# Bounding the Mapper Graph Interleaving Distance with a Loss Function

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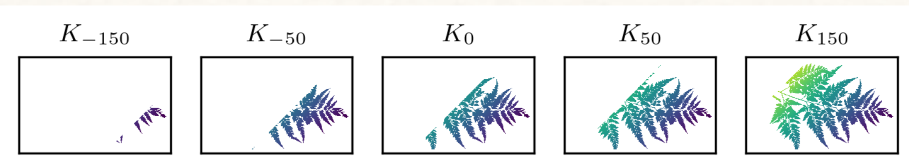
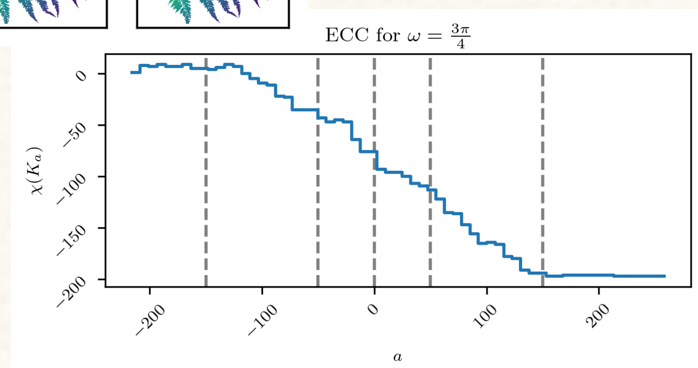
# Classes of Topological Signatures

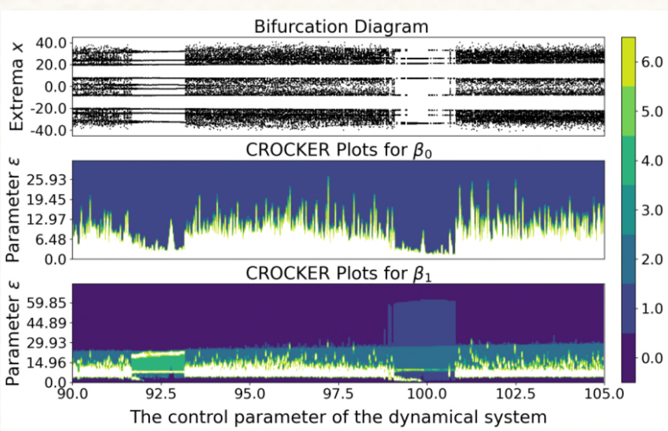
## Algebraic

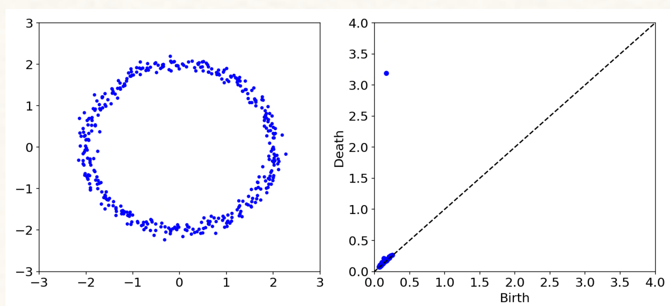



### Euler Characteristic Curve

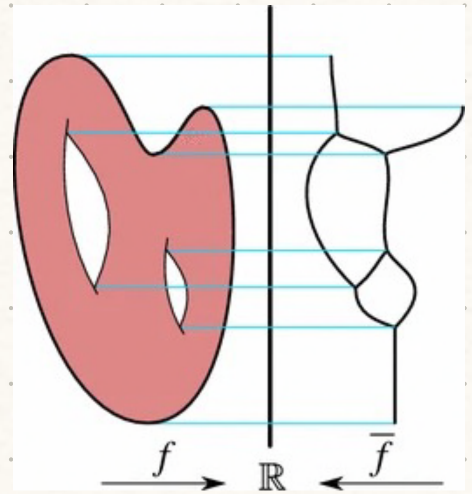



### CROCKER Plots



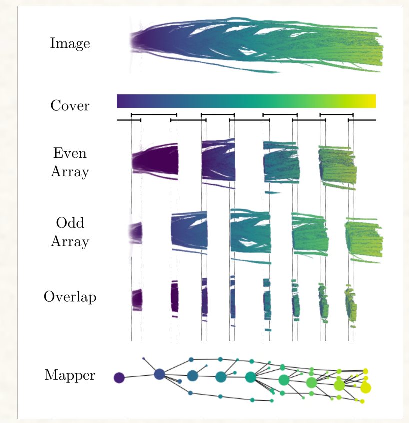



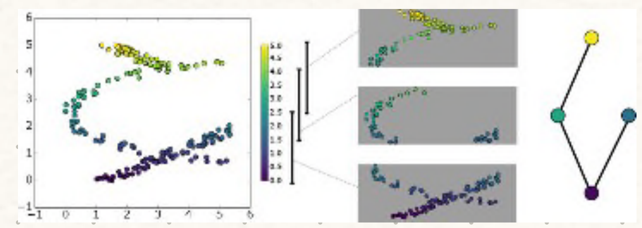

### Persistent Homology



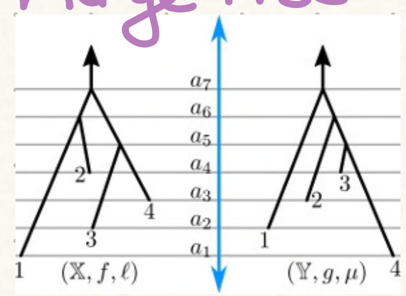

### Reeb Graph

### Mapper Graph



### Merge Tree




## Graph-Based

# Mapper Input

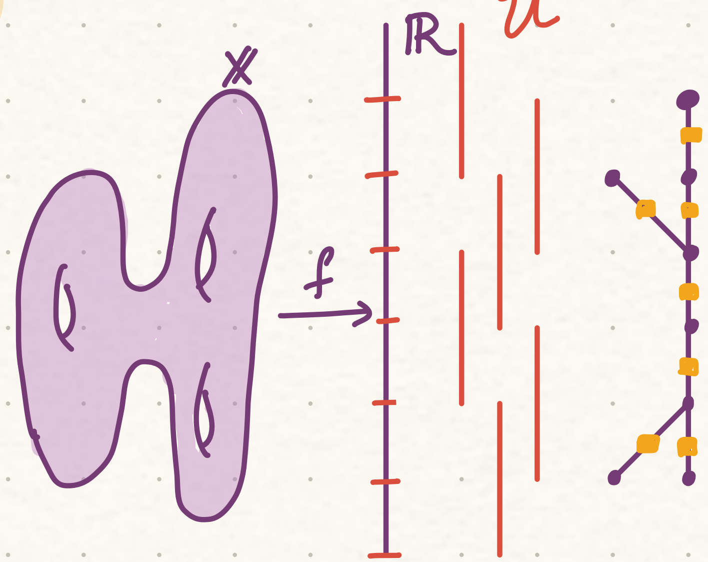
## Given data

* $f: \mathbb{X} \longrightarrow \mathbb{R}^d$
* cover $\mathcal{U} = \{U_i \subseteq \mathbb{R}^d\}$

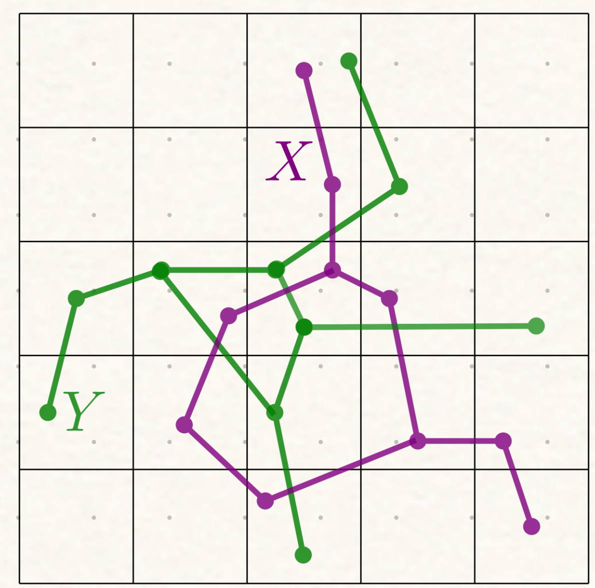
## Categorical mapper is functor

$$F: \mathrm{Open}(\mathrm{Nrv}(\mathcal{U})) \longrightarrow \mathrm{Set}$$
$$A \longmapsto \pi_0 f^{-1}(|A|)$$

$d=1$



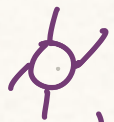

$d=2$




Singh, Mémoli, Carlsson. ***Topological methods for the analysis of high dimensional data sets and 3d object recognition.*** *Point Based Graphics at EuroCG, 2007*

# The Goal

A distance on mapper graphs

$d($  $,$  $)$

that is computable.

Distance:

$d : X \times X \longrightarrow \mathbb{R}$

- $d(x,y) = 0 \iff x = y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(x,z)$

# Interleaving Distance

Apologies for omissions
Please let me know!

## Persistence Modules

$F : (\mathbb{R}^d, \leq) \longrightarrow \mathrm{Vect}$

Chazal, Cohen-Steiner, Glisse, Guibas, Oudot, Lesnick, Blumberg, Bjerkevik, Botnan, Kerber, Landi, Bauer, Hiraoka, Scoccola, Escolar ... et al.

