Pseudotopological Foundations of Topological Data Analysis

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The Geometric Realization of AATRN
IMSI
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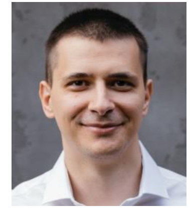

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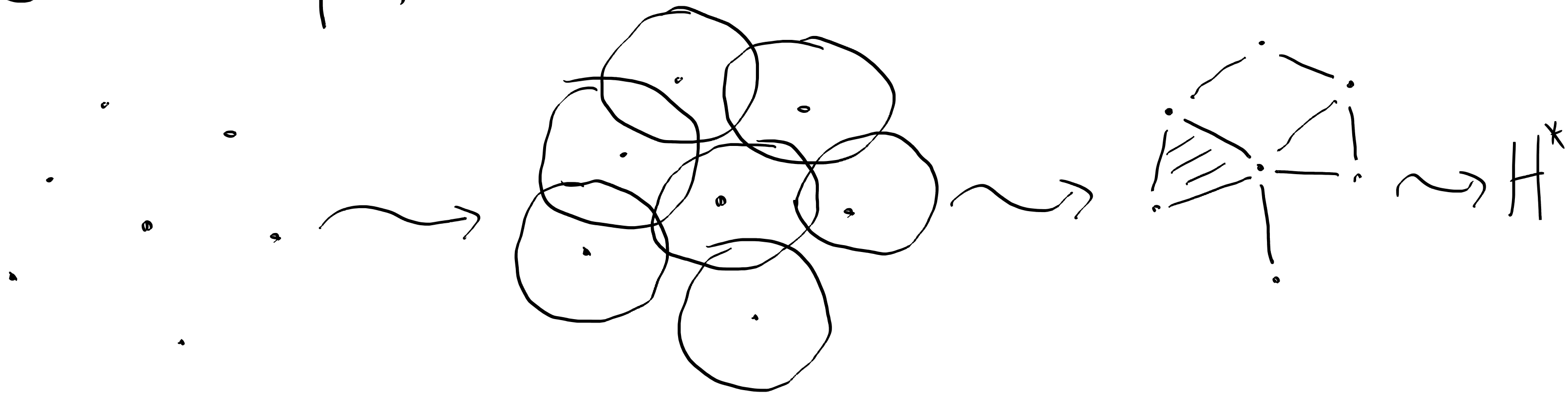
Basic Constructions in TDA

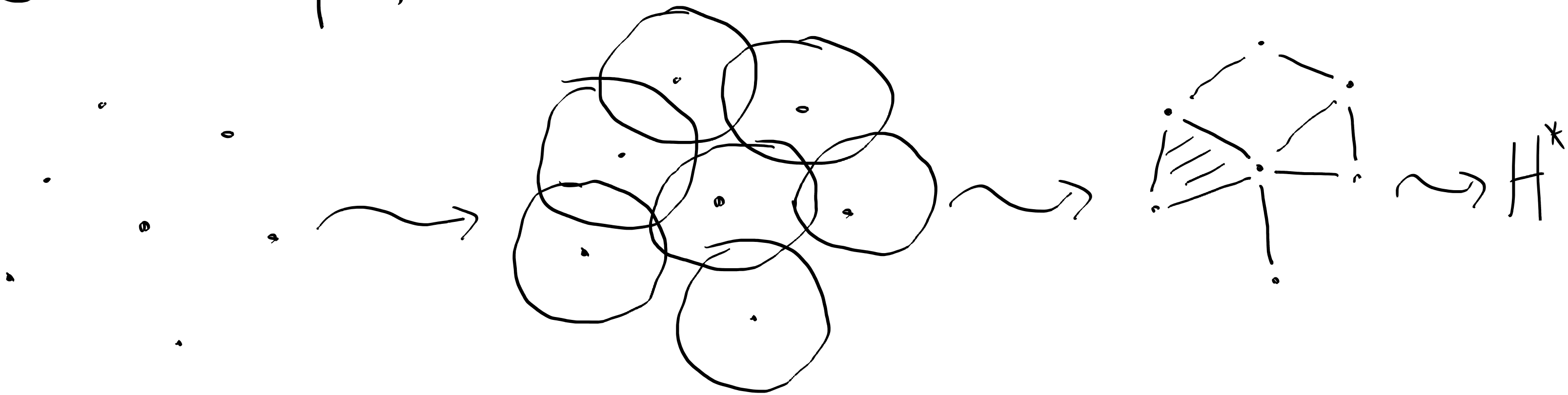
① Čech Complex, $r > 0$:



What is surprising here?

