

Bounding the Mapper Graph Interleaving Distance with a Loss Function

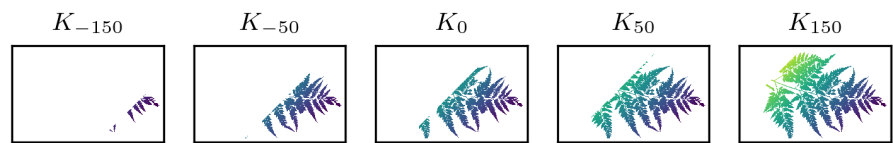
Liz Munch

Joint
with: Erin W Chambers, Ishika Ghosh, Sarah Percival, Bei Wang

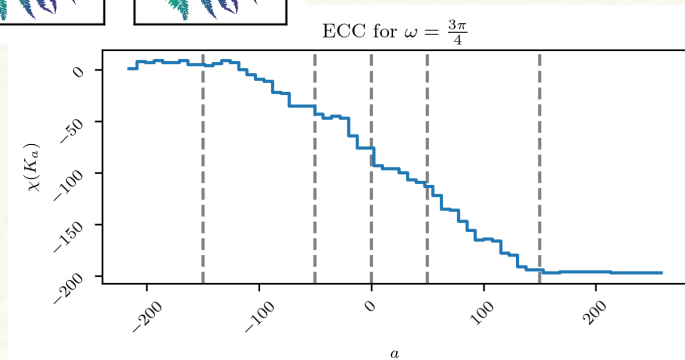
Michigan State University
Depts of CMSE & Math

Classes of Topological Signatures

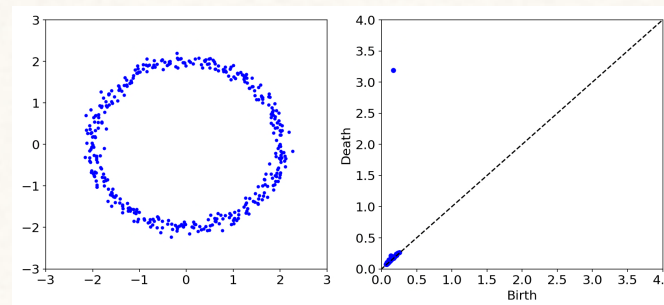
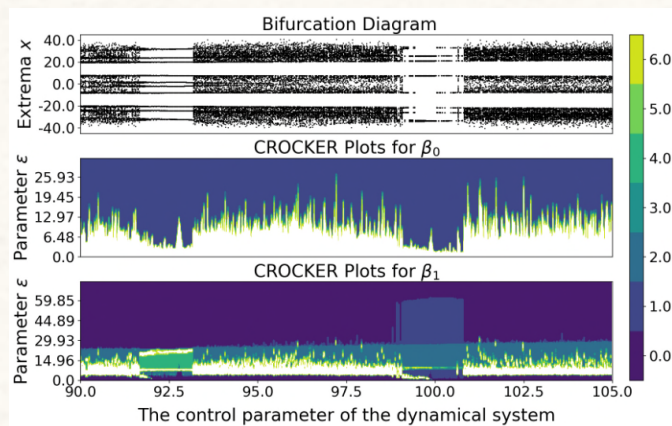
Algebraic



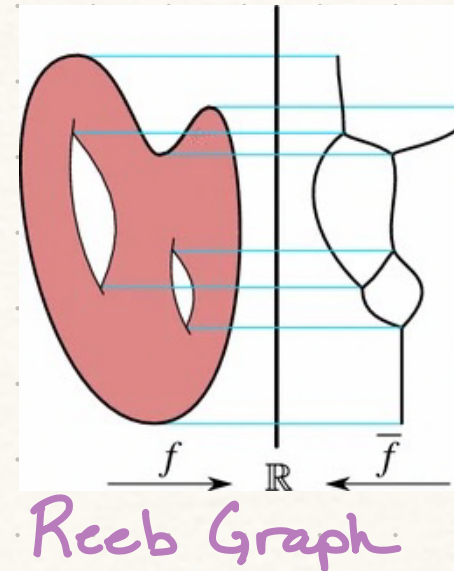
Euler Characteristic Curve



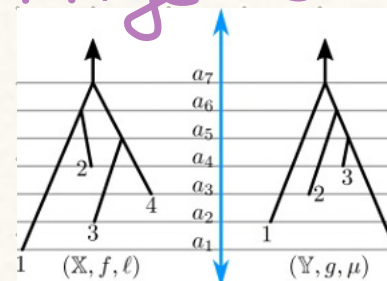
CROCKER Plots



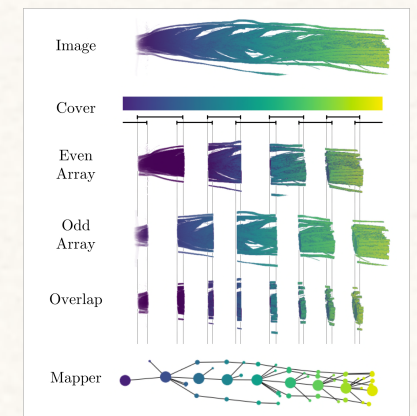
Persistent Homology



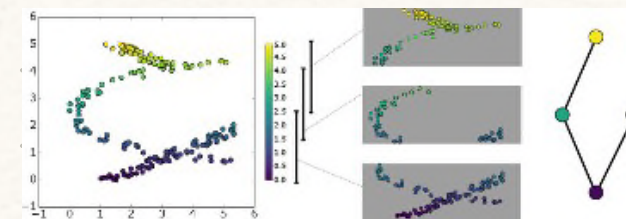
Merge Tree



Mapper Graph



Graph-Based



Mapper Input

Given data

- * $f: X \rightarrow \mathbb{R}^d$
- * cover $\mathcal{U} = \{U_i \subseteq \mathbb{R}^d\}$

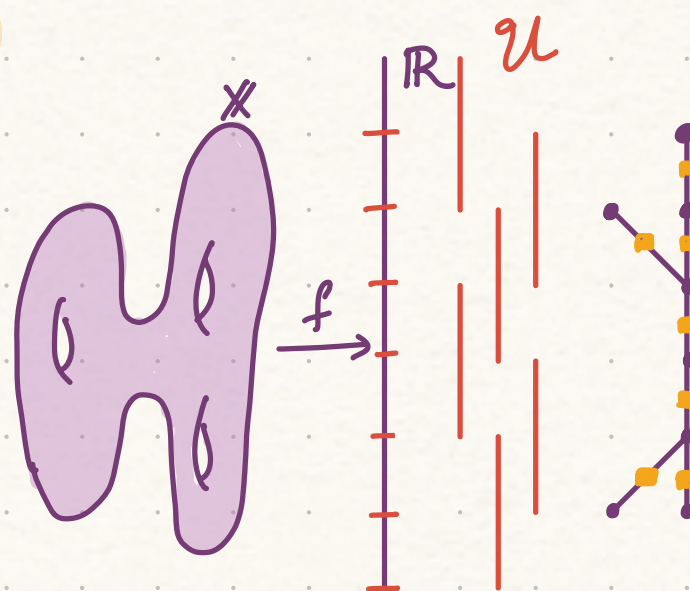
Categorical mapper is functor

$$F: \text{Open}(\text{Nrv}(\mathcal{U})) \rightarrow \text{Set}$$

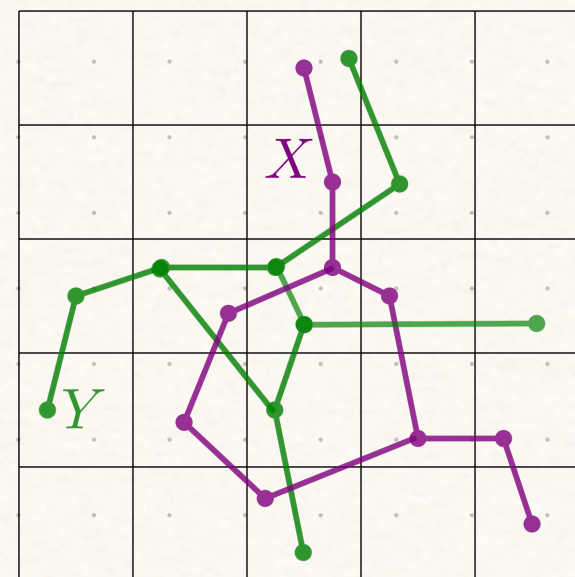
$$A \mapsto \pi_0 f^{-1}(|A|)$$

Singh, Mémoli, Carlsson. **Topological methods for the analysis of high dimensional data sets and 3d object recognition.** Point Based Graphics at EuroCG, 2007

$d=1$



$d=2$



The Goal

A distance on mapper graphs

$$d(\phi, \psi)$$

that is computable.

Distance:

$$d: X \times X \longrightarrow \mathbb{R}$$

- $d(x, y) = 0 \iff x = y$

- $d(x, y) = d(y, x)$

- $d(x, z) \leq d(x, y) + d(y, z)$

Interleaving Distance

Apologies for omissions
Please let me know!

Persistence Modules

$$F: (\mathbb{R}^d, \leq) \rightarrow \text{Vect}$$

Chazal
Cohen-Steiner
Glisse
Guibas
Oudot

Lesnick
Blumberg
Bjerkevik
Botnan
Kerber

Landi
Bauer
Hradka
Scocuda
Escobar
... et al.