## Merge Trees

$F : (\mathbb{R}, \leq) \longrightarrow \mathrm{Set}$

Morozov, Beketayev, Weber, Stefanou, Touli, Y. Wang, EM

## Functor categories

$F : \mathcal{P} \to \mathcal{C}$

Bubenik, Scott, deSilva

## Zigzags

$F : \cdot \swarrow \cdot \searrow \cdot \swarrow \cdot \searrow \cdot \longrightarrow \mathrm{Vect}$

W Kim, Mémoli, Botnan, Lesnick, Bjerkevik

## Reeb Graph

$F : \mathrm{Int} \longrightarrow \mathrm{Set}$

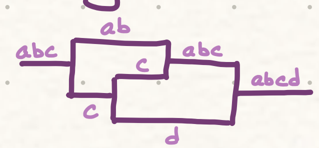
Curry, de Silva, EM, Patel, Bauer

## Category with a flow

$(\mathcal{C}, \tau : \mathbb{R} \to \mathrm{End}(\mathcal{C}))$

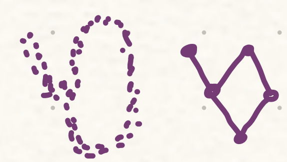
Stefanou, deSilva, EM

## Formigrams



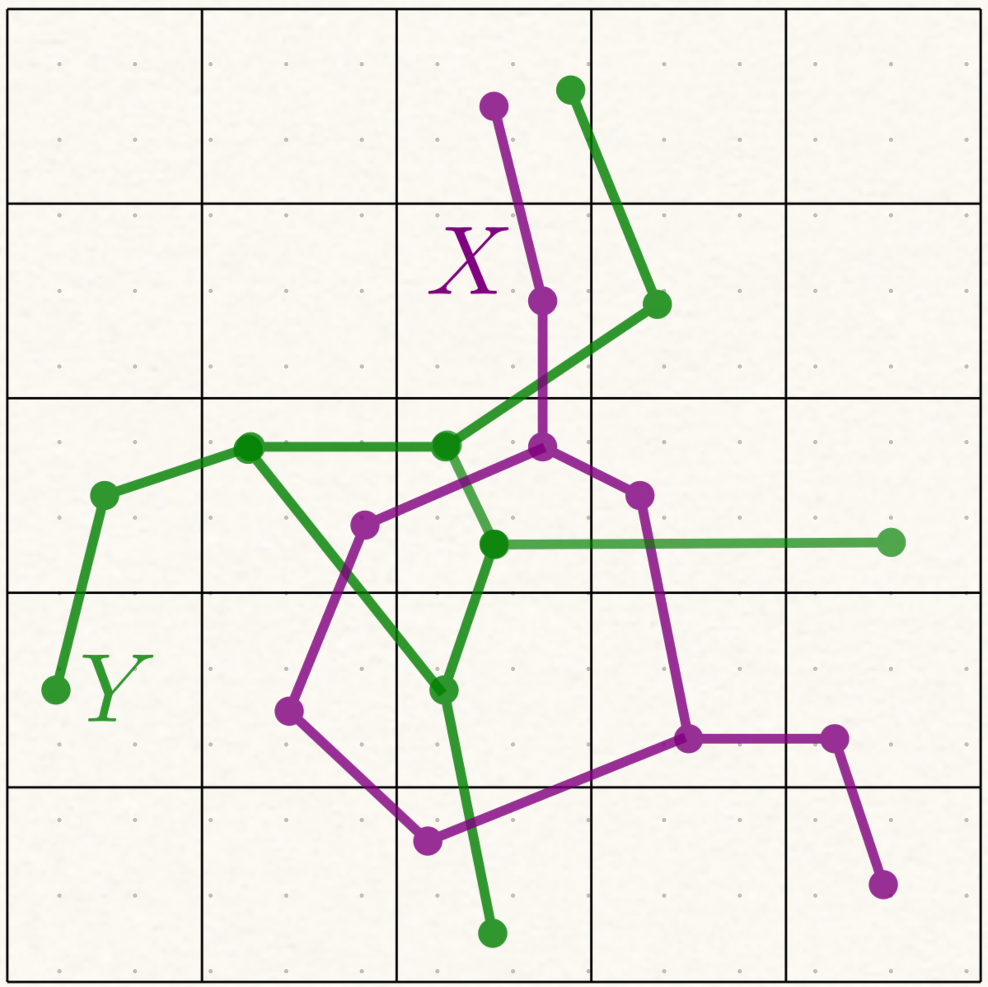

W. Kim, Mémoli, Stefanou

## Mapper Graphs



Dey, Y. Wang, Mémoli, B. Wang, EM, Brown, Bobrowski

# The Plan




## The Setup

* Discretize cover of $\mathbb{R}^d$
* Encode mapper as cosheaf
* Define Interleaving Distance

## The Punchline

* Given not-quite-an-interleaving $\varphi, \psi$
* Compute loss function $L(\varphi, \psi)$
* Use to build interleaving $\Phi, \Psi$
* Use to bound the interleaving dist.

## The Setup:

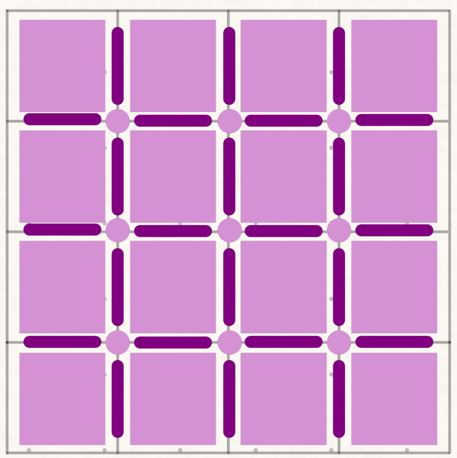
Grids
Mapper
Interleaving

# The Grid

Data: $f: \mathbb{X} \to \mathbb{R}^d$

$K$ is the cubical complex of $\mathbb{R}^d$ w/ sides $\delta$

Face relation $\sigma \leq \tau$



$K$ for $d = 2$

# The Grid

Data: $f: \mathbb{X} \to \mathbb{R}^d$

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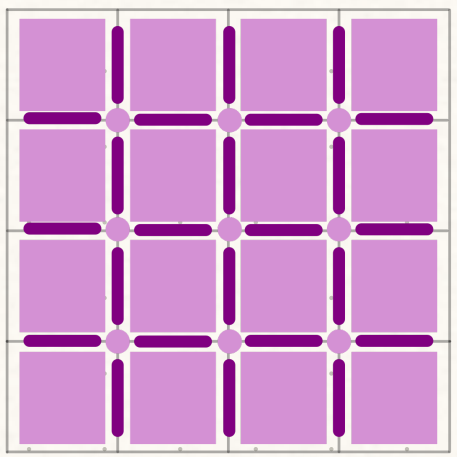
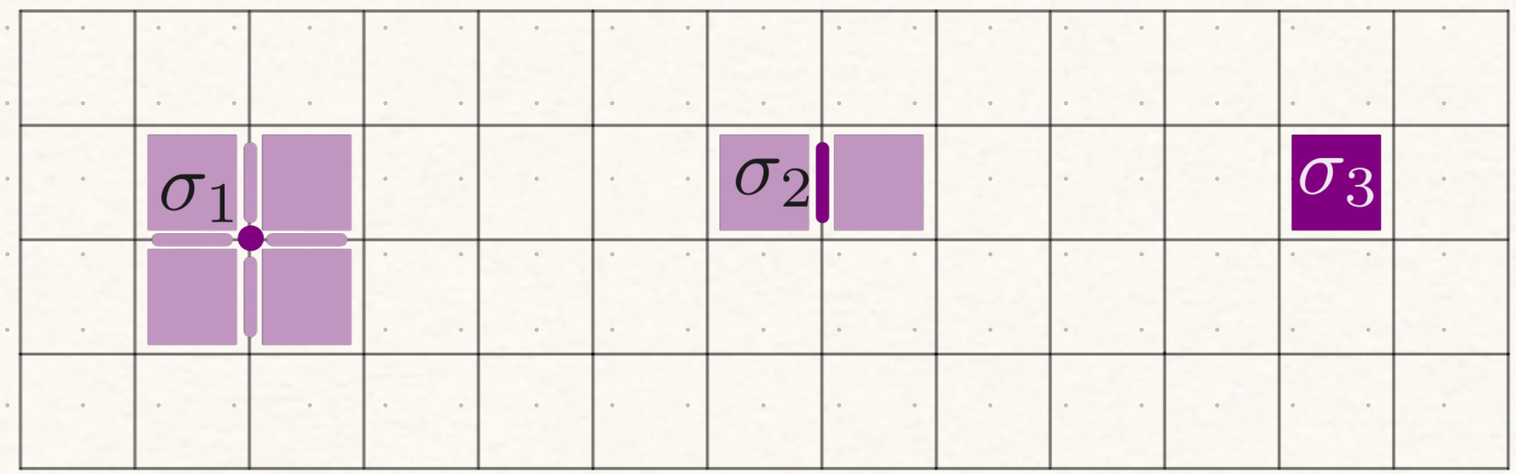


$K$ for $d=2$

Cover

$\mathcal{U} = \{ u_\sigma \mid u_\sigma = \bigcup_{\tau \geq \sigma} |\tau| \}$



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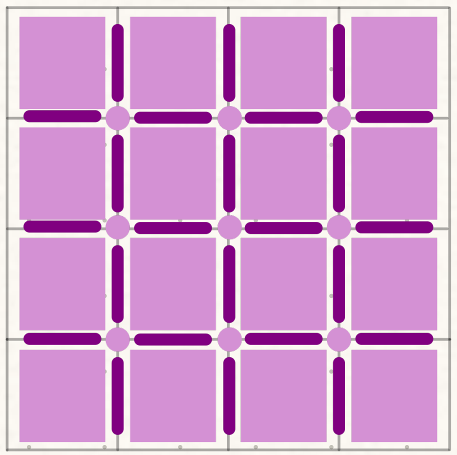


$K$ for $d = 2$

**Cover**

$$\mathcal{U} = \{ u_\sigma \mid u_\sigma = \bigcup_{\tau \geq \sigma} |\tau| \}$$

**Alexandroff Topology**

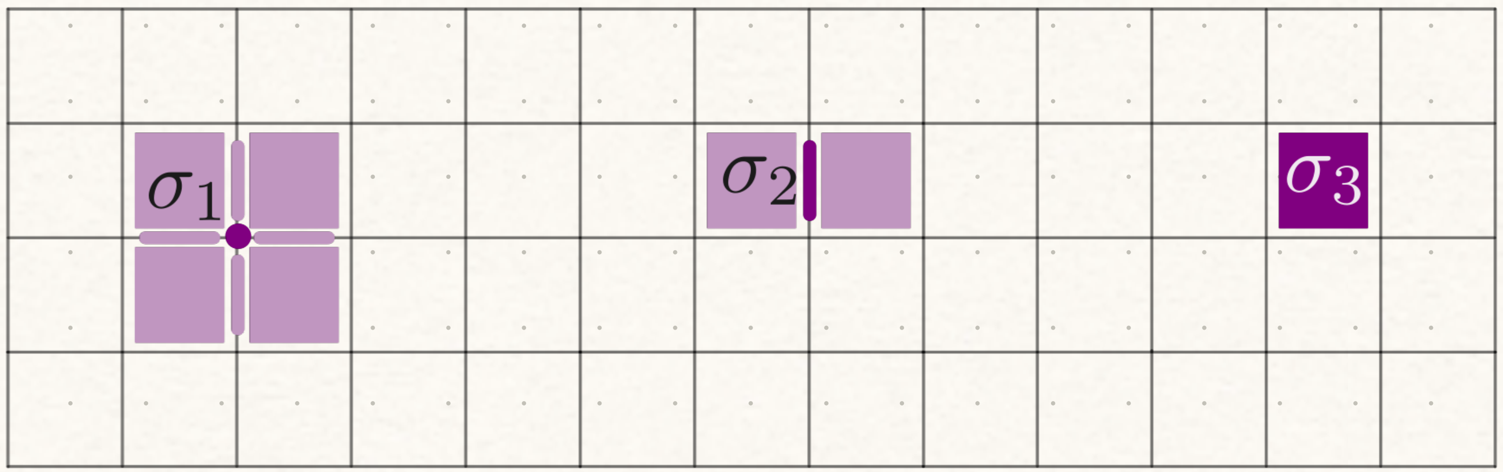
Up-set $u_\sigma \mapsto u_\sigma^{\uparrow} = \{ u_\tau \mid u_\tau \geq u_\sigma \}$

