than the more standard directional transform
  – see Onus et al 2024
- Similarly, using extended persistent homology allows for interesting insights into the vineyards.
  – see Turner et al 2022

Future work

- The radial transform has interesting connections to more traditional transforms
  ↳ medial axis Edelsbrunner, Stephenson & Thoresen '25
  & symmetry sets Bruce & Giblin
- Do knots appear in the radial transform of real data?
- Can we apply knot/link invariants to vineyards?
  ↳ or (perhaps too optimistic) can we apply statistical techniques on vineyards to braids?