Merge Trees

$$F: (\mathbb{R}, \leq) \rightarrow \text{Set}$$



Morozov
Beketayev
Weber
Stefanow

Touli
Y. Wang
EM

Functor categories

$$F: \mathcal{P} \rightarrow \mathcal{C}$$

Bubenik
Scott
deSilva

Zigzags

$$F: \downarrow \downarrow \downarrow \downarrow \rightarrow \text{Vect}$$

W. Kim
Mémoli
Botnan
Lesnick
Bjerkevik

Reeb Graph

$$F: \text{Int} \rightarrow \text{Set}$$



Curry
de Silva
EM

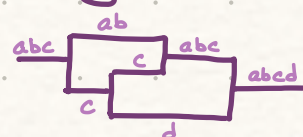
Patel
Bauer

Category with a flow

$$(\mathcal{C}, \tau: \mathbb{R} \rightarrow \text{End}(\mathcal{C}))$$

Stefanow
deSilva
EM

Formigrams



W. Kim
Mémoli
Stefanow

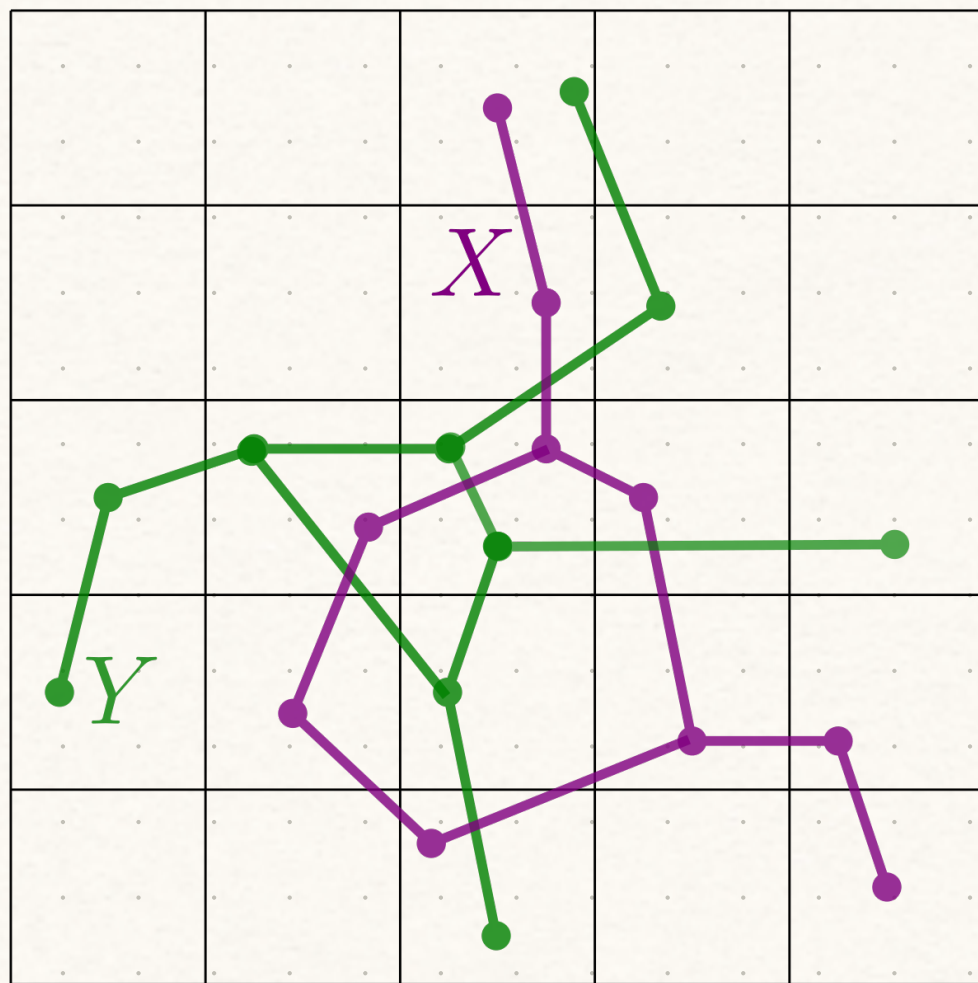
Mapper Graphs



Dey
Y. Wang
Mémoli

B. Wang
EM
Brown
Bobrowski

The Plan



The Setup

- * Discretize cover of $\mathbb{R}^d$
- * Encode mapper as cosheaf
- * Define Interleaving Distance

The Punchline

- * Given not-quite-an-interleaving φ, ψ
- * Compute loss function $L(\varphi, \psi)$
- * Use to build interleaving Φ, Ψ
- * Use to bound the interleaving dist.

The Setup:

Grids

Mapper

Interleaving

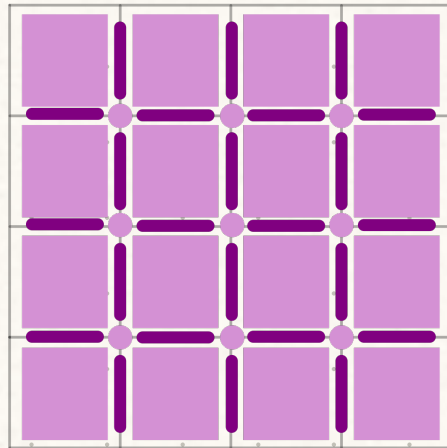
The Grid

Data: $f: X \rightarrow \mathbb{R}^d$

K is the cubical
complex of $\mathbb{R}^d$

w/ sides δ

Face relation $\sigma \leq \tau$

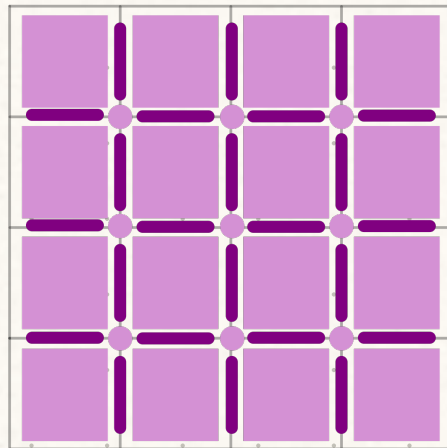


K for $d=2$

The Grid

Data: $f: X \rightarrow \mathbb{R}^d$

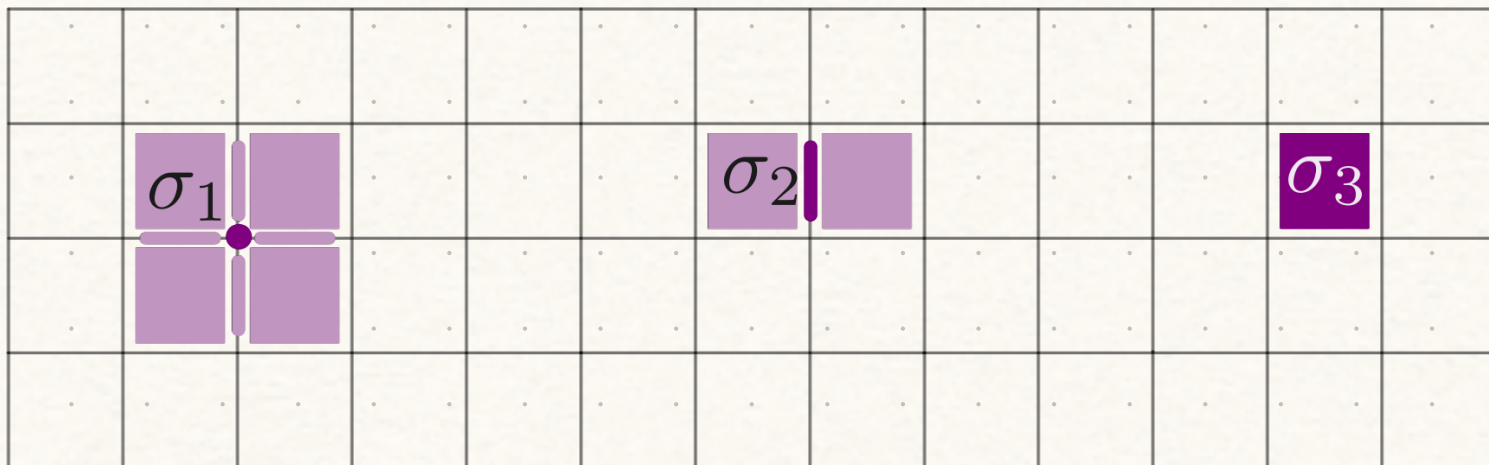
K is the cubical
complex of $\mathbb{R}^d$
w/ sides δ
Face relation $\sigma \leq \tau$



K for $d=2$

Cover

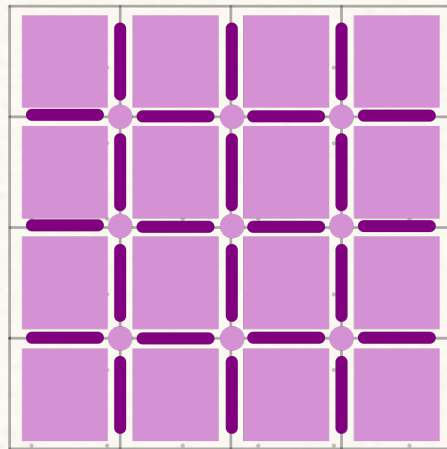
$$\mathcal{U} = \{u_\sigma \mid u_\sigma = \bigcup_{\tau \geq \sigma} |\tau|\}$$



The Grid

Data: $f: X \rightarrow \mathbb{R}^d$

K is the cubical complex of $\mathbb{R}^d$
w/ sides δ
Face relation $\sigma \leq \tau$



K for $d=2$

Cover

$$\mathcal{U} = \{u_\sigma \mid u_\sigma = \bigcup_{\tau \geq \sigma} \tau\}$$

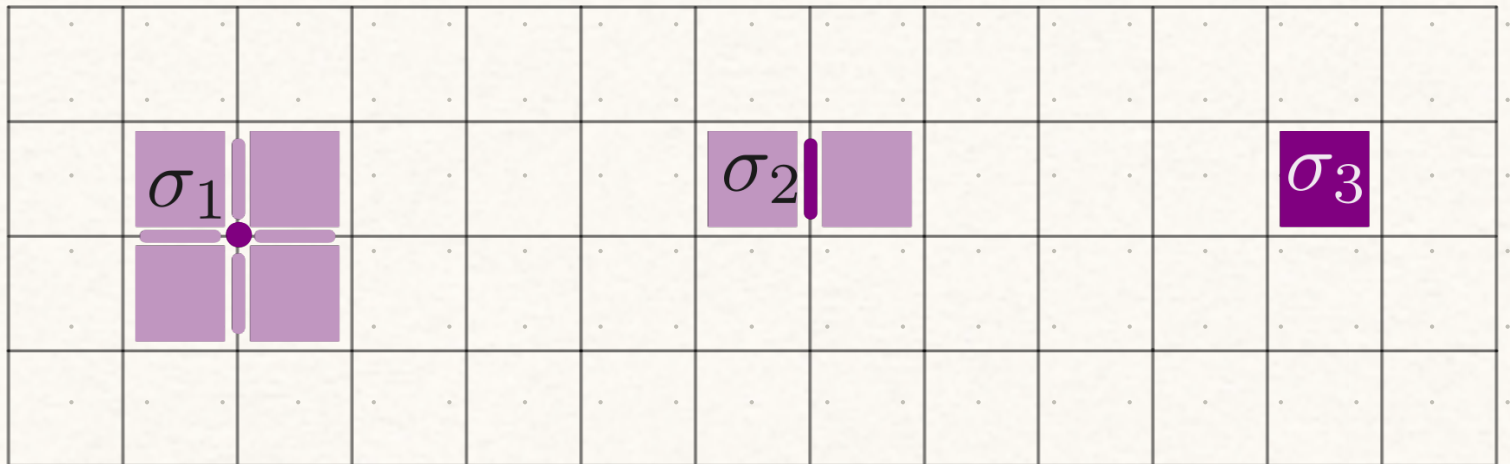
Alexandroff Topology

Up-set $u_\sigma \mapsto u_\sigma^\uparrow = \{u_\tau \mid u_\tau \geq u_\sigma\}$
Down-set $u_\sigma \mapsto u_\sigma^\downarrow = \{u_\tau \mid u_\tau \leq u_\sigma\}$

open sets

Open sets in $\mathcal{U}$

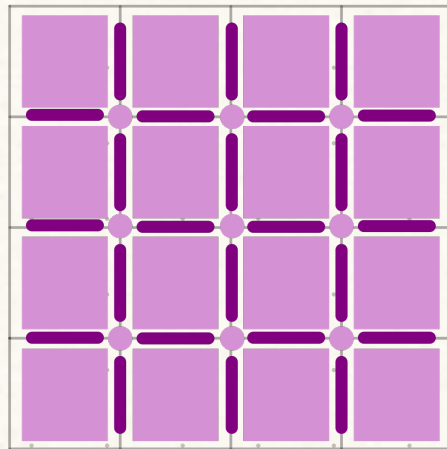
$$S_\sigma = \{u_\sigma\}^\downarrow$$



The Grid

Data: $f: X \rightarrow \mathbb{R}^d$

K is the cubical complex of $\mathbb{R}^d$
w/ sides δ
Face relation $\sigma \leq \tau$