Down-set $u_\sigma \mapsto u_\sigma^{\downarrow} = \{ u_\tau \mid u_\tau \leq u_\sigma \}$

open sets

**Open sets in $\mathcal{U}$**

$$S_\sigma = \{ u_\sigma \}^{\downarrow}$$

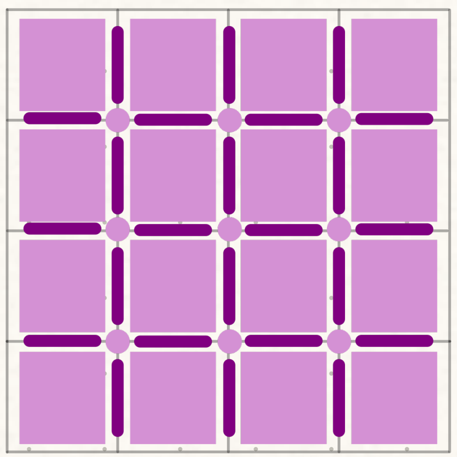


# The Grid

Data: $f: \mathbb{X} \to \mathbb{R}^d$

K is the cubical complex of $\mathbb{R}^d$ w/ sides $\delta$

Face relation $\sigma \leq \tau$



K for $d=2$

## Cover

$$\mathcal{U} = \{ U_\sigma \mid U_\sigma = \bigcup_{\tau \geq \sigma} |\tau| \}$$

## Alexandroff Topology

Up-set $U_\sigma \mapsto U_\sigma^\uparrow = \{U_\tau \mid U_\tau \geq U_\sigma\}$

Down-set $U_\sigma \mapsto U_\sigma^\downarrow = \{U_\tau \mid U_\tau \leq U_\sigma\}$

open sets

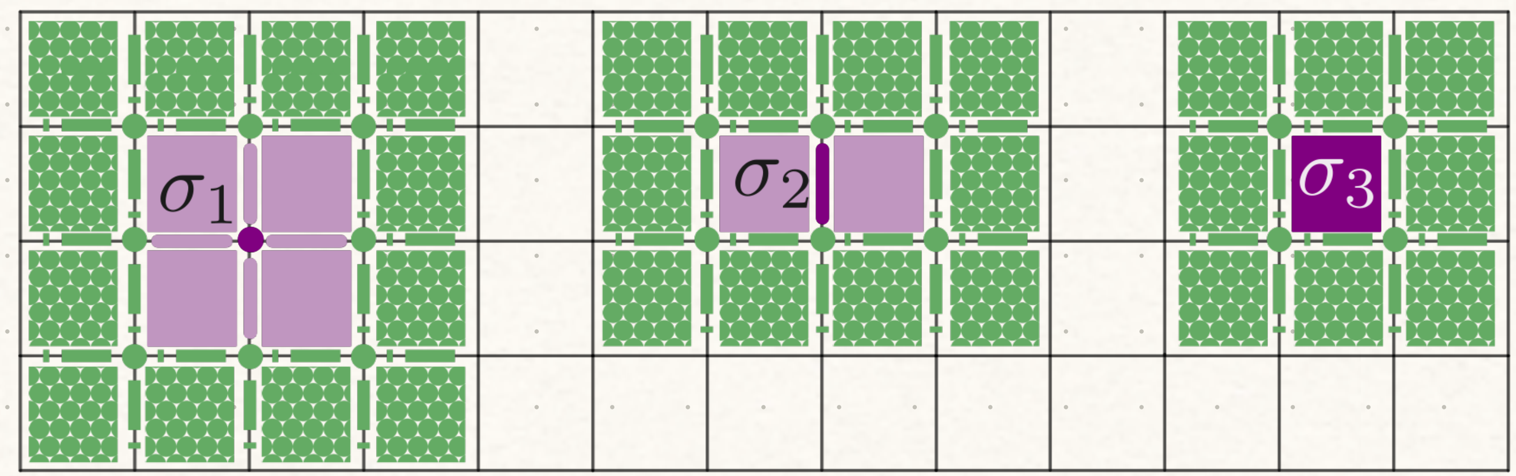
## Open sets in $\mathcal{U}$

$$S_\sigma = \{U_\sigma\}^\downarrow$$

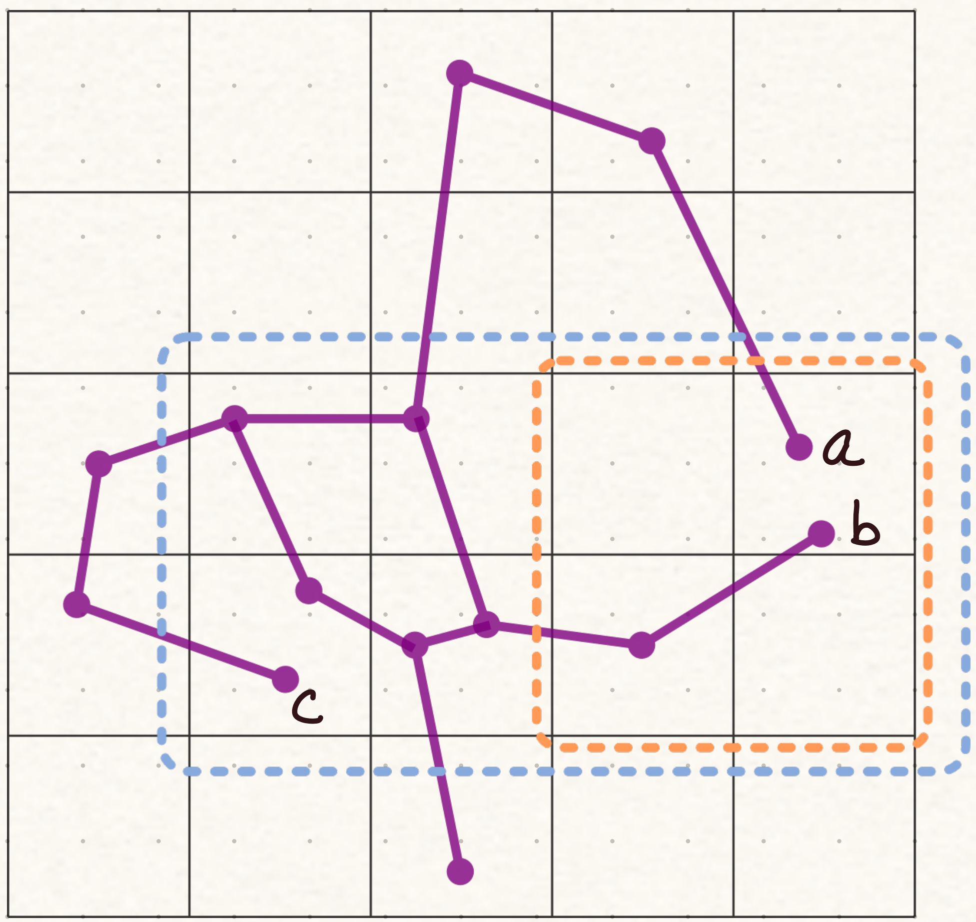
## Thickening

The n-thickening of S

$$S^n = \begin{cases} S & \text{if } n=0 \\ \left(S^{n-1}\right)^{\downarrow\uparrow} & n \geq 1 \end{cases}$$



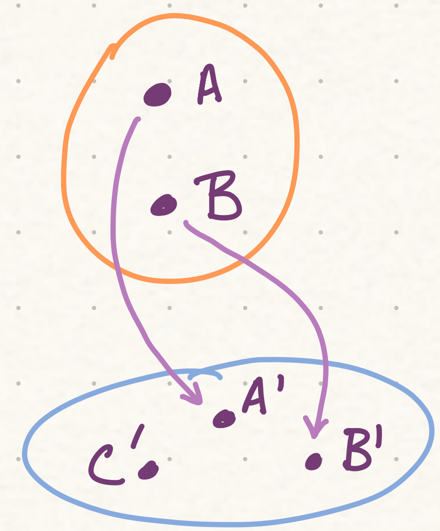
# Cosheaf Representation




$$F : \mathrm{Open}(\mathcal{U}) \longrightarrow \mathrm{Set}$$

$$\begin{tikzcd} S \arrow[d, phantom, "\cap\!|"] & \pi_0 f^{-1}(|S|) \arrow[d, "F{[\subseteq]}"] \\ T & \pi_0 f^{-1}(|T|) \end{tikzcd}$$

$$\begin{tikzcd} F(S) \arrow[d] \\ F(T) \end{tikzcd}$$



# The Interleaving Distance

## Definition

Given cosheaves $F, G : \mathrm{Open}(\mathcal{U}) \longrightarrow \mathrm{Set}$ and fix $n \geq 0$

An $n$-interleaving is a pair of natural transformations

$$\varphi : F \Rightarrow G \circ (-)^n \qquad \psi : G \Rightarrow F \circ (-)^n$$

Such that

$$\begin{tikzcd} F(S) \arrow[rr, "F[S\subseteq S^{2n}]"] \arrow[dr, "\varphi_S"'] & & F(S^{2n}) \\ & G(S^n) \arrow[ur, "\psi_{S^n}"'] & \end{tikzcd} \qquad \begin{tikzcd} & F(S^n) \arrow[dr, "\varphi_{S^n}"] & \\ G(S) \arrow[ur, "\psi_S"] \arrow[rr, "G[S\subseteq S^{2n}]"'] & & G(S^{2n}) \end{tikzcd}$$

Commute.

The interleaving distance is

$$d_I(F, G) = \min\{ n \geq 0 \mid \exists\ n\text{-interleaving} \}.$$

# Complexity

Persistence modules

Merge trees

Reeb graphs

| type/$\delta$ | $\mathbf{Z} \to \mathbf{Vec}$ | $\mathbf{Z} \to \mathbf{Set}$ | $\mathbf{Z}^2 \to \mathbf{Vec}$ | $\mathbf{Z}^2 \to \mathbf{Set}$ | $\mathbf{Z}^{L,C} \to \mathbf{Vec}_{\mathbb{Z}/2\mathbb{Z}}$ |
|---|---|---|---|---|---|
| $\delta = 0$ | $\mathcal{O}(n^\omega)$ | $\mathcal{O}(n)$ | $\mathcal{O}(n^6)$ | GI-complete | $\mathcal{O}(n^6)$ |
| $\delta \geq 1$ | $\mathcal{O}(n^\omega)$ | NP-complete | CI-hard | NP-complete | NP-complete |

Table 1: The complexity of checking for $\delta$-interleavings between modules $M$ and $M'$. If the target category is **Vec** then $n = \dim M + \dim M'$, and if the target category is **Set** then $n = |M| + |M'|$. Here $\omega$ is the matrix multiplication exponent.