Basic Constructions in TDA

① Čech Complex, $r>0$:

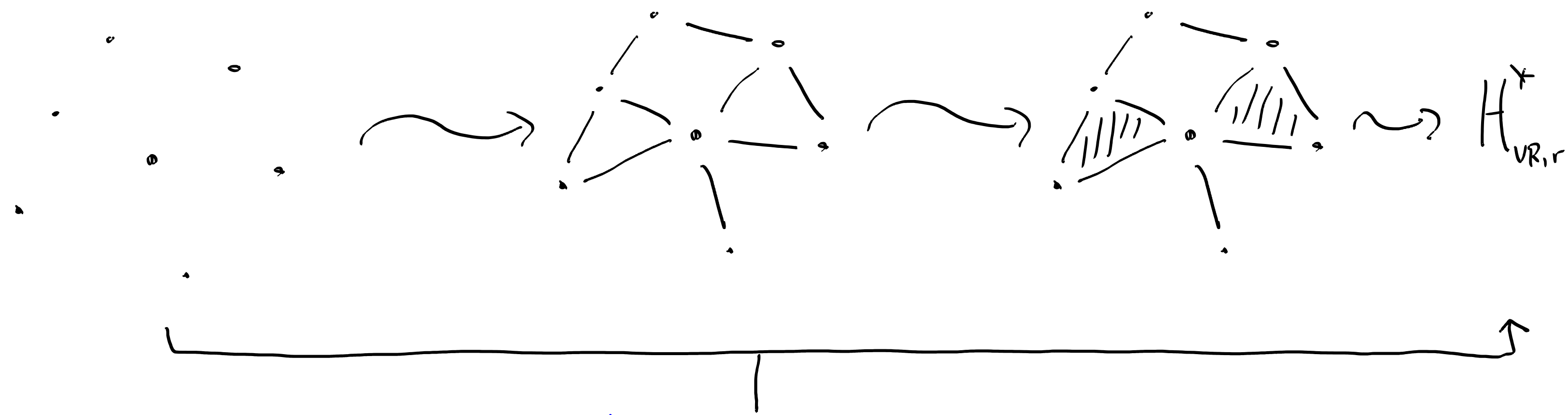


What is surprising here?

¡¡¡NOT FUNCTORIAL!!!

Basic Constructions in TDA

② Vietoris-Rips Complex, $r > 0$:



Functorial in Met (metric spaces and 1-Lipschitz maps)

This is better, but...

Met Has Problems

Functoriality of $F : \mathcal{C} \to \mathcal{D}$ tells us that for

$$f : X \to Y, \quad g : Y \to Z,$$

1) $F(g \circ f) = F(g) \circ F(f)$, and
2) $F(1_X) = 1_{F(X)}$

This is most useful when combined into:

$$g \circ f = 1_X \implies F(g) \circ F(f) = 1_{F(X)}$$

What happens with functors from Met?

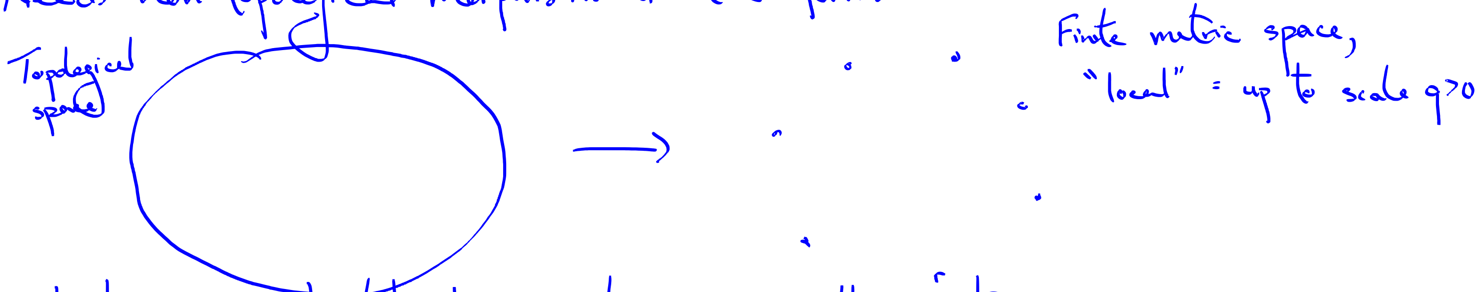
Met Has Problems

- $g \circ f = 1_X \Rightarrow F(g) \circ F(f) = 1_{F(X)}$ is still true, but it's vacuous for any interesting function $f: X \to Y$
- Suppose $f: X \to Y \in$ Met but f is not an isometry. Then $g: Y \to X$ s.t. $g \circ f = 1$ is NOT in Met.
- This makes it difficult to use functoriality in Met to prove theorems.