K for $d=2$

Cover

$$\mathcal{U} = \{u_\sigma \mid u_\sigma = \bigcup_{\tau \geq \sigma} u_\tau\}$$

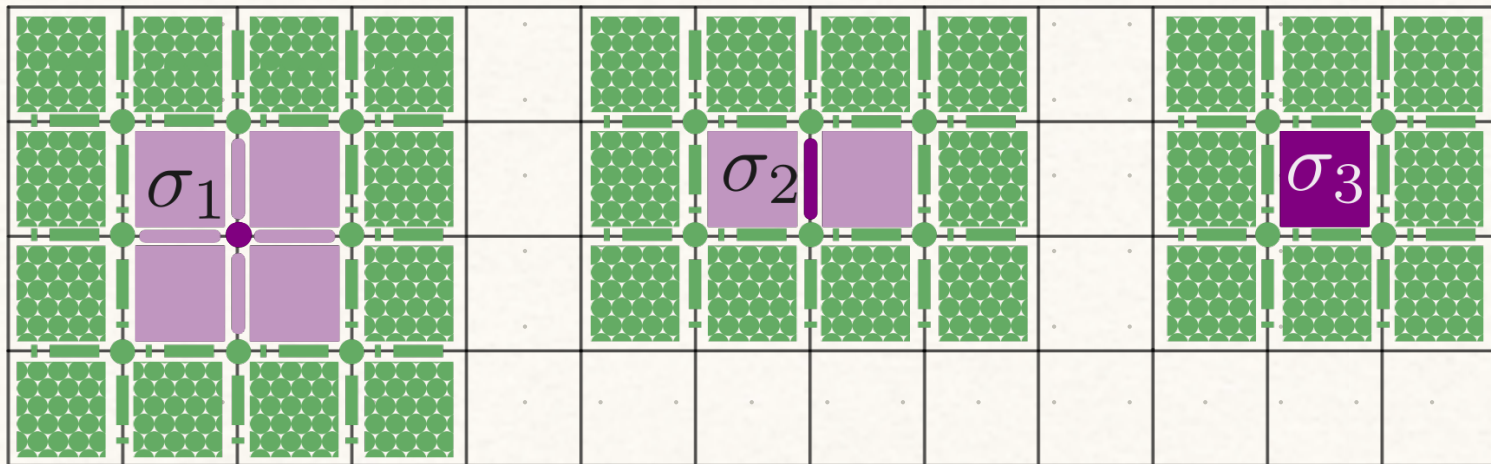
Alexandroff Topology

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Open sets in $\mathcal{U}$

$$S_\sigma = \{u_\sigma\}^\downarrow$$

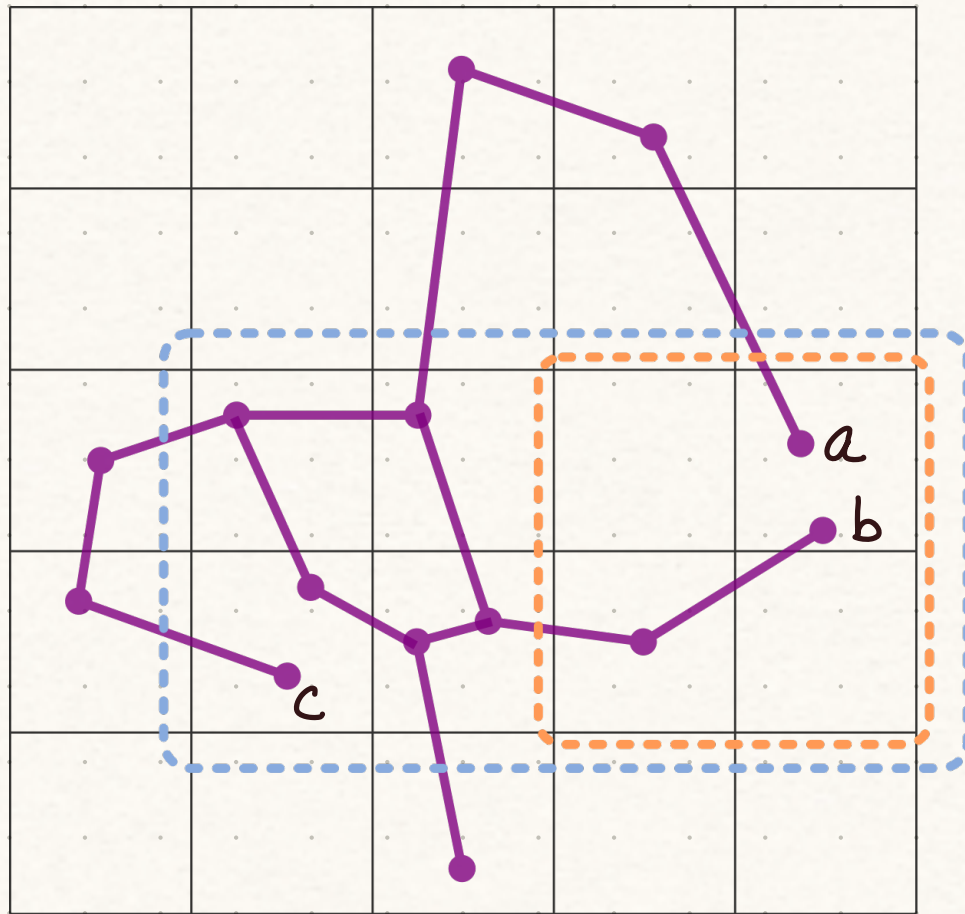
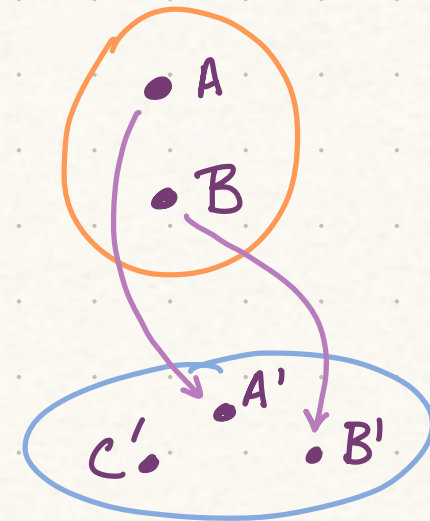


Thickening

The n -thickening of

$$S^n = \begin{cases} S & \text{if } n=0 \\ (S^{n-1})^{\downarrow\uparrow} & n \geq 1 \end{cases}$$

Cache Representation


$$\begin{array}{ccc}
 F: \text{Open}(\mathcal{U}) & \longrightarrow & \text{Set} \\
 S & & \pi_0 f^{-1}(|S|) \\
 \cap & & \downarrow F[\leq] \\
 T & & \pi_0 f^{-1}(|T|)
 \end{array}$$
$$F(S)$$
$$F(T)$$


The Interleaving Distance

Definition

Given cosheaves $F, G : \text{Open}(\mathcal{U}) \rightarrow \text{Set}$ and fix $n \geq 0$
 An n -interleaving is a pair of natural transformations

$$\varphi : F \Rightarrow G \circ (-)^{\sim} \quad \psi : G \Rightarrow F \circ (-)^{\sim}$$

Such that

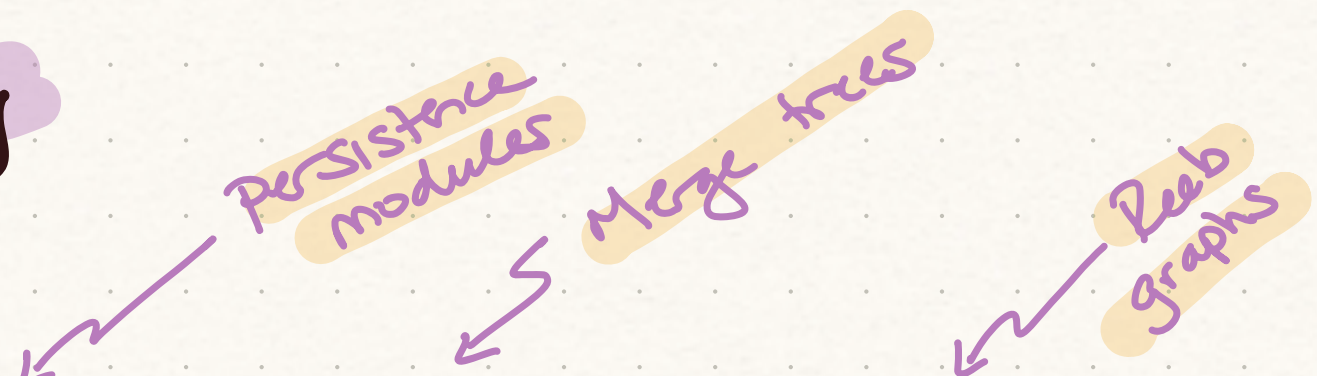
$$\begin{array}{ccccc}
 F(S) & \xrightarrow{F[S \subseteq S^{2n}]} & F(S^{2n}) & & F(S^n) \\
 \searrow \varphi_S & & \nearrow \psi_{S^n} & \nearrow \psi_S & \searrow \varphi_{S^n} \\
 & & G(S^n) & & G(S) \\
 & & & \xrightarrow{G[S \subseteq S^{2n}]} & G(S^{2n})
 \end{array}$$

Commutate.

The interleaving distance is

$$d_I(F, G) = \min \{ n \geq 0 \mid \exists n\text{-interleaving} \}.$$

Complexity



type/ δ	$\mathbf{Z} \rightarrow \mathbf{Vec}$	$\mathbf{Z} \rightarrow \mathbf{Set}$	$\mathbf{Z}^2 \rightarrow \mathbf{Vec}$	$\mathbf{Z}^2 \rightarrow \mathbf{Set}$	$\mathbf{Z}^{L,C} \rightarrow \mathbf{Vec}_{\mathbb{Z}/2\mathbb{Z}}$
$\delta = 0$	$\mathcal{O}(n^\omega)$	$\mathcal{O}(n)$	$\mathcal{O}(n^6)$	GI-complete	$\mathcal{O}(n^6)$
$\delta \geq 1$	$\mathcal{O}(n^\omega)$	NP-complete	CI-hard	NP-complete	NP-complete

Table 1: The complexity of checking for δ -interleavings between modules M and M' . If the target category is $\mathbf{Vec}$ then $n = \dim M + \dim M'$, and if the target category is $\mathbf{Set}$ then $n = |M| + |M'|$. Here ω is the matrix multiplication exponent.

The
Loss
Function

n-Assignment

Definition

An **unnatural transformation** is a collection of maps

$$\varphi = \{ \varphi_u: F(S) \longrightarrow G^n(T) \\ \forall u \in \text{Open}(K) \}$$

Definition

An **n-assignment** is a pair of unnatural transformations

$$\varphi: F \Rightarrow G^n \\ \psi: G \Rightarrow F^n$$

$$\begin{array}{ccc} F(S) & \xrightarrow{F[\varepsilon]} & F(T) \\ \varphi_u \downarrow & \begin{array}{c} \text{---} \text{?} \text{---} \\ \text{---} \end{array} & \downarrow \varphi_v \\ G(S^n) & \xrightarrow{G[\varepsilon]} & G(T^n) \end{array}$$

n-Assignment

Definition

An **unnatural transformation** is a collection of maps

$$\psi = \{ \psi_u: F(S) \longrightarrow G^n(T) \\ \forall u \in \text{Open}(K) \}$$

$$\begin{array}{ccc} F(S) & \xrightarrow{F[\varepsilon]} & F(T) \\ \psi_u \downarrow & \text{?} & \downarrow \psi_v \\ G(S^n) & \xrightarrow{G[\varepsilon]} & G(T^n) \end{array}$$

Definition

An **n-assignment** is a pair of unnatural transformations

$$\begin{aligned} \varphi: F &\Rightarrow G^n \\ \psi: G &\Rightarrow F^n \end{aligned}$$

Warning:

Un-natural Transformations

Natural Transformations

Four Diagrams

Natural Transformation

$$\begin{array}{ccc}
 F(S) & \xrightarrow{F[\subseteq]} & F(T) \\
 \varphi_S \searrow & & \searrow \varphi_T \\
 & G(S^n) \xrightarrow{G[\subseteq]} G(T^n) & \\
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & F(S^n) \xrightarrow{F[\subseteq]} F(T^n) \\
 & \nearrow \psi_S & \nearrow \psi_T \\
 G(S) & \xrightarrow{G[\subseteq]} & G(T)
 \end{array}$$

Notation:

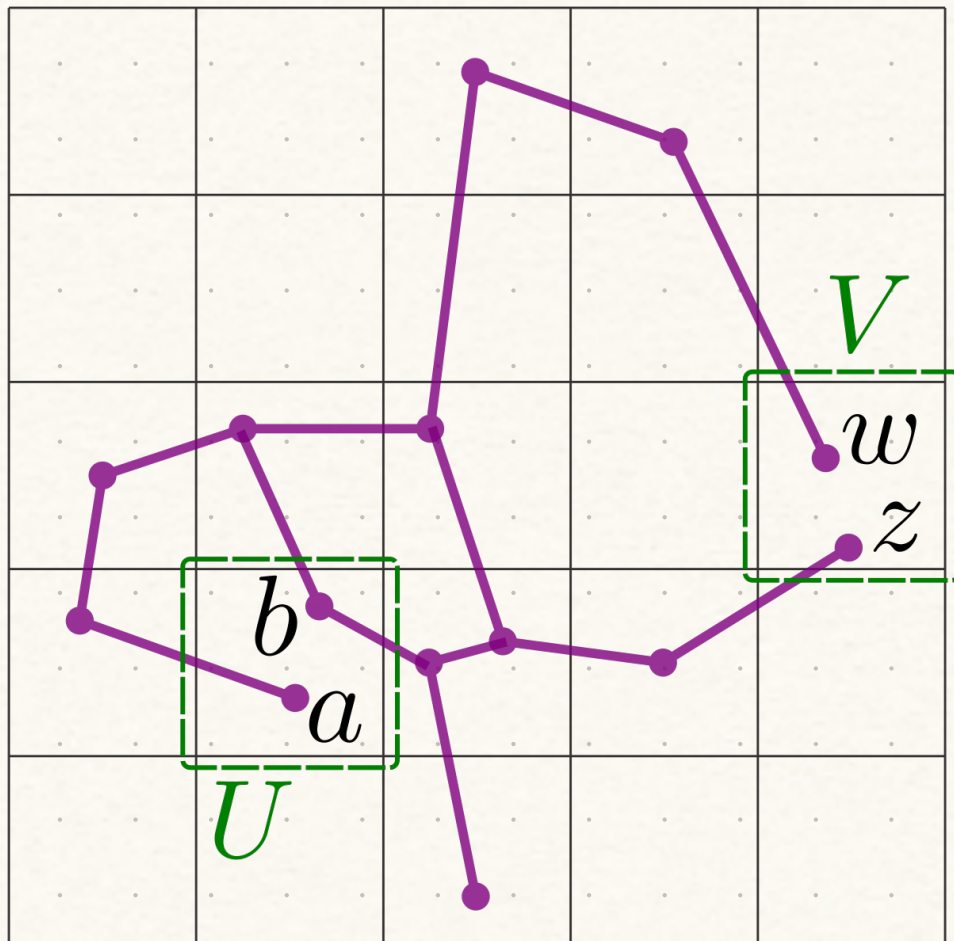
$$\begin{array}{c}
 \square_{\varphi}(S, T) \\
 \nabla_{\varphi, \psi}(S)
 \end{array}$$

$$\begin{array}{c}
 \square_{\psi}(S, T) \\
 \Delta_{\varphi, \psi}(S)
 \end{array}$$

Inter-leaving

$$\begin{array}{ccc}
 F(S) & \xrightarrow{F[S \subseteq S^{2n}]} & F(S^{2n}) \\
 \varphi_S \searrow & & \nearrow \psi_{S^n} \\
 & G(S^n) & \\
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & F(S^n) \\
 & \nearrow \psi_S & \searrow \varphi_{S^n} \\
 G(S) & \xrightarrow{G[S \subseteq S^{2n}]} & G(S^{2n})
 \end{array}$$

Distance in $F(S)$



Distance in $F(S)$

$$d_S^F(A, B) = \min \{ k \geq 0 \mid F[S \leq S^k](A) = F[S \leq S^k](B) \}$$

* Set = ∞ if no k exists

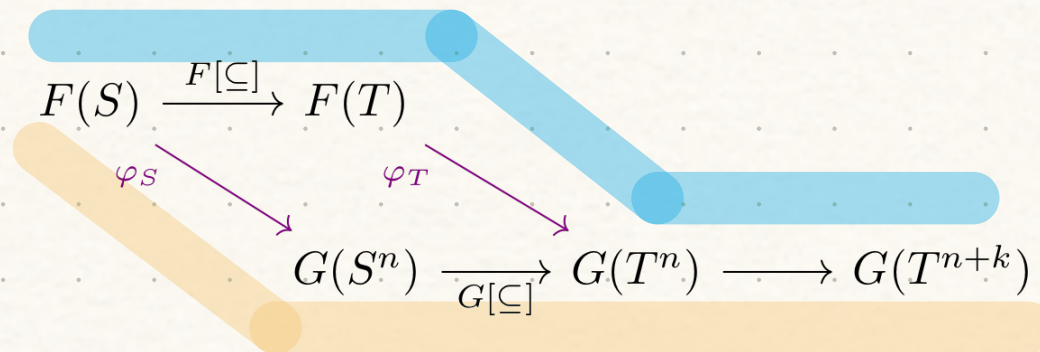
$$F(S) \xrightarrow{F[\leq]} F(S^k)$$

$$\begin{array}{ccc} A & \mapsto & C \\ B & \mapsto & C \end{array}$$

$\xleftarrow{\varepsilon_x}$ $d_u^F(A, B) = 1$ $d_v^F(w, z) = 2$

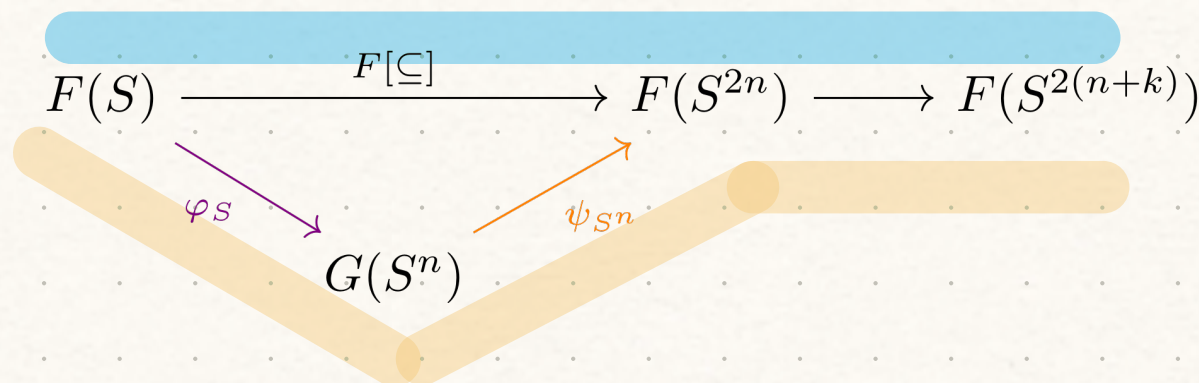
Loss

$$L_{\square}^{S,T}(\varphi) = \max_{\alpha \in F(S)} d_{T^n}^G(\varphi_T \circ F[S \subseteq T](\alpha), G[S^n \subseteq T^n] \circ \varphi_S(\alpha))$$



Net. Trans

$$L_{\nabla}^S(\varphi, \psi) = \max_{\alpha \in F(S)} \left[\frac{1}{2} \cdot d_{S^{2n}}^F(F[S \subseteq S^{2n}](\alpha), \psi_{S^n} \circ \varphi_S(\alpha)) \right]$$



Inter-leaving

Full Loss Function

Definition

Given n -assignment φ, ψ . Define

$$L_{\square}^{S,T}(\varphi) = \max_{\alpha \in F(S)} d_{T^n}^G(\varphi_T \circ F[S \subseteq T](\alpha), G[S^n \subseteq T^n] \circ \varphi_S(\alpha))$$

$$L_{\square}^{S,T}(\psi) = \max_{\alpha \in G(S)} d_{T^n}^F(\psi_T \circ G[S \subseteq T](\alpha), F[S^n \subseteq T^n] \circ \psi_S(\alpha))$$

$$L_{\nabla}^S(\varphi, \psi) = \max_{\alpha \in F(S)} \left[\frac{1}{2} \cdot d_{S^{2n}}^F(F[S \subseteq S^{2n}](\alpha), \psi_{S^n} \circ \varphi_S(\alpha)) \right]$$

$$L_{\triangle}^S(\varphi, \psi) = \max_{\alpha \in G(S)} \left[\frac{1}{2} \cdot d_{S^{2n}}^G(G[S \subseteq S^{2n}](\alpha), \varphi_{S^n} \circ \psi_S(\alpha)) \right].$$

Then the loss is

$$L(\varphi, \psi) = \max_{S \subseteq T} \left\{ L_{\square}^{S,T}, L_{\square}^{S,T}, L_{\triangle}^S, L_{\nabla}^S \right\}$$

Bound v. 1.0

Chambers, EM, Percival, B. Wang. **Bounding the interleaving distance for mapper graphs with a loss function.** JACT, 2025. *arXiv:2307.15130*

Theorem (Chambers, EM, Percival, B. Wang 2025)

Given n -assignment

$$\varphi = \{\varphi_s: F(s) \rightarrow G(s^n)\}$$

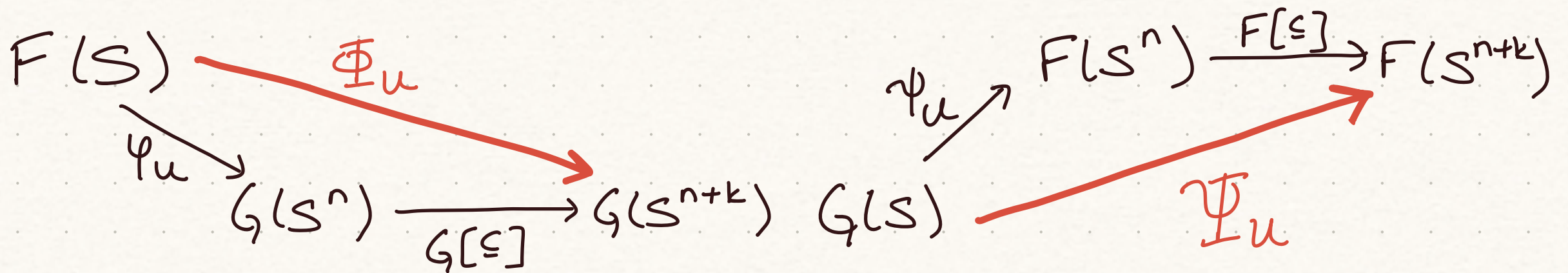
$$\psi = \{\psi_s: G(s) \rightarrow F(s^n)\}$$

$$d_I(F, G) \leq n + L(\varphi, \psi).$$

Proof Sketch

Given n -assignment φ, ψ with loss $k = L(\varphi, \psi)$

Define $(n+k)$ -assignment Φ and Ψ



n-Assignment

Definition

A **unnatural transformation** is a collection of maps

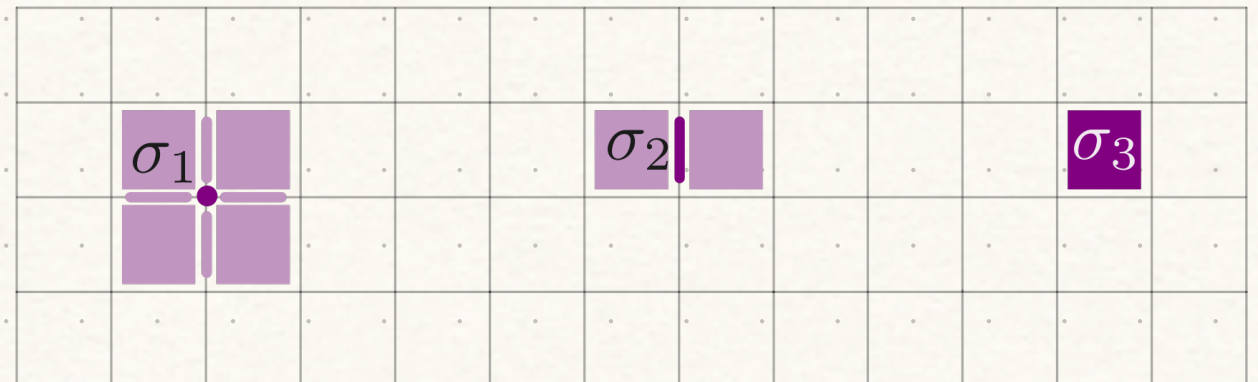
$$\varphi = \{ \varphi_u: F(S) \longrightarrow G^n(S) \mid \forall S \in \text{Ob}(\mathcal{U}) \}$$