*Bjerkevik and Botnan.* ***Computational Complexity of the Interleaving Distance****. SoCG 2018, arXiv:1712.04281*

# The Loss Function

# n-Assignment

**Definition**

An unnatural transformation is a collection of maps

$$\varphi = \{ \varphi_U : F(S) \longrightarrow G^n(T) \quad \forall\, U \in \mathrm{Open}(K) \}$$

**Definition**

An n-assignment is a pair of unnatural transformations

$$\varphi : F \Rightarrow G^n$$
$$\psi : G \Rightarrow F^n$$

$$\begin{tikzcd}
F(S) \arrow[r, "{F[\leq]}"] \arrow[d, "\varphi_U"'] \arrow[dr, phantom, "?"] & F(T) \arrow[d, "\varphi_V"] \\
G(S^n) \arrow[r, "{G[\leq]}"'] & G(T^n)
\end{tikzcd}$$

# n-Assignment

**Definition**

An unnatural transformation is a collection of maps

$$\varphi = \{ \varphi_u : F(S) \longrightarrow G^n(T) \quad \forall\, u \in \mathrm{Open}(K) \}$$

$$\begin{tikzcd}
F(S) \arrow[r, "{F[\leq]}"] \arrow[d, "\varphi_u"'] \arrow[dr, phantom, "?"] & F(T) \arrow[d, "\varphi_v"] \\
G(S^n) \arrow[r, "{G[\leq]}"'] & G(T^n)
\end{tikzcd}$$

**Definition**

An n-assignment is a pair of unnatural transformations

$$\varphi : F \Rightarrow G^n$$
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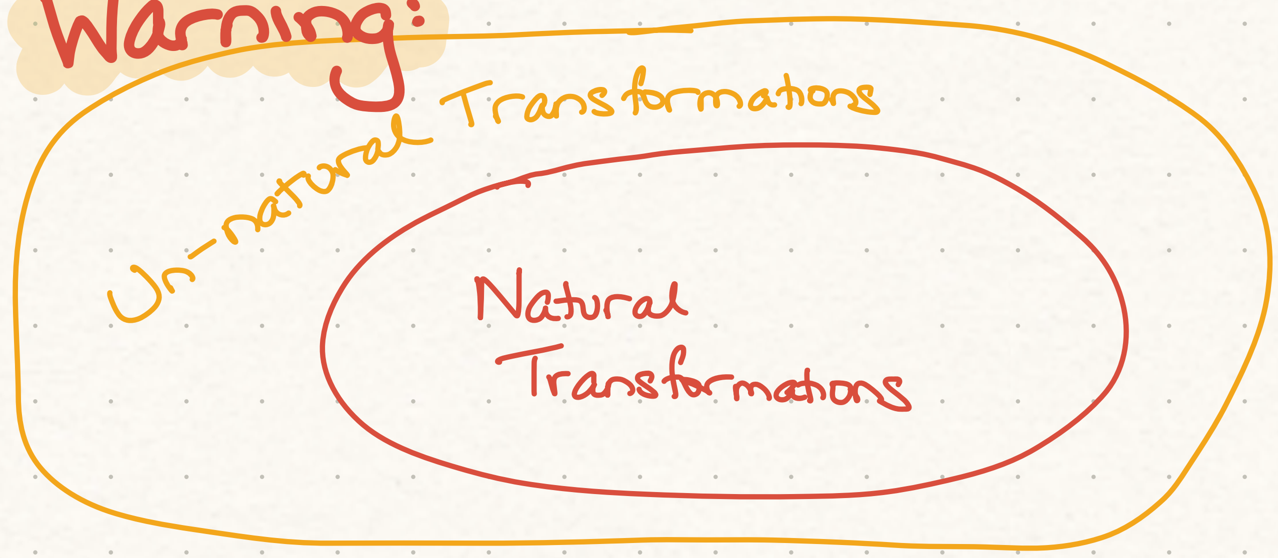
**Warning:**



# Four Diagrams

**Natural Transformation**

$$\begin{tikzcd}
F(S) \arrow[r, "F[\subseteq]"] \arrow[dr, "\varphi_S"'] & F(T) \arrow[dr, "\varphi_T"] & \\
& G(S^n) \arrow[r, "G[\subseteq]"'] & G(T^n)
\end{tikzcd}
\qquad
\begin{tikzcd}
& F(S^n) \arrow[r, "F[\subseteq]"] & F(T^n) \\
G(S) \arrow[ur, "\psi_S"] \arrow[r, "G[\subseteq]"'] & G(T) \arrow[ur, "\psi_T"] &
\end{tikzcd}$$

**Notation:**

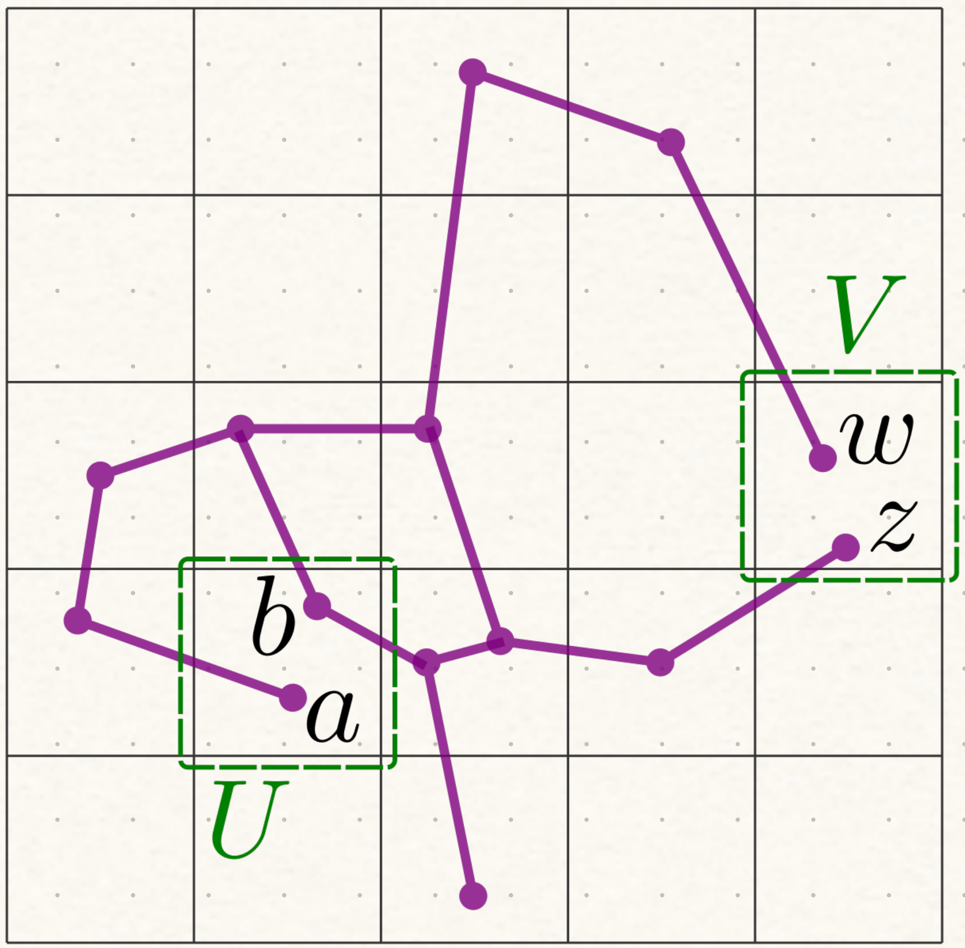
$\boxed{▱_{\varphi}(S,T)}$ $\qquad$ $\boxed{▱_{\psi}(S,T)}$

$\boxed{\nabla_{\varphi,\psi}(S)}$ $\qquad$ $\boxed{\Delta_{\varphi,\psi}(S)}$

**Interleaving**

$$\begin{tikzcd}
F(S) \arrow[rr, "F[S\subseteq S^{2n}]"] \arrow[dr, "\varphi_S"'] & & F(S^{2n}) \\
& G(S^n) \arrow[ur, "\psi_{S^n}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& F(S^n) \arrow[dr, "\varphi_{S^n}"] & \\
G(S) \arrow[ur, "\psi_S"] \arrow[rr, "G[S\subseteq S^{2n}]"'] & & G(S^{2n})
\end{tikzcd}$$

# Distance in $F(S)$




## Distance in $F(S)$

$$d^F_S(A,B) = \min\{k \geq 0 \mid F[S \leq S^k](A) = F[S \leq S^k]B\}$$

* set $= \infty$ if no $k$ exists

$$F(S) \xrightarrow{F[\leq]} F(S^k)$$

$A \mapsto C$

$B \mapsto C$

Ex

$d^F_u(A,B) = 1$ $\quad$ $d^F_V(w,z) = 2$

# Loss

Nat. Trans

$$L^{S,T}_{\square}(\varphi) = \max_{\alpha \in F(S)} d^G_{T^n}\left(\varphi_T \circ F[S \subseteq T](\alpha), G[S^n \subseteq T^n] \circ \varphi_S(\alpha)\right)$$

$$\begin{tikzcd}
F(S) \arrow[r, "F[\subseteq]"] \arrow[dr, "\varphi_S"'] & F(T) \arrow[dr, "\varphi_T"] & & \\
& G(S^n) \arrow[r, "G[\subseteq]"'] & G(T^n) \arrow[r] & G(T^{n+k})
\end{tikzcd}$$

---

Interleaving

$$L^S_{\triangledown}(\varphi, \psi) = \max_{\alpha \in F(S)} \left[ \frac{1}{2} \cdot d^F_{S^{2n}}\left(F[S \subseteq S^{2n}](\alpha), \psi_{S^n} \circ \varphi_S(\alpha)\right) \right]$$

$$\begin{tikzcd}
F(S) \arrow[rr, "F[\subseteq]"] \arrow[dr, "\varphi_S"'] & & F(S^{2n}) \arrow[r] & F(S^{2(n+k)}) \\
& G(S^n) \arrow[ur, "\psi_{S^n}"'] & &
\end{tikzcd}$$

# Full Loss Function

## Definition

Given n-assignment $\varphi, \psi$. Define

$$L_{\boxbslash}^{S,T}(\varphi) = \max_{\alpha \in F(S)} d_{T^n}^{G}(\varphi_T \circ F[S \subseteq T](\alpha), G[S^n \subseteq T^n] \circ \varphi_S(\alpha))$$

$$L_{\boxslash}^{S,T}(\psi) = \max_{\alpha \in G(S)} d_{T^n}^{F}(\psi_T \circ G[S \subseteq T](\alpha), F[S^n \subseteq T^n] \circ \psi_S(\alpha))$$

$$L_{\triangledown}^{S}(\varphi, \psi) = \max_{\alpha \in F(S)} \left[ \frac{1}{2} \cdot d_{S^{2n}}^{F}(F[S \subseteq S^{2n}](\alpha), \psi_{S^n} \circ \varphi_S(\alpha)) \right]$$

$$L_{\triangle}^{S}(\varphi, \psi) = \max_{\alpha \in G(S)} \left[ \frac{1}{2} \cdot d_{S^{2n}}^{G}(G[S \subseteq S^{2n}](\alpha), \varphi_{S^n} \circ \psi_S(\alpha)) \right].$$