Towards Better Categories for TDA

What characteristics should a "TDA-category" have?

1. It should contain Top as a subcategory
2. It should contain (reflexive) graphs as a subcategory
3. Morphisms should respect "local" structure
4. Objects should include data which defines what "local" structure means
5. Needs non-topological morphisms of the form

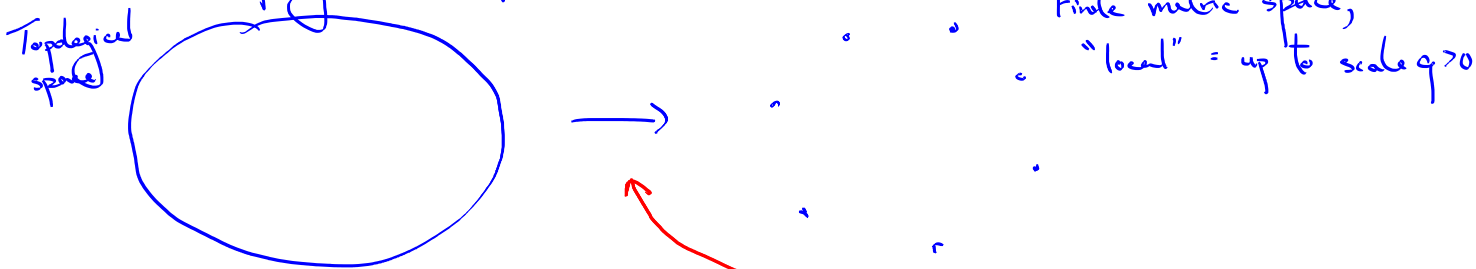


6. Evaluation maps should be morphisms in the category

Towards Better Categories for TDA

Let's examine

5. Needs non-topological morphisms of the form

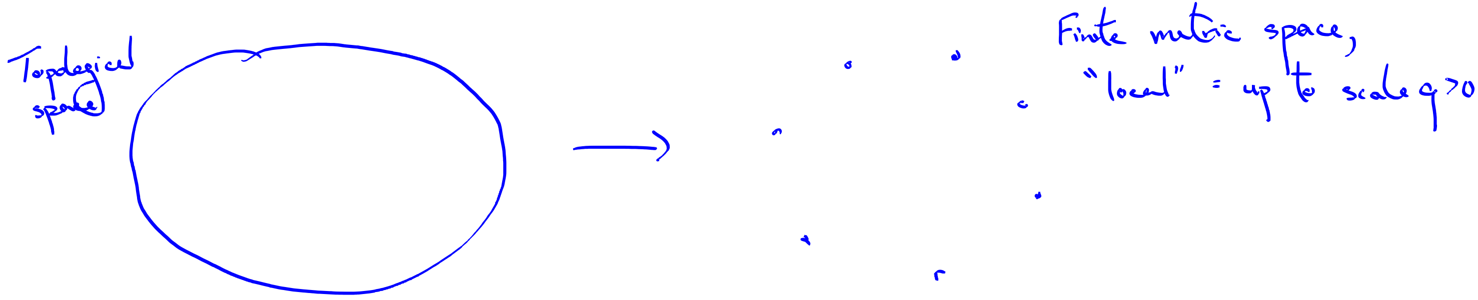


Recall ε-δ definition of continuity:

A function $f:(X,d_X)\to(Y,d_Y)$ is continuous iff $\forall \varepsilon>0, \forall x\in X$
$\exists\, \delta_{\varepsilon,x}>0$ s.t. $d_X(x,x')<\delta_{x,\varepsilon} \Rightarrow d_Y(f(x),f(x'))<\varepsilon$

Can this be adapted to admit a function like ?

Towards Better Categories for TDA



A function $f:(X,d_X)\to(Y,d_Y)$ is (p,q)-continuous iff $\forall \varepsilon>0, \forall x\in X$ $\exists\, \delta_{\varepsilon,x}>0$ s.t. $d_X(x,x') < p + \delta_{x,\varepsilon} \implies d_Y(f(x),f(x')) < q + \varepsilon$

Towards Better Categories for TDA

First candidate - Mesoscopic spaces (Mes)

Ob = Triples (X, d_X, p), i.e. a metric space (X, d_X) a preferred scale $p \geq 0$

$\text{Hom}((X, d_X, p), (Y, d_Y, q)) = (p, q)$ - continuous functions

Satisfies several of our requirements, but evaluation maps are not necessarily continuous and doesn't contain Top or all reflexive graphs

Which categories contain Top?