Definition

A **n-assignment** is a pair of unnatural transformations

$$\begin{aligned} \varphi: F &\Rightarrow G^n \\ \psi: G &\Rightarrow F^n \end{aligned}$$

$$\begin{array}{ccc} F(S) & \xrightarrow{F[\varepsilon]} & F(T) \\ \varphi_S \downarrow & \text{?} & \downarrow \varphi \\ G(S^n) & \xrightarrow{G[\varepsilon]} & G(T^n) \end{array}$$



Basis

$\wedge$ n-Assignment

Definition

A ^{basis} $\wedge$ unnatural transformation is a collection of maps

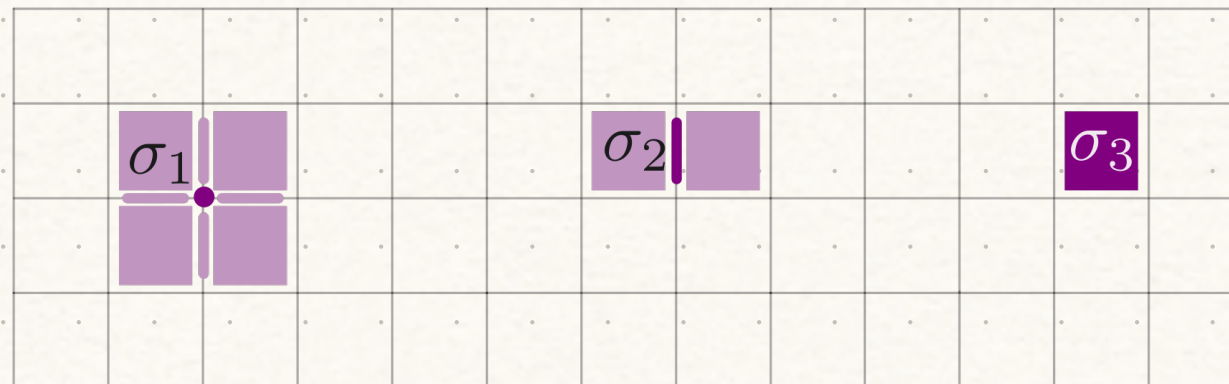
$$\varphi = \{ \varphi_{u_\sigma}: F(S_\sigma) \longrightarrow G^n(S_\sigma) \mid \forall \sigma \in K \}$$

Definition

A ^{basis} $\wedge$ n-assignment is a pair of $\wedge$ unnatural transformations

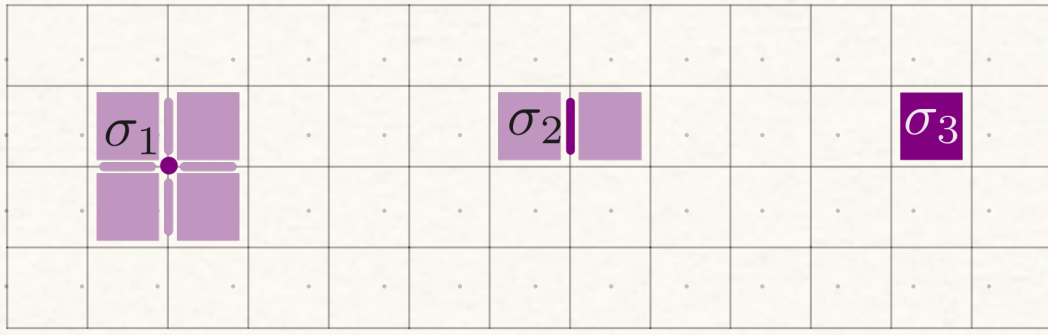
$$\begin{aligned} \varphi: F &\Rightarrow G^n \\ \psi: G &\Rightarrow F^n \end{aligned}$$

$$\begin{array}{ccc} F(S) & \xrightarrow{F[\varepsilon]} & F(T) \\ \varphi_S \downarrow & \text{?} & \downarrow \varphi \\ G(S^n) & \xrightarrow{G[\varepsilon]} & G(T^n) \end{array}$$

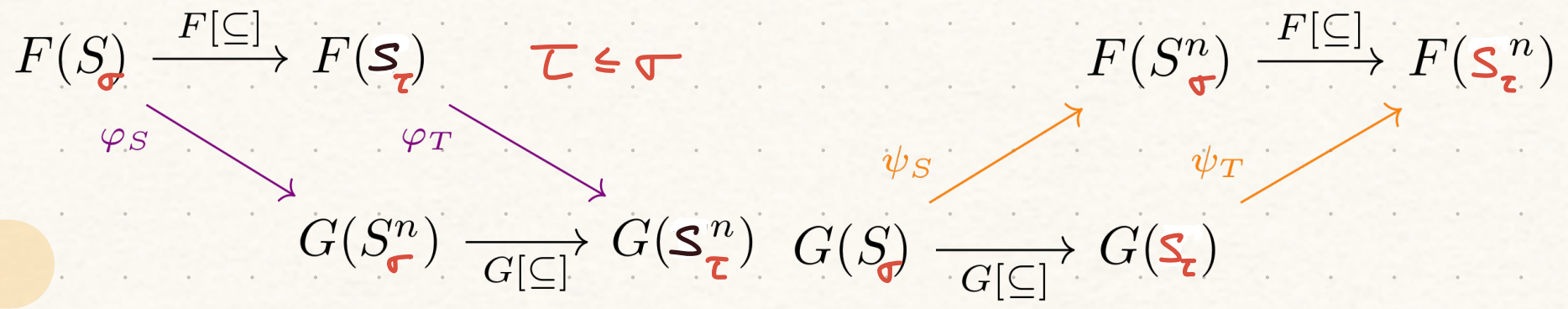


Four Diagrams

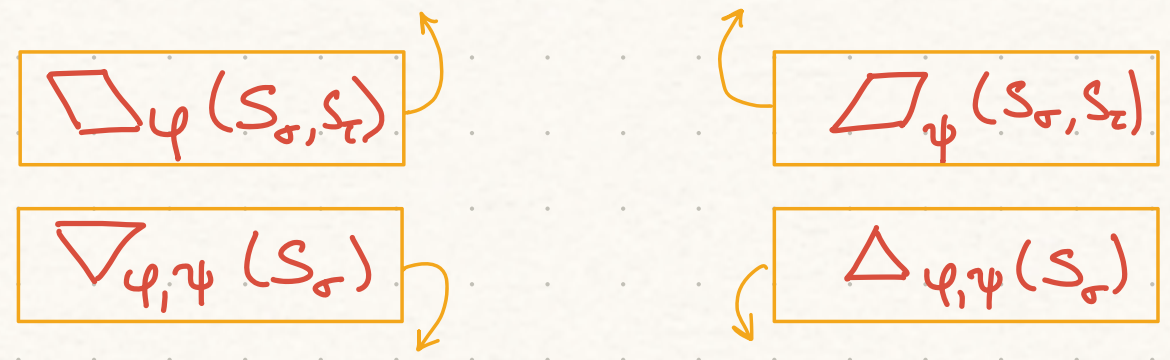
Basis



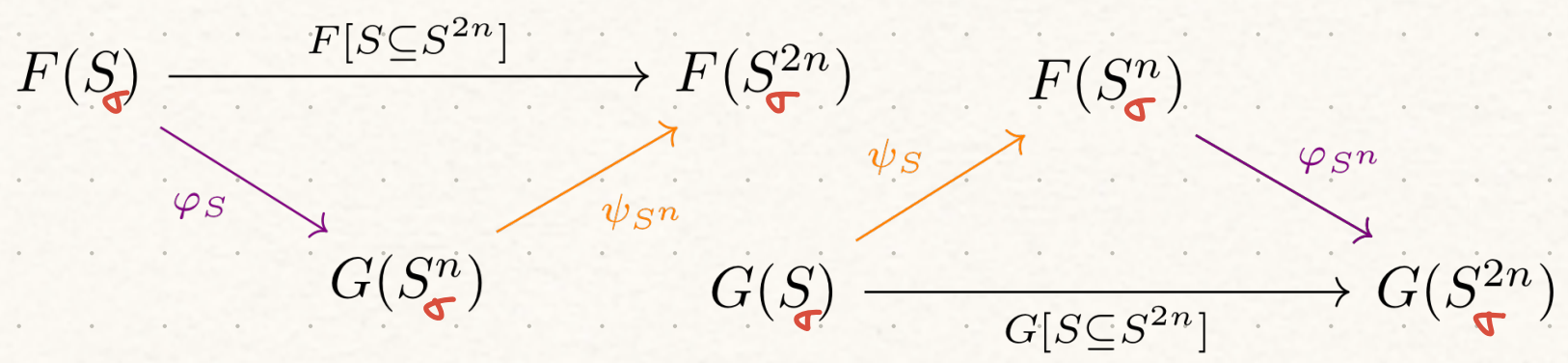
Natural Transformation



Notation:



Inter-leaving

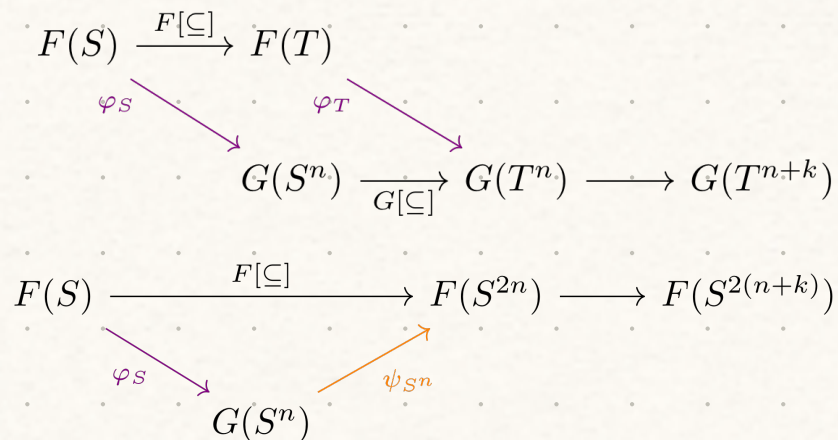
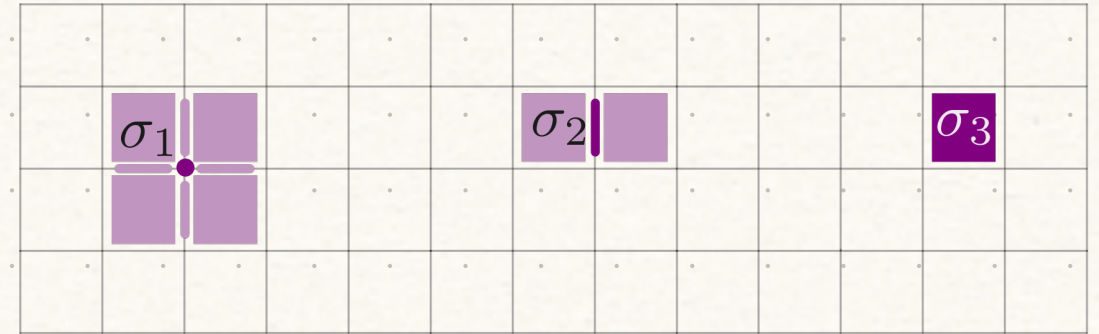