Then the loss is

$$L(\varphi, \psi) = \max_{S \subseteq T} \left\{ L_{\boxbslash}^{S,T}, L_{\boxslash}^{S,T}, L_{\triangle}^{S}, L_{\triangledown}^{S} \right\}$$

# Bound v. 1.0

Chambers, EM, Percival, B. Wang. **Bounding the interleaving distance for mapper graphs with a loss function.** JACT, 2025. *arXiv:2307.15130*

**Theorem** (Chambers, EM, Percival, B. Wang 2025)

Given $n$-assignment
$$\varphi = \{\varphi_s : F(s) \to G(s^n)\}$$
$$\psi = \{\psi_s : G(s) \to F(s^n)\}$$

$$d_I(F,G) \leq n + L(\varphi, \psi).$$

# Proof Sketch

Given $n$-assignment $\varphi, \psi$ with loss $k = L(\varphi, \psi)$

Define $(n+k)$-assignment $\Phi$ and $\Psi$

$$\begin{tikzcd}
F(S) \arrow[dr, "\varphi_u"'] \arrow[drr, "\Phi_u"] & & \\
& G(S^n) \arrow[r, "G[\subseteq]"'] & G(S^{n+k})
\end{tikzcd}
\qquad
\begin{tikzcd}
& F(S^n) \arrow[r, "F[\subseteq]"] & F(S^{n+k}) \\
G(S) \arrow[ur, "\psi_u"] \arrow[urr, "\Psi_u"'] & &
\end{tikzcd}$$

# n-Assignment

**Definition**

A unnatral transformation is a collection of maps

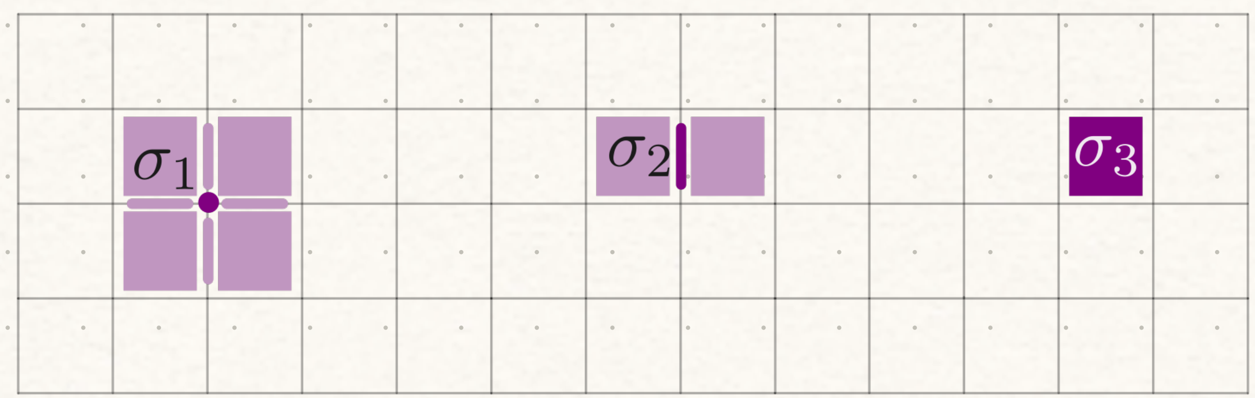
$$\varphi = \{ \varphi_u : F(S) \longrightarrow G^n(S) \quad \forall\, S \in \mathrm{Open}(\mathcal{U}) \}$$

**Definition**

A n-assignment is a pair of unnatural transformations

$$\varphi : F \Rightarrow G^n$$
$$\psi : G \Rightarrow F^n$$

$$\begin{tikzcd}
F(S) \arrow[r, "{F[\subseteq]}"] \arrow[d, "\varphi_S"'] \arrow[dr, phantom, "?"] & F(T) \arrow[d, "\varphi"] \\
G(S^n) \arrow[r, "{G[\subseteq]}"'] & G(T^n)
\end{tikzcd}$$



# Basis n-Assignment

**Definition**

A basis unnatural transformation is a collection of maps

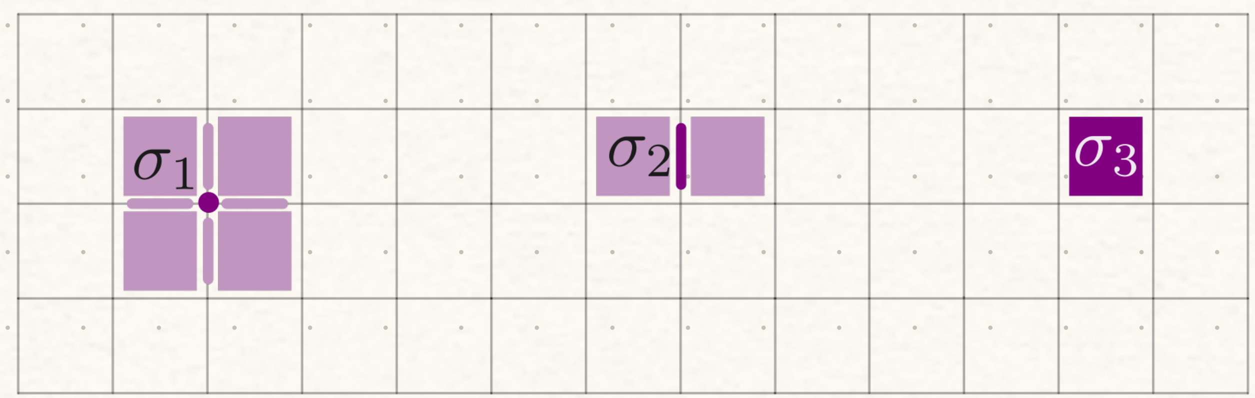
$$\varphi = \{ \varphi_{u_\sigma} : F(S_\sigma) \longrightarrow G^n(S_\sigma) \quad \forall\, \sigma \in K \}$$

**Definition**

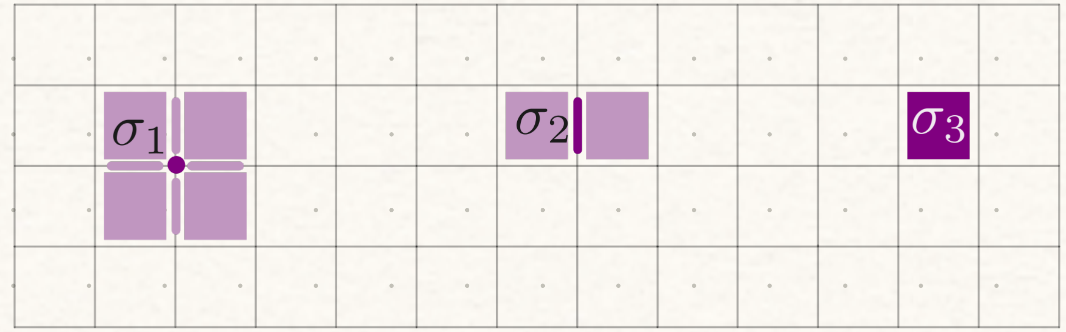
A basis n-assignment is a pair of basis unnatural transformations

$$\varphi : F \Rightarrow G^n$$
$$\psi : G \Rightarrow F^n$$

$$\begin{tikzcd}
F(S) \arrow[r, "{F[\leq]}"] \arrow[d, "\varphi_S"'] \arrow[dr, phantom, "?"] & F(T) \arrow[d, "\varphi"] \\
G(S^n) \arrow[r, "{G[\leq]}"'] & G(T^n)
\end{tikzcd}$$



# Four Basis Diagrams




$\tau \leq \sigma$

**Natural Transformation**

$$\begin{tikzcd}
F(S_\sigma) \arrow[r, "{F[\subseteq]}"] \arrow[dr, "\varphi_S"'] & F(S_\tau) \arrow[dr, "\varphi_T"'] & \\
& G(S^n_\sigma) \arrow[r, "{G[\subseteq]}"'] & G(S^n_\tau)
\end{tikzcd}
\qquad
\begin{tikzcd}
& F(S^n_\sigma) \arrow[r, "{F[\subseteq]}"] & F(S^n_\tau) \\
G(S_\sigma) \arrow[ur, "\psi_S"] \arrow[r, "{G[\subseteq]}"'] & G(S_\tau) \arrow[ur, "\psi_T"] &
\end{tikzcd}$$

**Notation:**

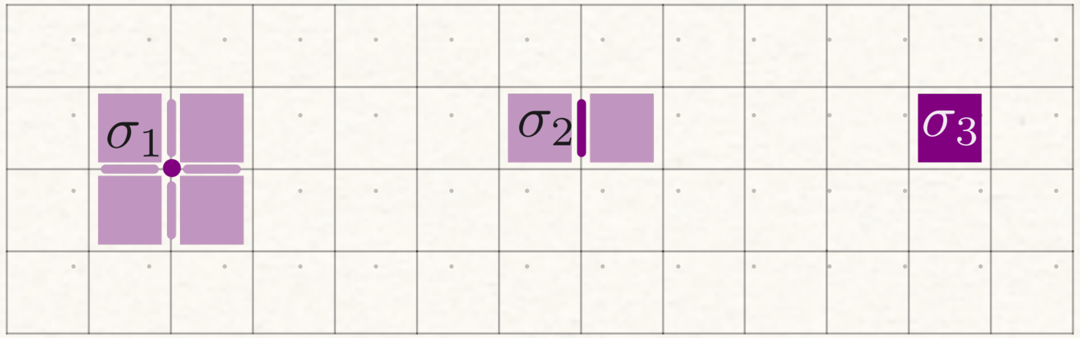
$\boxed{\square_{\varphi}(S_\sigma, S_\tau)}$ $\qquad$ $\boxed{\square_{\psi}(S_\sigma, S_\tau)}$

$\boxed{\nabla_{\varphi,\psi}(S_\sigma)}$ $\qquad$ $\boxed{\Delta_{\varphi,\psi}(S_\sigma)}$

**Interleaving**

$$\begin{tikzcd}
F(S_\sigma) \arrow[rr, "{F[S\subseteq S^{2n}]}"] \arrow[dr, "\varphi_S"'] & & F(S^{2n}_\sigma) \\
& G(S^n_\sigma) \arrow[ur, "\psi_{S^n}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& F(S^n_\sigma) \arrow[dr, "\varphi_{S^n}"] & \\
G(S_\sigma) \arrow[ur, "\psi_S"] \arrow[rr, "{G[S\subseteq S^{2n}]}"'] & & G(S^{2n}_\sigma)
\end{tikzcd}$$

