Geometric construction of Kashiwara crystals on multiparameter persistence

Yasu Hiraoka
(joint work with Kohei Yahiro)

Kyoto University
&
Inamori Research Institute for Science

Content

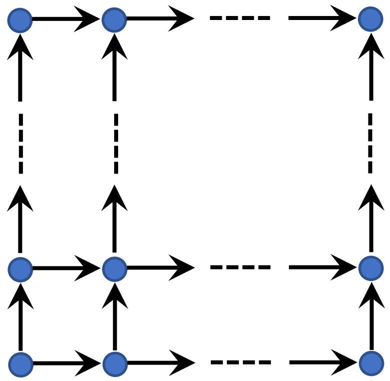
1. Irreducible decompositions of varieties of multi-persistence
   (not indecomposable decomposition)

2. Show good combinatorial/geometric properties on irreducible decomposition, called Kashiwara crystal

Today's talk is more focused on geometric representation theory than on persistence theory

Multi-parameter persistence

- 2-parameter persistence module



rep on a bound quiver

: commutative relations

(Krull-Schmidt) : indecomposable,

- Problem: is too big (containing non-intervals w/ continuous parameters), i.e., indecomposable decomposition (KS) is too fine a classification to characterize multi-parameter persistence.

- One approach: interval approximation, generalized persistence, signed barcode (Asashiba et al, Kim-Mémoli, Botnan-Oppermann-Oudot)

Representation variety

- Dimension vector

- Representation variety
  (affine algebraic variety)

  polynomial

  Ex)

  Here,

  quadratic polynomials

- Group acts on (as changes of bases)

  e.g., Carlsson & Zomorodian (2009) study classification & parameterization of multi-persistence by applying geometric invariant theory to under this group action.

Irreducible decomposition

- A nonempty algebraic set is irreducible

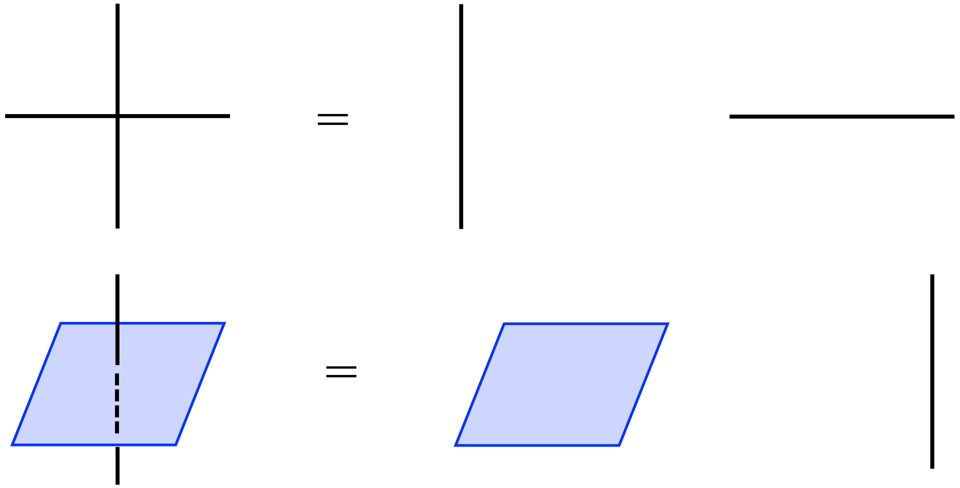
  by algebraic sets implies or

- Irreducible decomposition: Every algebraic set can be uniquely expressed as

  (finite union)

  where are irreducible and for . Each is called an irreducible component.

  Ex)



- : The set of all irreducible components in .
- The main result of this talk shows a combinatorial structure (Kashiwara crystal) on .

Kashiwara crystal

: A bound quiver, : undirected graph of , : adjacent matrix of

- A generalized Cartan matrix , : # of vertices
- Root data of
  - : complex vector space, (: simple root)
  - are linearly independent - -

weight lattice

- A Kashiwara crystal is a datum (crystal basis)

Kashiwara operators

for such that

(1)

(2)

(3)

(4)

Remark on geometric representation theory

- quiver Cartan matrix Lie algebra

  quantum group

- quiver Ringel-Hall algebra over (with)

  [ ]

  Theorem 1 (Ringel): for

  Theorem 2 (Lusztig): for arbitrary and

  : Grothendieck group of a category of perverse sheaves of a rep variety (i.e., geometric generalization of Theorem 1)