Towards Better Categories for TDA

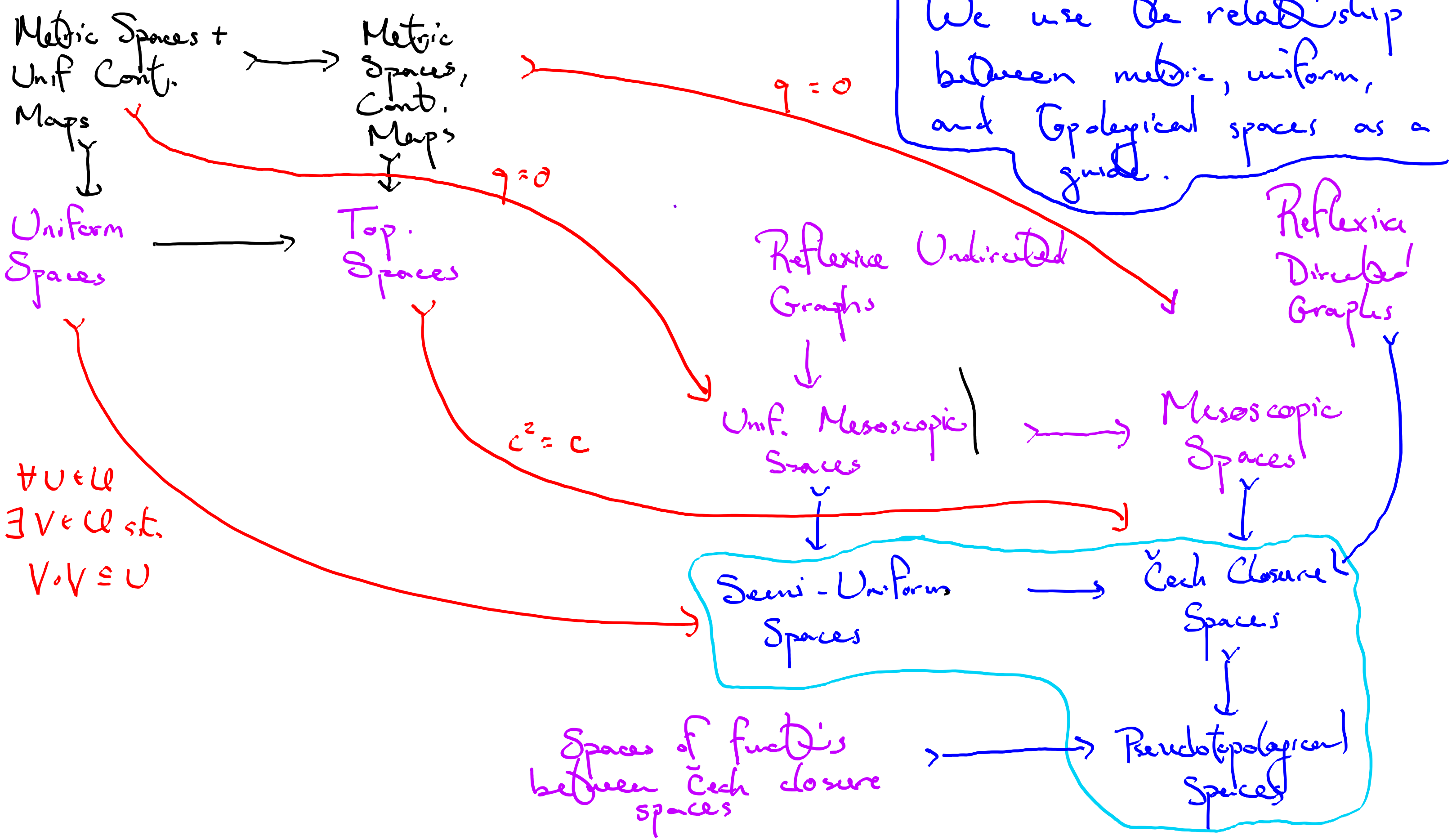
Lots of options

$$
\begin{tikzcd}
\text{Topological spaces} \arrow[r] & \text{Čech closure spaces} \arrow[r] & \text{Pseudotopological spaces} \arrow[r] & \text{Limit spaces} \arrow[r] & \text{Gen. convergence spaces} \\
\text{Uniform spaces} \arrow[r] \arrow[dr] & \text{Semi-uniform spaces} \arrow[r] & \text{Semi-uniform limit spaces} \arrow[r] & \text{Semi-uniform convergence spaces} & \\
\text{Proximity spaces} \arrow[u] & \text{Nearness spaces} & & &
\end{tikzcd}
$$

Others...!