# Loss Function




## Definition

The basis loss function for n-assignments $\varphi, \psi$ is

$$L_B(\varphi,\psi) = \max_{\sigma \leq \tau} \left\{ L_{\reflectbox{▱}}^{S_\tau, S_\sigma}, L_{▱}^{S_\tau, S_\sigma}, L_{\triangle}^{S_\sigma}, L_{\triangledown}^{S_\sigma} \right\}$$

$$\begin{tikzcd}
F(S) \arrow[r, "F[\subseteq]"] \arrow[dr, "\varphi_S"'] & F(T) \arrow[dr, "\varphi_T"'] & & \\
& G(S^n) \arrow[r, "G[\subseteq]"'] & G(T^n) \arrow[r] & G(T^{n+k})
\end{tikzcd}$$

$$\begin{tikzcd}
F(S) \arrow[rr, "F[\subseteq]"] \arrow[dr, "\varphi_S"'] & & F(S^{2n}) \arrow[r] & F(S^{2(n+k)}) \\
& G(S^n) \arrow[ur, "\psi_{S^n}"'] & &
\end{tikzcd}$$

$$L_{\reflectbox{▱}}^{S,T}(\varphi) = \max_{\alpha \in F(S)} d^G_{T^n}(\varphi_T \circ F[S \subseteq T](\alpha), G[S^n \subseteq T^n] \circ \varphi_S(\alpha))$$

$$L_{▱}^{S,T}(\psi) = \max_{\alpha \in G(S)} d^F_{T^n}(\psi_T \circ G[S \subseteq T](\alpha), F[S^n \subseteq T^n] \circ \psi_S(\alpha))$$

$$L_{\triangledown}^{S}(\varphi,\psi) = \max_{\alpha \in F(S)} \left\lceil \frac{1}{2} \cdot d^F_{S^{2n}}(F[S \subseteq S^{2n}](\alpha), \psi_{S^n} \circ \varphi_S(\alpha)) \right\rceil$$

$$L_{\triangle}^{S}(\varphi,\psi) = \max_{\alpha \in G(S)} \left\lceil \frac{1}{2} \cdot d^G_{S^{2n}}(G[S \subseteq S^{2n}](\alpha), \varphi_{S^n} \circ \psi_S(\alpha)) \right\rceil.$$

# Main Theorem

**Theorem** Chambers, EM, Percival, B. Wang 2025

Given basis $n$- assignments $\varphi = \{ \varphi_{u_\sigma} : F(S_\sigma) \to G^n(S_\sigma) \mid \sigma \in K \}$

$\psi = \{ \psi_{u_\sigma} : G(S_\sigma) \to F^n(S_\sigma) \mid \sigma \in K \}$

$$d_I(F, G) \leq n + L_B(\varphi, \psi)$$

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$$d_I(F, G) \leq n + \underline{\underline{L_B(\varphi, \psi)}}$$

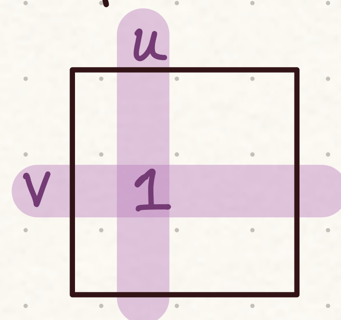
This is in P!!!!!!

Computation

# Integer Program

* Store n-assignment maps as binary matrices

  $\varphi(u) = v$ ↳




* Loss computation
  ↳ matrix multiplication

* Treat $\varphi, \psi$ matrices as variables
  ↳ optimal loss

* Binary search over $n$

| | Loss Term | Diagram | Matrix Multiplication | Eval. |
|---|---|---|---|---|
| Edge-vertex Parallelogram | $L^{S_\tau,S_\sigma}_{\reflectbox{▱}}$ | $\begin{tikzcd} F(S_\tau) \arrow[r, "F[\subseteq]"] \arrow[dr, "\varphi_{S_\tau}"'] & F(S_\sigma) \arrow[dr, "\varphi_{S_\sigma}"] & \\ & G^n(S_\tau) \arrow[r, "G[\subseteq]"'] & G^n(S_\sigma) \end{tikzcd}$ | $D^V_{G^n}\left(M^V_\varphi \cdot B^\uparrow_F - B^\uparrow_{G^n} \cdot M^E_\varphi\right)$<br>$D^V_{G^n}\left(M^V_\varphi \cdot B^\downarrow_F - B^\downarrow_{G^n} \cdot M^E_\varphi\right)$ | $\max\limits_{x_{ij} \in A} x_{ij}$ |
| | $L^{S_\tau,S_\sigma}_{▱}$ | $\begin{tikzcd} & F^n(S_\tau) \arrow[r, "F[\subseteq]"] & F^n(S_\sigma) \\ G(S_\tau) \arrow[ur, "\psi_{S_\tau}"] \arrow[r, "G[\subseteq]"'] & G(S_\sigma) \arrow[ur, "\psi_{S_\sigma}"'] & \end{tikzcd}$ | $D^V_{F^n}\left(M^V_\psi \cdot B^\uparrow_G - B^\uparrow_{F^n} \cdot M^E_\psi\right)$<br>$D^V_{F^n}\left(M^V_\psi \cdot B^\downarrow_G - B^\downarrow_{F^n} \cdot M^E_\psi\right)$ | |
| Thickening Parallelogram | $L^{S_\rho,S^n_\rho}_{\reflectbox{▱}}$ | $\begin{tikzcd} F(S_\rho) \arrow[r, "F[\subseteq]"] \arrow[dr, "\varphi_{S_\rho}"'] & F^n(S_\rho) \arrow[dr, "\varphi_{S^n_\rho}"] & \\ & G^n(S_\rho) \arrow[r, "G[\subseteq]"'] & G^{2n}(S_\rho) \end{tikzcd}$ | $D^V_{G^n}\left(M^V_{\varphi^n} \cdot I^V_F - I^V_{G^n} \cdot M^V_\varphi\right)$<br>$D^E_{G^n}\left(M^E_{\varphi^n} \cdot I^E_F - I^E_{G^n} \cdot M^E_\varphi\right)$ | |
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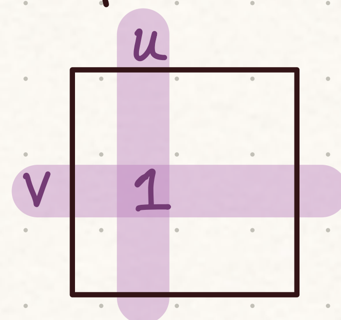
| Triangle | $L^{S_\rho}_{\triangledown}$ | $\begin{tikzcd} F(S_\rho) \arrow[rr, "F[\subseteq]"] \arrow[dr, "\varphi_{S_\rho}"'] & & F^{2n}(S_\rho) \\ & G^n(S_\rho) \arrow[ur, "\psi_{S^n_\rho}"'] & \end{tikzcd}$ | $D^V_{F^{2n}}\left(I^V_{F^n} \cdot I^V_F - M^V_{\psi^n} \cdot M^V_\varphi\right)$<br>$D^E_{F^{2n}}\left(I^E_{F^n} \cdot I^E_F - M^E_{\psi^n} \cdot M^E_\varphi\right)$ | $\max\limits_{x_{ij} \in A} \left\lceil \frac{x_{ij}}{2} \right\rceil$ |
| | $L^{S_\rho}_{\triangle}$ | $\begin{tikzcd} & F^n(S_\rho) \arrow[dr, "\varphi_{S^n_\rho}"] & \\ G(S_\rho) \arrow[ur, "\psi_{S_\rho}"] \arrow[rr, "G[\subseteq]"'] & & G^{2n}(S_\rho) \end{tikzcd}$ | $D^V_{G^{2n}}\left(I^V_{G^n} \cdot I^V_G - M^V_{\varphi^n} \cdot M^V_\psi\right)$<br>$D^E_{G^{2n}}\left(I^E_{G^n} \cdot I^E_G - M^E_{\varphi^n} \cdot M^E_\psi\right)$ | |

# Integer Program

* Store n-assignment maps as binary matrices
  $\varphi(u) = v$

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| | $L^{S_\tau,S_\sigma}_{\text{▱}}$ | $\begin{tikzcd} & F^n(S_\tau) \arrow[r, "F[\subseteq]"] & F^n(S_\sigma) \\ G(S_\tau) \arrow[ur, "\psi_{S_\tau}"] \arrow[r, "G[\subseteq]"'] & G(S_\sigma) \arrow[ur, "\psi_{S_\sigma}"'] & \end{tikzcd}$ | $D^V_{F^n}\left(M^V_\psi \cdot B^\uparrow_G - B^\uparrow_{F^n} \cdot M^E_\psi\right)$ <br> $D^V_{F^n}\left(M^V_\psi \cdot B^\downarrow_G - B^\downarrow_{F^n} \cdot M^E_\psi\right)$ | |
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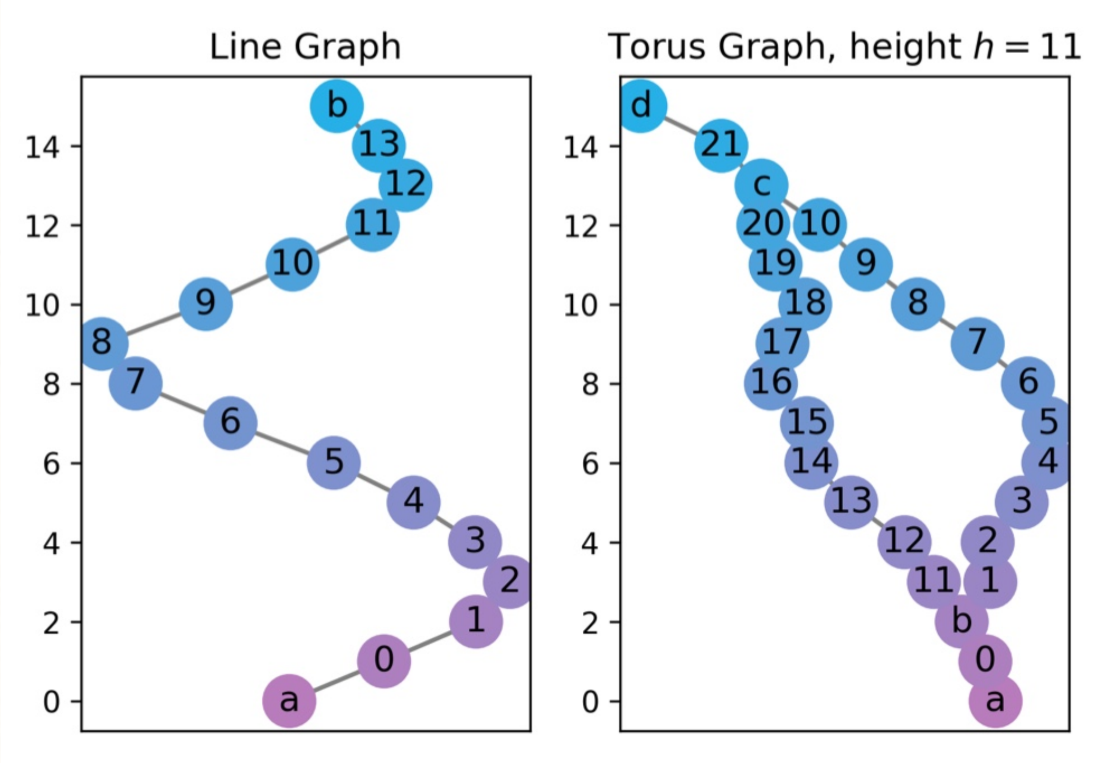
| | $L^{S_\rho}_{\bigtriangleup}$ | $\begin{tikzcd} & F^n(S_\rho) \arrow[dr, "\varphi_{S^n_\rho}"] & \\ G(S_\rho) \arrow[ur, "\psi_{S_\rho}"] \arrow[rr, "G[\subseteq]"'] & & G^{2n}(S_\rho) \end{tikzcd}$ | $D^V_{G^{2n}}\left(I^V_{G^n} \cdot I^V_G - M^V_{\varphi^n} \cdot M^V_\psi\right)$ <br> $D^E_{G^{2n}}\left(I^E_{G^n} \cdot I^E_G - M^E_{\varphi^n} \cdot M^E_\psi\right)$ | |