Loss Function

Definition

The **basis loss function** for n -assignments φ, ψ is

$$L_B(\varphi, \psi) = \max_{\sigma \leq \tau} \left\{ L_{\square}^{S_\tau, S_\sigma}, L_{\square}^{S_\tau, S_\sigma}, L_{\triangle}^{S_\sigma}, L_{\nabla}^{S_\sigma} \right\}$$



$$L_{\square}^{S,T}(\varphi) = \max_{\alpha \in F(S)} d_{T^n}^G(\varphi_T \circ F[S \subseteq T](\alpha), G[S^n \subseteq T^n] \circ \varphi_S(\alpha))$$

$$L_{\square}^{S,T}(\psi) = \max_{\alpha \in G(S)} d_{T^n}^F(\psi_T \circ G[S \subseteq T](\alpha), F[S^n \subseteq T^n] \circ \psi_S(\alpha))$$

$$L_{\nabla}^S(\varphi, \psi) = \max_{\alpha \in F(S)} \left[\frac{1}{2} \cdot d_{S^{2n}}^F(F[S \subseteq S^{2n}](\alpha), \psi_{S^n} \circ \varphi_S(\alpha)) \right]$$

$$L_{\triangle}^S(\varphi, \psi) = \max_{\alpha \in G(S)} \left[\frac{1}{2} \cdot d_{S^{2n}}^G(G[S \subseteq S^{2n}](\alpha), \varphi_{S^n} \circ \psi_S(\alpha)) \right]$$

Main Theorem

Theorem Chambers, EM, Perzval, B. Wang 2025

Given ^{basis} n -assignments $\varphi = \{ \varphi_{u_\sigma} : F(S_\sigma) \rightarrow G^n(S_\sigma) \mid \sigma \in K \}$
 $\psi = \{ \psi_{u_\sigma} : G(S_\sigma) \rightarrow F^n(S_\sigma) \mid \sigma \in K \}$

$$d_I(F, G) \leq n + L_B(\varphi, \psi)$$

Main Theorem

Theorem Chambers, EM, Perzval, B. Wang 2025

Given ^{basis} n -assignments $\varphi = \{ \varphi_{u_\sigma} : F(S_\sigma) \rightarrow G^n(S_\sigma) \mid \sigma \in K \}$
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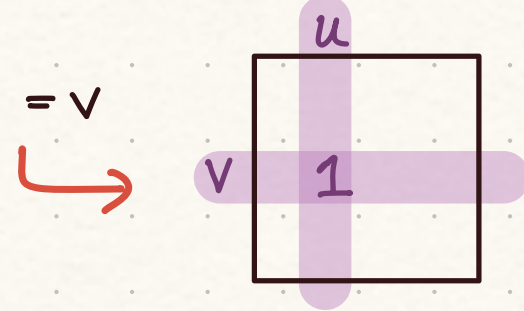
$$d_I(F, G) \leq n + \underline{L_B(\varphi, \psi)}$$

This is in P!!!!

Computation

Integer Program

* Store n-assignment maps as binary matrices
 $\varphi(u) = v$



* Loss computation
 $\hookrightarrow$ matrix multiplication

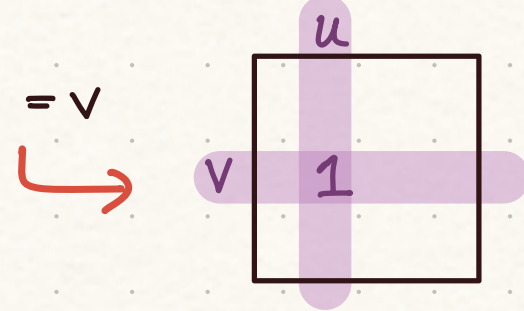
* Treat φ, ψ matrices as variables
 $\hookrightarrow$ optimal loss

* Binary search over n

	Loss Term	Diagram	Matrix Multiplication	Eval.
Edge-vertex Parallelogram	$L_{\square}^{S_\tau, S_\sigma}$	$ \begin{array}{ccc} F(S_\tau) & \xrightarrow{F[\subseteq]} & F(S_\sigma) \\ & \searrow \varphi_{S_\tau} & \searrow \varphi_{S_\sigma} \\ & & G^n(S_\tau) \xrightarrow{G[\subseteq]} G^n(S_\sigma) \end{array} $	$ \begin{aligned} & D_{G^n}^V \left(M_\varphi^V \cdot B_F^\uparrow - B_{G^n}^\uparrow \cdot M_\varphi^E \right) \\ & D_{G^n}^V \left(M_\varphi^V \cdot B_F^\downarrow - B_{G^n}^\downarrow \cdot M_\varphi^E \right) \end{aligned} $	$\max_{x_{ij} \in A} x_{ij}$
	$L_{\square}^{S_\tau, S_\sigma}$	$ \begin{array}{ccc} & F^n(S_\tau) \xrightarrow{F[\subseteq]} F^n(S_\sigma) & \\ \nearrow \psi_{S_\tau} & & \nearrow \psi_{S_\sigma} \\ G(S_\tau) & \xrightarrow{G[\subseteq]} & G(S_\sigma) \end{array} $	$ \begin{aligned} & D_{F^n}^V \left(M_\psi^V \cdot B_G^\uparrow - B_{F^n}^\uparrow \cdot M_\psi^E \right) \\ & D_{F^n}^V \left(M_\psi^V \cdot B_G^\downarrow - B_{F^n}^\downarrow \cdot M_\psi^E \right) \end{aligned} $	
Thickening Parallelogram	$L_{\square}^{S_\rho, S_\rho^n}$	$ \begin{array}{ccc} F(S_\rho) & \xrightarrow{F[\subseteq]} & F^n(S_\rho) \\ & \searrow \varphi_{S_\rho} & \searrow \varphi_{S_\rho^n} \\ & & G^n(S_\rho) \xrightarrow{G[\subseteq]} G^{2n}(S_\rho) \end{array} $	$ \begin{aligned} & D_{G^n}^V \left(M_{\varphi^n}^V \cdot I_F^V - I_{G^n}^V \cdot M_\varphi^V \right) \\ & D_{G^n}^E \left(M_{\varphi^n}^E \cdot I_F^E - I_{G^n}^E \cdot M_\varphi^E \right) \end{aligned} $	$\max_{x_{ij} \in A} x_{ij}$
	$L_{\square}^{S_\rho, S_\rho^n}$	$ \begin{array}{ccc} & F^n(S_\rho) \xrightarrow{F[\subseteq]} F^{2n}(S_\rho) & \\ \nearrow \psi_{S_\rho} & & \nearrow \psi_{S_\rho^n} \\ G(S_\rho) & \xrightarrow{G[\subseteq]} & G(S_\rho^n) \end{array} $	$ \begin{aligned} & D_{F^n}^V \left(M_{\psi^n}^V \cdot I_G^V - I_{F^n}^V \cdot M_\psi^V \right) \\ & D_{F^n}^E \left(M_{\psi^n}^E \cdot I_G^E - I_{F^n}^E \cdot M_\psi^E \right) \end{aligned} $	
Triangle	$L_{\nabla}^{S_\rho}$	$ \begin{array}{ccc} F(S_\rho) & \xrightarrow{F[\subseteq]} & F^{2n}(S_\rho) \\ & \searrow \varphi_{S_\rho} & \searrow \psi_{S_\rho^n} \\ & & G^n(S_\rho) \end{array} $	$ \begin{aligned} & D_{F^{2n}}^V \left(I_{F^n}^V \cdot I_F^V - M_{\psi^n}^V \cdot M_\varphi^V \right) \\ & D_{F^{2n}}^E \left(I_{F^n}^E \cdot I_F^E - M_{\psi^n}^E \cdot M_\varphi^E \right) \end{aligned} $	$\max_{x_{ij} \in A} \left\lceil \frac{x_{ij}}{2} \right\rceil$
	$L_{\triangle}^{S_\rho}$	$ \begin{array}{ccc} & F^n(S_\rho) & \\ \nearrow \psi_{S_\rho} & & \searrow \varphi_{S_\rho^n} \\ G(S_\rho) & \xrightarrow{G[\subseteq]} & G^{2n}(S_\rho) \end{array} $	$ \begin{aligned} & D_{G^{2n}}^V \left(I_{G^n}^V \cdot I_G^V - M_{\varphi^n}^V \cdot M_\psi^V \right) \\ & D_{G^{2n}}^E \left(I_{G^n}^E \cdot I_G^E - M_{\varphi^n}^E \cdot M_\psi^E \right) \end{aligned} $	