- quiver Kashiwara crystal

- quiver double quiver rep variety (under preprojective relations)

  the set of irr components of ,

  Theorem 3 (Kashiwara-Saito): (i.e., geometric construction of the crystal)

Kashiwara crystal of

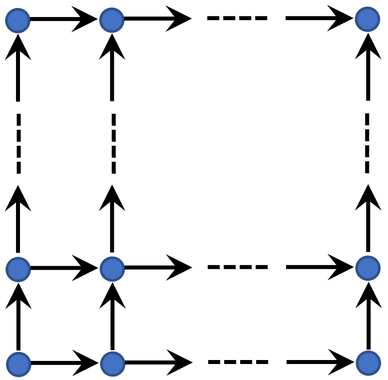
#1. Can we construct a crystal of on in a similar way to Kashiwara-Saito?
(Do irr. decompositions in multi-persistence have good combinatorial structures?)

Note: Not preprojective relations, but commutative relations

#2. Can we construct a quantum group of which gives as its crystal?

Main result (answer to #1)

, dimension vectors



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(1) short exact for

(2)

(3) for

i.e., the moduli variety of all extensions of persistence modules

- For , the Kashiwara operators are defined by

Theorem (H. and Yahiro, arXiv:2504.14844): give a Kashiwara crystal.

Kashiwara crystal of 1-parameter persistence

- For each , is a vector space, hence irreducible.

Theorem:

Intuition of :

- Given a generic , we study whether we can construct the left comm. diagram.
-
- For , hence 0.
- For , comm. diagram w/ generic .

Irreducible components of 2 x 2 persistence

Prop: For ,

.

For ,

.

Theorem:

Intuition of the Kashiwara operators in 2 x 2 persistence

- Generic rep of : ,

- :

generic rep

Future work

- Can we construct a quantum group of multi-persistence?
    - quantum group of the bound quiver giving the Kashiwara crystal

- From representation variety to indecomposable decompositions of multi-persistence
    - geometric characterization of interval approximations?

- Assign (generic) invariants as irreducible components
    - discriminative power
    - crystal graph and stability

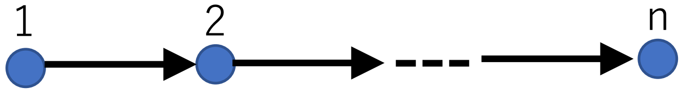
Reference: H. and Yahiro, arXiv:2504.14844

Thank you!

1-parameter persistence

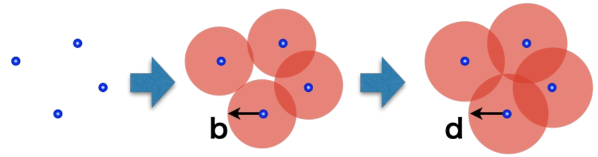
- 1-parameter persistence module:

representation on A_n-quiver



Filtration

Persistent homology



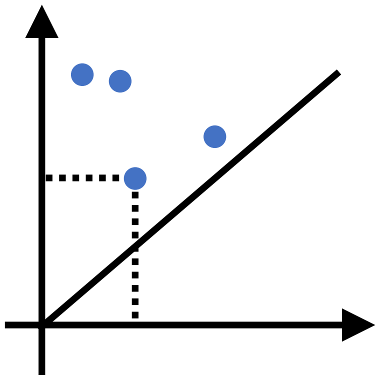
- (Krull-Schmidt) : indecomposable,

(Gabriel)

$$I[b,d] : 0 \to \cdots \to 0 \to \overset{\text{at}}{\mathbb{k}} \to \cdots \to \overset{\text{at}}{\mathbb{k}} \to 0 \to \cdots \to 0$$

expresses the birth & death of a generator

- Persistence diagram:



Remark on Kashiwara crystal

- Cartan matrix Lie algebra universal enveloping algebra (over)
  quantum group (over)

- A crystal basis (Kashiwara crystal) of an integrable weight module
  def
  1. -submodule generates over
  2. is a -basis of
  3. for
  4. iff
  5. ,,,,

can be viewed as a basis of at

Character formula:

- : the irr. highest weight -module, : the highest weight vector

  is a (unique) crystal basis

-

  is a crystal basis of

Intuition of the Kashiwara operators in 2 x 2 persistence

- Generic rep of : ,

- :

generic rep

not generic rep