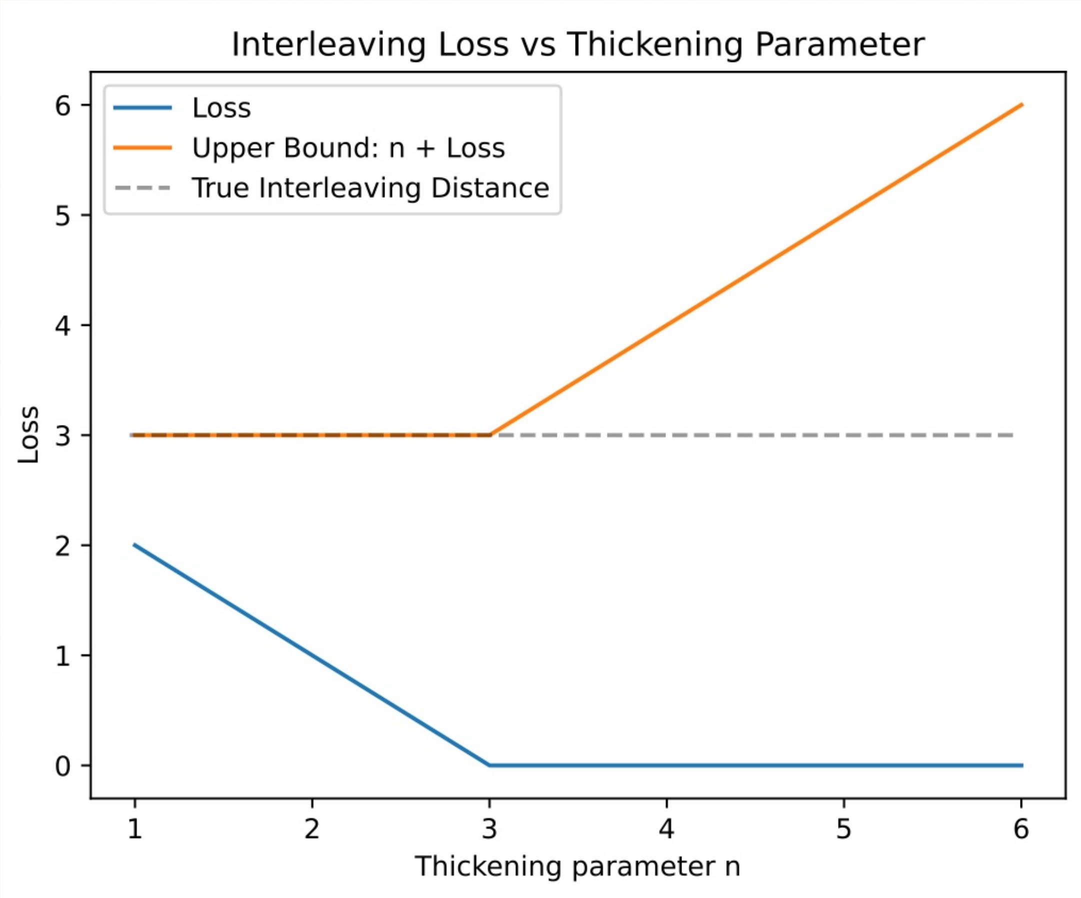


Details: Ishika Ghosh

Chambers, Ghosh, EM, Percival, B. Wang. **Towards an Optimal Bound for the Interleaving Distance on Mapper Graphs.** *arXiv:2504.03865*