Integer Program

* Store n-assignment maps as binary matrices
 $\varphi(u) = v$



* Loss computation
 $\hookrightarrow$ matrix multiplication

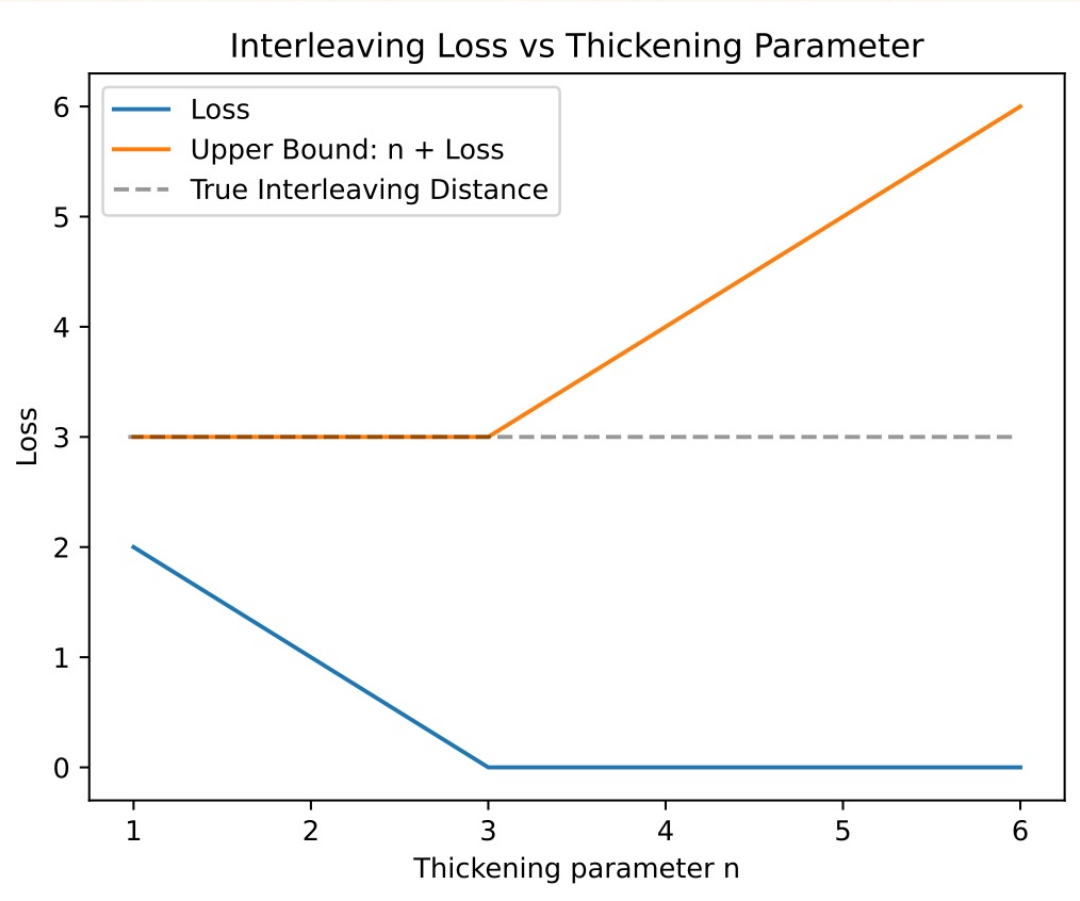
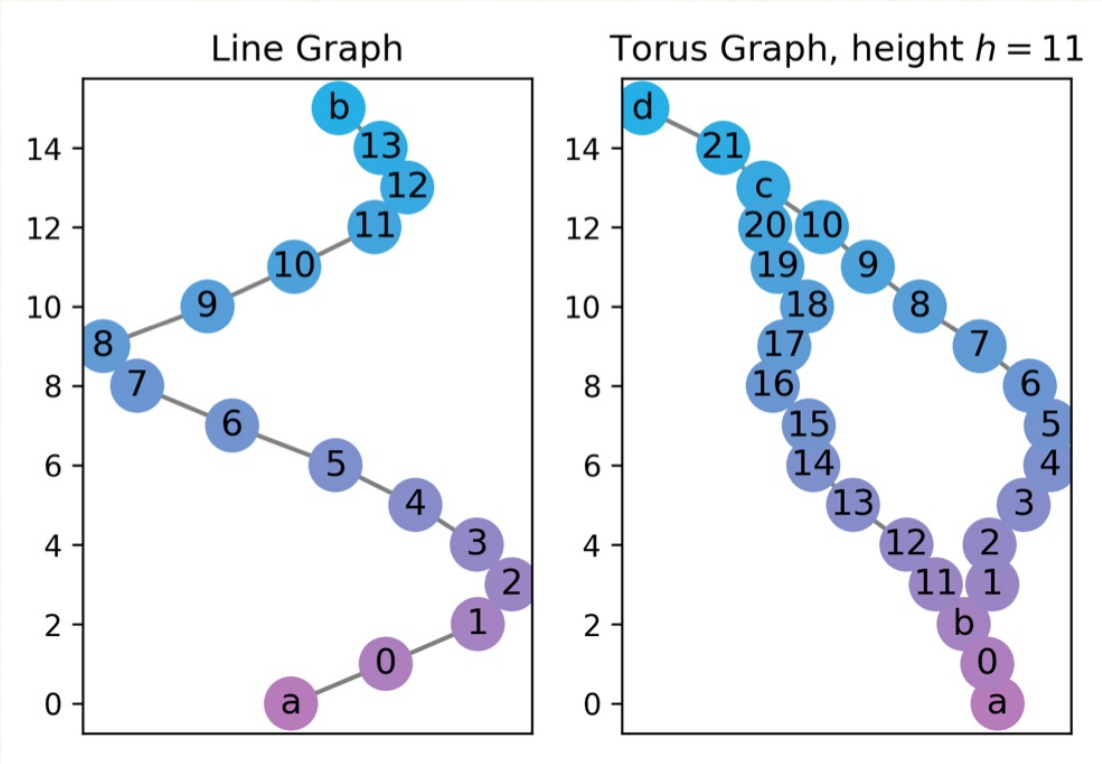
* Treat φ, ψ matrices as variables
 $\hookrightarrow$ optimal loss

* Binary search over n

	Loss Term	Diagram	Matrix Multiplication	Eval.
Edge-vertex Parallelogram	$L_{\nabla}^{S_\tau, S_\sigma}$	$ \begin{array}{ccc} F(S_\tau) & \xrightarrow{F[\subseteq]} & F(S_\sigma) \\ & \searrow \varphi_{S_\tau} & \searrow \varphi_{S_\sigma} \\ & & G^n(S_\tau) \xrightarrow{G[\subseteq]} G^n(S_\sigma) \end{array} $	$ \begin{array}{l} D_{G^n}^V \left(M_\varphi^V \cdot B_F^\uparrow - B_{G^n}^\uparrow \cdot M_\varphi^E \right) \\ D_{G^n}^V \left(M_\varphi^V \cdot B_F^\downarrow - B_{G^n}^\downarrow \cdot M_\varphi^E \right) \end{array} $	$\max_{x_{ij} \in A} x_{ij}$
	$L_{\square}^{S_\tau, S_\sigma}$	$ \begin{array}{ccc} & F^n(S_\tau) \xrightarrow{F[\subseteq]} F^n(S_\sigma) & \\ \psi_{S_\tau} \nearrow & & \searrow \psi_{S_\sigma} \\ G(S_\tau) \xrightarrow{G[\subseteq]} G(S_\sigma) & & \end{array} $	$ \begin{array}{l} D_{F^n}^V \left(M_\psi^V \cdot B_G^\uparrow - B_{F^n}^\uparrow \cdot M_\psi^E \right) \\ D_{F^n}^V \left(M_\psi^V \cdot B_G^\downarrow - B_{F^n}^\downarrow \cdot M_\psi^E \right) \end{array} $	
Thickening Parallelogram	$L_{\nabla}^{S_\rho, S_\rho^n}$	$ \begin{array}{ccc} F(S_\rho) & \xrightarrow{F[\subseteq]} & F^n(S_\rho) \\ & \searrow \varphi_{S_\rho} & \searrow \varphi_{S_\rho^n} \\ & & G^n(S_\rho) \xrightarrow{G[\subseteq]} G^{2n}(S_\rho) \end{array} $	$ \begin{array}{l} D_{G^n}^V \left(M_{\varphi^n}^V \cdot I_F^V - I_{G^n}^V \cdot M_\varphi^V \right) \\ D_{G^n}^E \left(M_{\varphi^n}^E \cdot I_F^E - I_{G^n}^E \cdot M_\varphi^E \right) \end{array} $	$\max_{x_{ij} \in A} \left\lceil \frac{x_{ij}}{2} \right\rceil$
	$L_{\square}^{S_\rho, S_\rho^n}$	$ \begin{array}{ccc} & F^n(S_\rho) \xrightarrow{F[\subseteq]} F^{2n}(S_\rho) & \\ \psi_{S_\rho} \nearrow & & \searrow \psi_{S_\rho^n} \\ G(S_\rho) \xrightarrow{G[\subseteq]} G(S_\rho^n) & & \end{array} $	$ \begin{array}{l} D_{F^n}^V \left(M_{\psi^n}^V \cdot I_G^V - I_{F^n}^V \cdot M_\psi^V \right) \\ D_{F^n}^E \left(M_{\psi^n}^E \cdot I_G^E - I_{F^n}^E \cdot M_\psi^E \right) \end{array} $	
Triangle	$L_{\nabla}^{S_\rho}$	$ \begin{array}{ccc} F(S_\rho) & \xrightarrow{F[\subseteq]} & F^{2n}(S_\rho) \\ & \searrow \varphi_{S_\rho} & \searrow \psi_{S_\rho^n} \\ & & G^n(S_\rho) \end{array} $	$ \begin{array}{l} D_{F^{2n}}^V \left(I_{F^n}^V \cdot I_F^V - M_{\psi^n}^V \cdot M_\varphi^V \right) \\ D_{F^{2n}}^E \left(I_{F^n}^E \cdot I_F^E - M_{\psi^n}^E \cdot M_\varphi^E \right) \end{array} $	$\max_{x_{ij} \in A} \left\lceil \frac{x_{ij}}{2} \right\rceil$
	$L_{\triangle}^{S_\rho}$	$ \begin{array}{ccc} & F^n(S_\rho) & \\ \psi_{S_\rho} \nearrow & & \searrow \varphi_{S_\rho^n} \\ G(S_\rho) \xrightarrow{G[\subseteq]} G^{2n}(S_\rho) & & \end{array} $	$ \begin{array}{l} D_{G^{2n}}^V \left(I_{G^n}^V \cdot I_G^V - M_{\varphi^n}^V \cdot M_\psi^V \right) \\ D_{G^{2n}}^E \left(I_{G^n}^E \cdot I_G^E - M_{\varphi^n}^E \cdot M_\psi^E \right) \end{array} $	

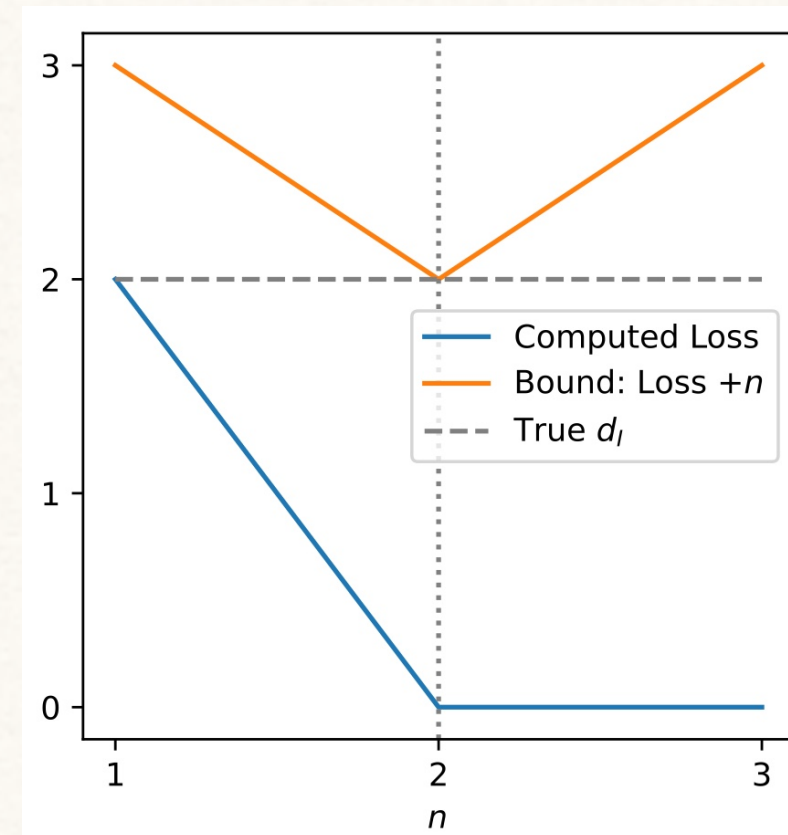
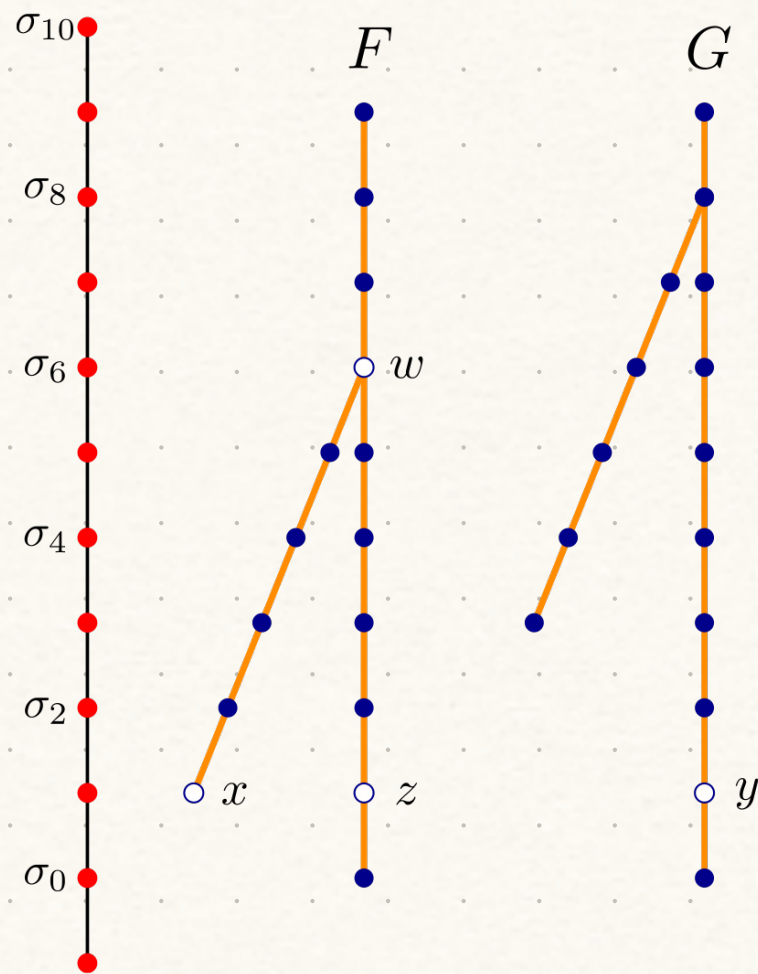
Chambers, Ghosh, EM, Percival, B. Wang. **Towards an Optimal Bound for the Interleaving Distance on Mapper Graphs.** *arXiv:2504.03865*