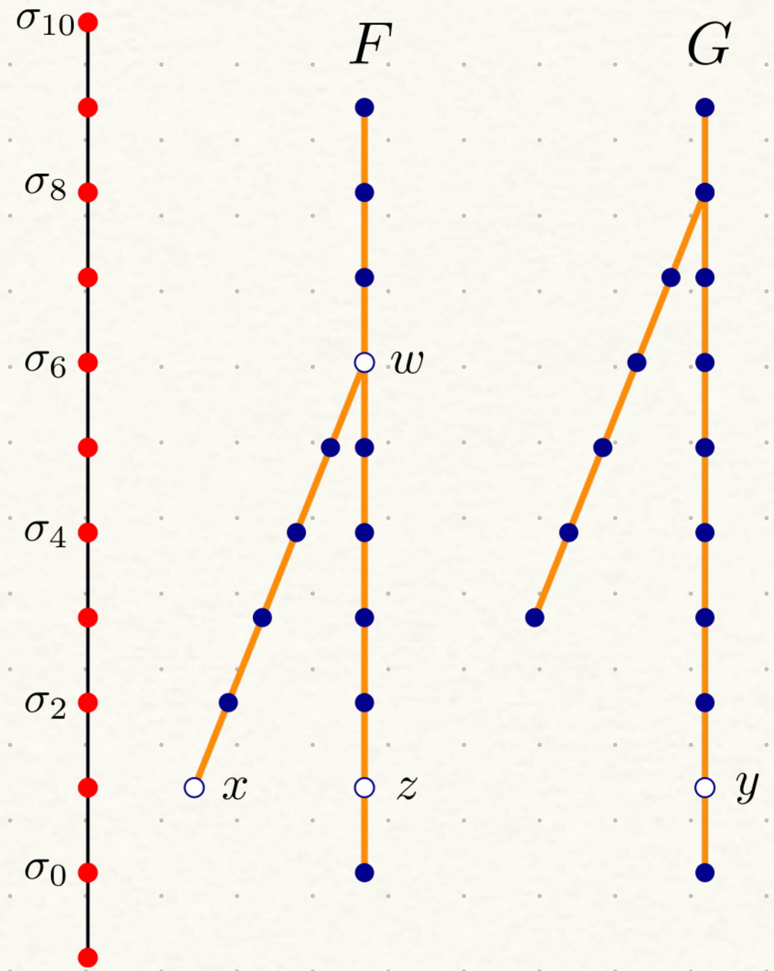
# Ex: Torus and Line

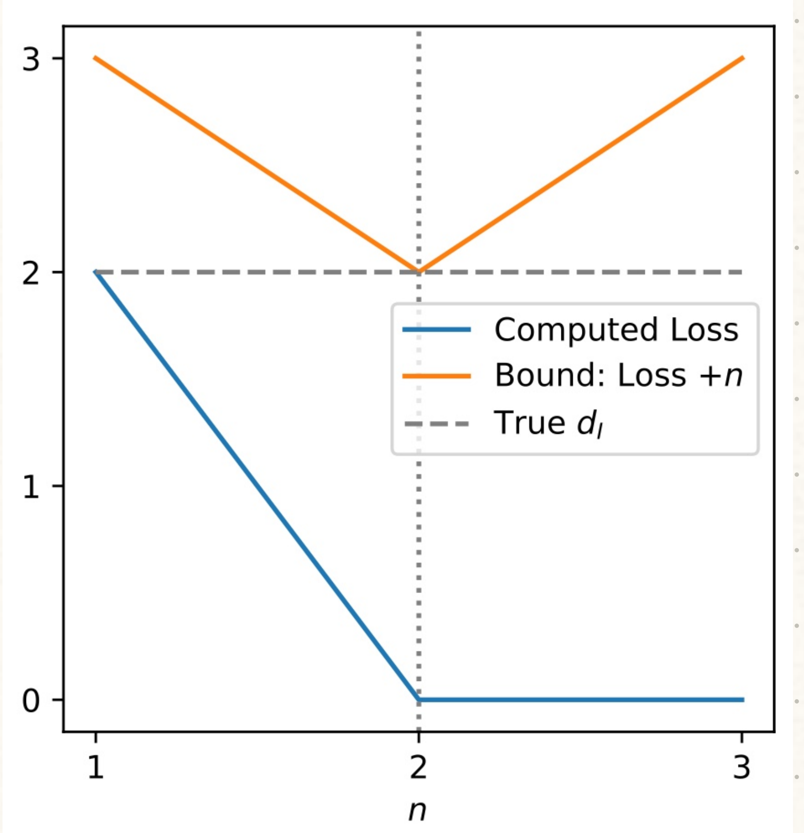



$$d_I(F,G) \leq n + L(\varphi,\psi)$$

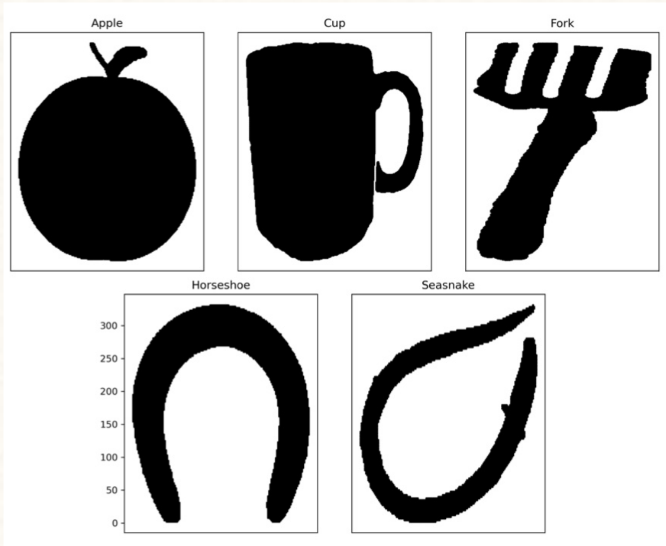
# Binary Search is Necessary

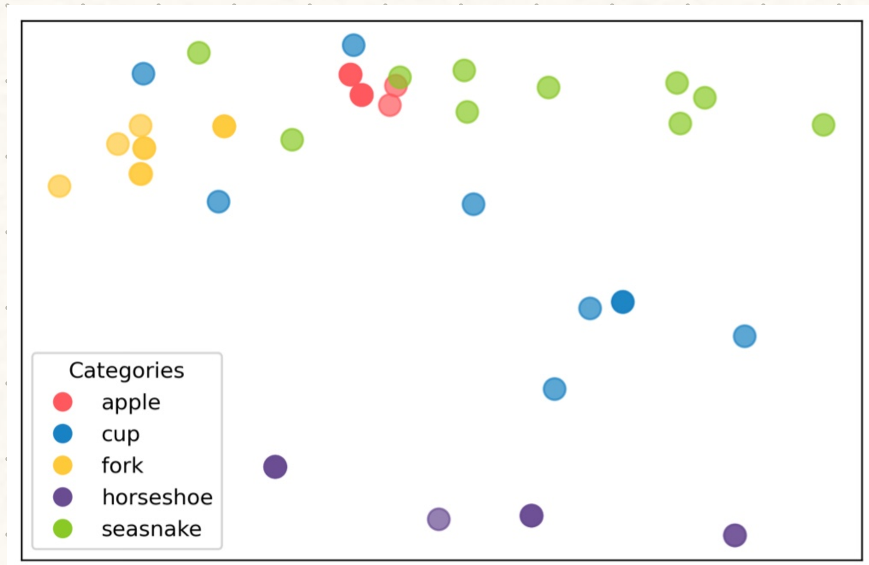


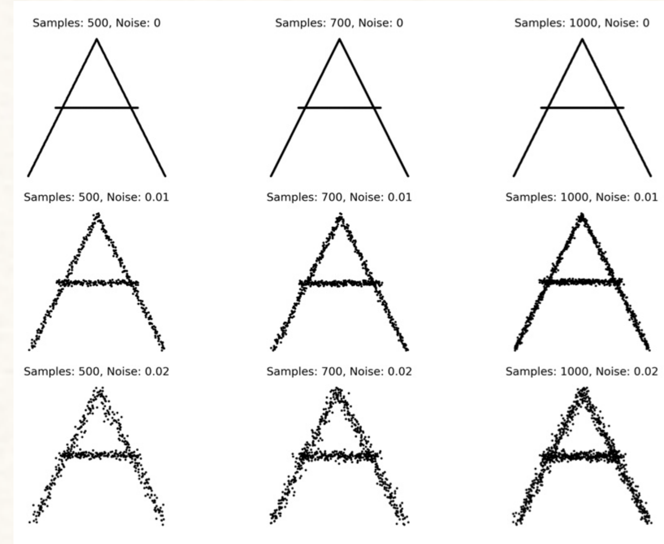

$$d_I(F,G) \leq n + L(\varphi, \psi)$$

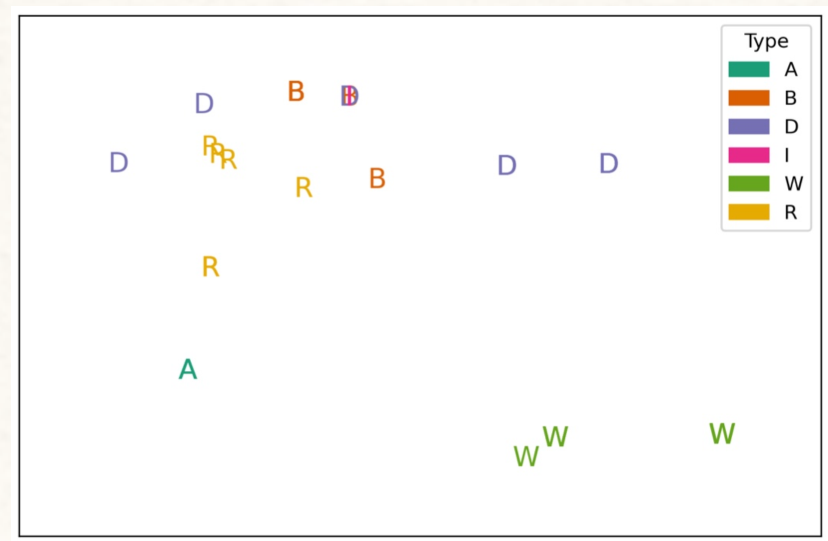
# More Examples




MPEG-7




Alphabet

Wrap-Up

# TL;DR

## What we did

* Fix $F, G : \mathrm{Open}(K) \longrightarrow \mathrm{Set}$

* Given $n$-assignment $\varphi, \psi$

  Compute $L_B(\varphi, \psi)$

* Use ILP to get best $\varphi, \psi$

* Binary search over $n$

* Provide explicit $(n + L_B(\varphi, \psi))$-interleaving $\Phi, \Psi$

# TL;DR

## What we did

* Fix $F, G : \mathrm{Open}(K) \longrightarrow \mathbf{Set}$
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* Use ILP to get best $\varphi, \psi$
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* Provide explicit $(n + L_B(\varphi, \psi))$-interleaving $\Phi, \Psi$

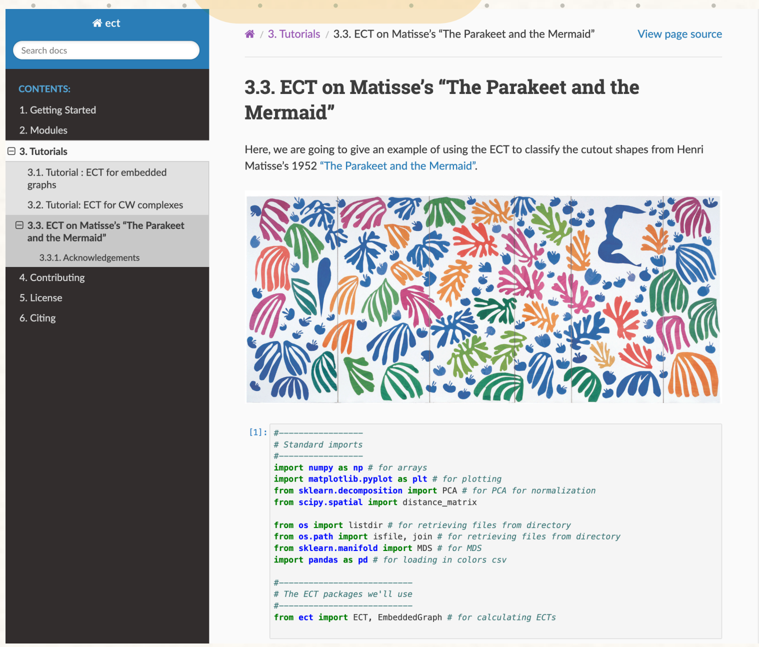
## Future Work

* Extension to other signatures
  - Reeb graphs
  - Multiparameter Persistence

    Astrid Olave's Poster!
* Additional theoretical guarantees

# Open-Source Code

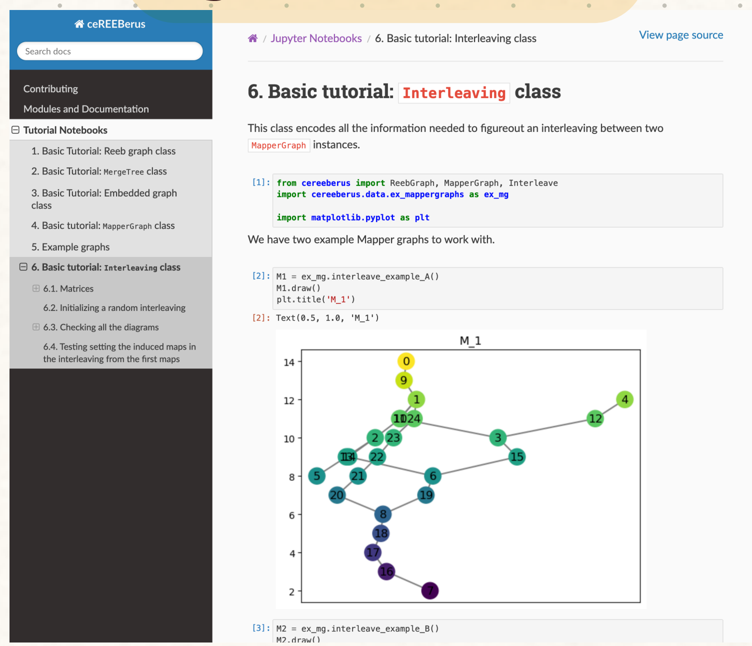
## ect




pip install ect

munchlab.github.io/ect

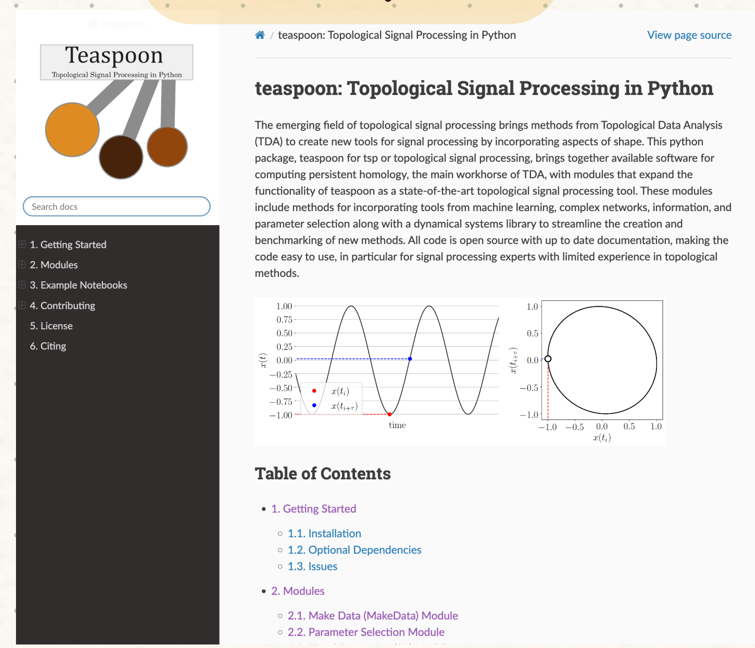
## ceREEBerus




pip install cereeberus

munchlab.github.io/cereeberus

## teaspoon




pip install teaspoon

teaspoontda.github.io/teaspoon

# Thanks!

To my funding sources

To my research group

To my collaborators

Erin Chambers

Ishika Ghosh

Sarah Percival

Bei Wang

- EC, EM, SP, BW. **Bounding the interleaving distance for mapper graphs with a loss function.** JACT, 2025. *arXiv:2307.15130*
- EC, IG, EM, SP, BW. **Towards an Optimal Bound for the Interleaving Distance on Mapper Graphs.** *arXiv:2504.03865*

MunchLab 2024-2025

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