Ex: Torus and Line



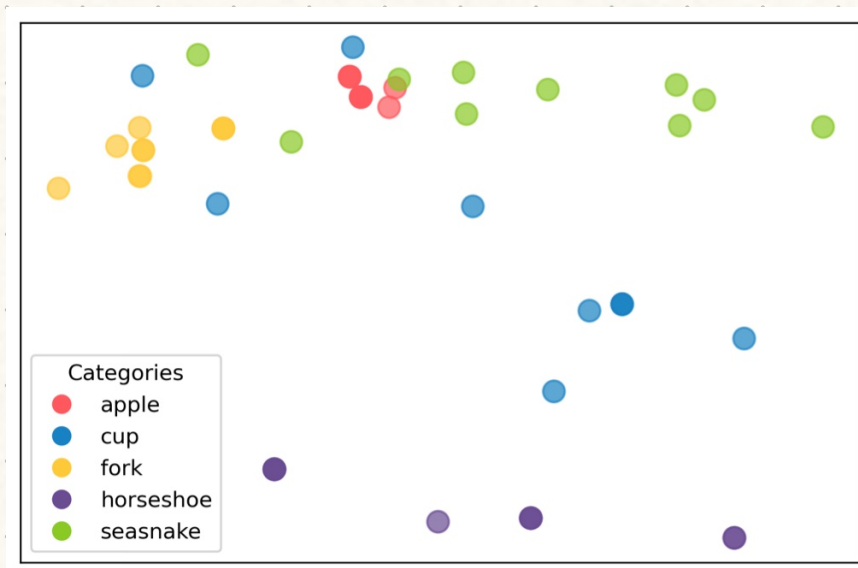
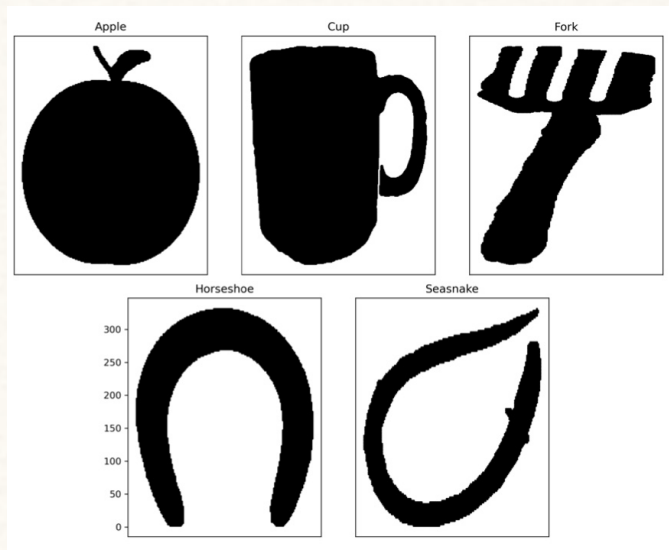
$$d_I(F, G) \leq n + L(\varphi, \psi)$$

Binary Search is Necessary

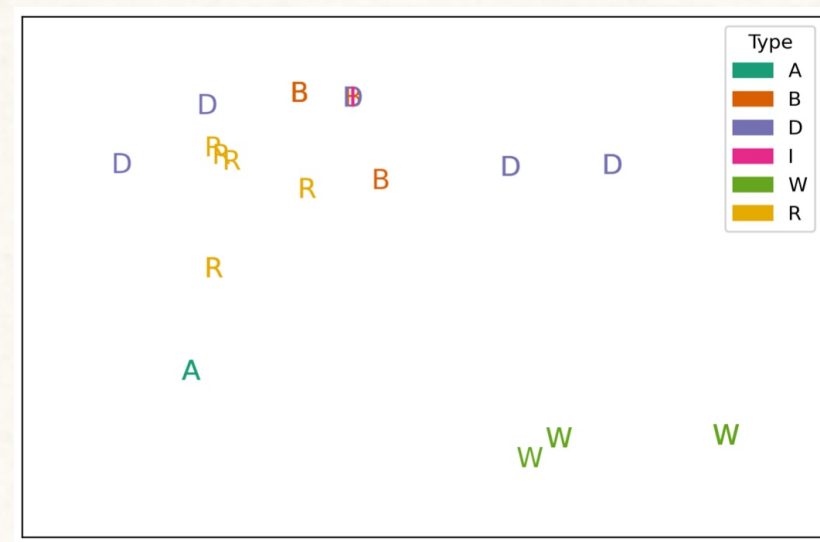
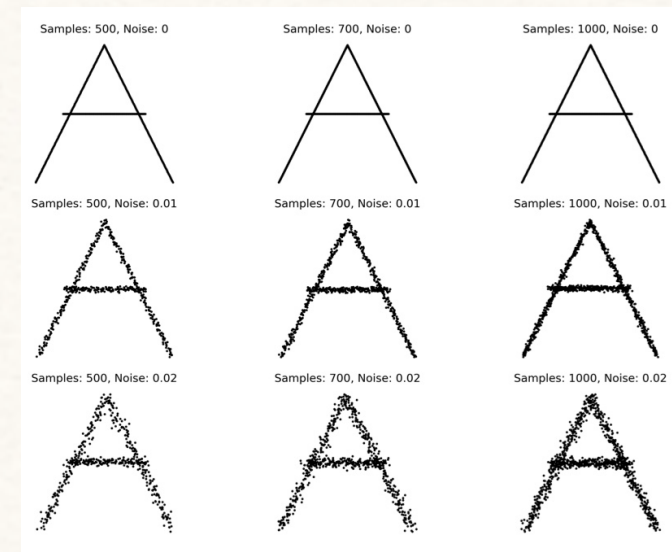


$$d_{\mathcal{I}}(F, G) \leq n + L(\varphi, \psi)$$

More Examples



MPEG-7



Alphabet

Wrap-Up

TL;DR

What we did

- * Fix $F, G: \text{Open}(K) \rightarrow \text{Set}$

- * Given n -assignment φ, ψ

compute $L_B(\varphi, \psi)$

- * Use ILP to get best φ, ψ

- * Binary search over n

- * Provide explicit $(n + L_B(\varphi, \psi))$ -
interleaving Φ, Ψ

TL;DR

What we did

- * Fix $F, G: \text{Open}(K) \rightarrow \text{Set}$

- * Given n -assignment φ, ψ

compute $L_B(\varphi, \psi)$

- * Use ILP to get best φ, ψ

- * Binary search over n

- * Provide explicit $(n + L_B(\varphi, \psi))$ -
interleaving Φ, Ψ

Future Work

- * Extension to other signatures

 - Reeb graphs

 - Multiparameter Persistence

Astrid Olave's
Poster !

- * Additional theoretical guarantees

Open-Source Code

ect

ect

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2. Modules
3. Tutorials
3.1. Tutorial: ECT for embedded graphs
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3.3.1. Acknowledgements
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3.3. ECT on Matisse's "The Parakeet and the Mermaid"

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Here, we are going to give an example of using the ECT to classify the cutout shapes from Henri Matisse's 1952 "The Parakeet and the Mermaid".



```
[1]: #-----  
# Standard imports  
#-----  
import numpy as np # for arrays  
import matplotlib.pyplot as plt # for plotting  
from sklearn.decomposition import PCA # for PCA for normalization  
from scipy.spatial import distance_matrix  
  
from os import listdir # for retrieving files from directory  
from os.path import isfile, join # for retrieving files from directory  
from sklearn.manifold import MDS # for MDS  
import pandas as pd # for loading in colors csv  
  
#-----  
# The ECT packages we'll use  
#-----  
from ect import ECT, EmbeddedGraph # for calculating ECTs
```

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Contributing
Modules and Documentation
Tutorial Notebooks
1. Basic Tutorial: Reeb graph class
2. Basic Tutorial: MergeTree class
3. Basic Tutorial: Embedded graph class
4. Basic tutorial: MapperGraph class
5. Example graphs
6. Basic tutorial: InterLeaving class
6.1. Matrices
6.2. Initializing a random interleaving
6.3. Checking all the diagrams
6.4. Testing setting the induced maps in the interleaving from the first maps

6. Basic tutorial: InterLeaving class

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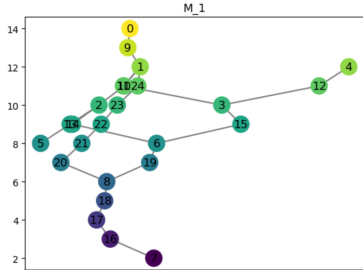
This class encodes all the information needed to figureout an interleaving between two `MapperGraph` instances.

```
[1]: from cereberus import ReebGraph, MapperGraph, InterLeave  
import cereberus.data.ex_mappergraphs as ex_mg  
  
import matplotlib.pyplot as plt
```

We have two example Mapper graphs to work with.

```
[2]: M1 = ex_mg.interleave_example_A()  
M1.draw()  
plt.title('M_1')
```

```
[2]: Text(0.5, 1.0, 'M_1')
```



```
[3]: M2 = ex_mg.interleave_example_B()  
M2.draw()
```

teaspoon

Teaspoon

Topological Signal Processing in Python

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6. Citing

teaspoon: Topological Signal Processing in Python

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teaspoon: Topological Signal Processing in Python

The emerging field of topological signal processing brings methods from Topological Data Analysis (TDA) to create new tools for signal processing by incorporating aspects of shape. This python package, teaspoon for tsp or topological signal processing, brings together available software for computing persistent homology, the main workhorse of TDA, with modules that expand the functionality of teaspoon as a state-of-the-art topological signal processing tool. These modules include methods for incorporating tools from machine learning, complex networks, information, and parameter selection along with a dynamical systems library to streamline the creation and benchmarking of new methods. All code is open source with up to date documentation, making the code easy to use, in particular for signal processing experts with limited experience in topological methods.

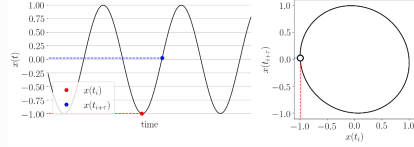


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 - 1.2. Optional Dependencies
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 - 2.1. Make Data (MakeData) Module
 - 2.2. Parameter Selection Module

pip install ect
munchlab.github.io/ect

pip install cereberus
munchlab.github.io/cereberus

pip install teaspoon
teaspoontda.github.io/teaspoon

Thanks!

To my collaborators

Erin Chambers

Ishika Ghosh

Sarah Percival

Bei Wang

- EC, EM, SP, BW. **Bounding the interleaving distance for mapper graphs with a loss function.** JACT, 2025. [arXiv:2307.15130](#)
- EC, IG, EM, SP, BW. **Towards an Optimal Bound for the Interleaving Distance on Mapper Graphs.** [arXiv:2504.03865](#)



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To my funding sources



To my research group



MunchLab 2024-2025