(Adapted from Preuss '02)

We use the relationship between metric, uniform, and topological spaces as a guide.



Our Choices

Semi-Uniform Spaces - Well-adapted for VR-cohomology (R, 2020), also implicit in earlier work by Plaut et al. (2007-2016)

Čech Closure Spaces - Well-adapted for Čech cohomology (Palacios 2019) and sheaf theory (R, 2025)

Pseudotopological Spaces - Well-adapted for homotopy theory (R, 2023, Treviño-Marroquín, 2025, Ebel + Kapulkin 2025)

Related Work

Hausmann (1994), Metric cohomology (VR homology theory for $r=0$)

Adams and collaborators (2013 - present) computations of VR and Čech homology at given scales for circles, ellipses, spheres, etc.

Bartholdi, Schick, Smale, and Smale (2013)
Smale and Smale (2015)

$\Big\}$ Hodge theory at a scale

Related Work

<u>Homotopy Theories for graphs (Discrete Homotopy):</u>

Plaut et al (2007 - ...) Metric spaces at scale, discrete source space

Barcelo et al. (2006- ...), Kapulkin and Carranza (20),
"cubical" homotopy for graphs

Digital homotopy (1980s- ...), Lupton, Scoville, Oprea, Staecker (2019-...)

Dochtermann (2009), Chih and Scull (2022) - X-homotopy

Grigor'yan et al (incl. Rolando Jimenez) (2013 - ...) - Path homology

Čech Closure Spaces

Closure spaces are determined by the "neighborhood filters" of every point.

Formally, a closure space is a pair (X, c) where X is a set and $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is a function such that

(1) $c(\emptyset) = \emptyset$

(2) $A \subset c(A)$

(3) $c(A \cup B) = c(A) \cup c(B)$

If (4) $c^2 = c$ then c induces a topology $\to \{ A \mid \exists B \subset X$ with $A = X \setminus c(B) \}$

Continuous Functions on Closure Spaces

A function $f : (X, c_X) \to (Y, c_Y)$ is continuous iff
$\forall A \subset X$, $f(c_X(A)) \subset c_Y(f(A))$

Thm (Rieser '21) Let (X, d_X) and (Y, d_Y) be metric spaces. Then for any $p, q \geq 0$, a function $f : X \to Y$ is (p,q)-continuous iff it is continuous as a map of closure spaces $f : (X, c_p) \to (Y, c_q)$.

Examples: Graphs and Digraphs

Let $G = (V, E)$ be a (possibly directed) graph

For a vertex $v \in V$, define

$$c(v) = \{ v' \mid v = v' \text{ or } (v, v') \in E \}$$

For $A \subset V$, define $c(A) = \bigcup_{v \in A} c(v)$.

Then (V, c) is a closure space.

Examples: Mesoscopic Spaces

(Rieser '21)

Let (X,d) be a metric space, and let $\rho \geq 0$.
For $A \subset X$, define

$$c_\rho(A) = \{ x \in X \mid d(x,A) \leq \rho \}$$

Then (X, c_ρ) is a closure space, and (X, c_0) is a topological space with the usual metric topology.

Interiors and Interior Covers

Given a subset $A \subset (X, c)$, we define the interior of A by

$$i(A) = X - c(X - A)$$

U is a neighborhood of A iff $A \subset i(U)$

A cover $\mathcal{U}$ is an interior cover iff $X = \bigcup_{U \in \mathcal{U}} i(U)$

Neighborhoods and Closure Structures

Thm (Cech '66) For every $x \in X$, suppose that $\mathcal{N}(x)$ is a collection of sets such that

(1) $\mathcal{N}(x) \neq \emptyset$

(2) $\forall U \in \mathcal{N}(x)$, $x \in U$

(3) $\forall U_1, U_2 \in \mathcal{N}(x)$, $\exists U \in \mathcal{N}(x)$ s.t. $U \subset U_1 \cap U_2$

Then $\exists!$ closure structure c on X s.t. $\forall x \in X$, $\mathcal{N}(x)$ generates the neighborhood filter of x. Furthermore,

$$c(A) = \{ x \in X \mid U \in \mathcal{N}(x) \Rightarrow U \cap A \neq \emptyset \}$$

What Can We Do?

1) Homotopy Theory (really in Pseudotopological Spaces)
2) Čech cohomology and sheaf theory
3) Singular and VR-homology (VR-cohomology optimally in semi-uniform spaces)
4) Simplify Known Results
5) Persistence and Stability

Pseudotopological Spaces

<u>Def</u> A pseudotopological space is a set X together with a relation $\lambda \subset X \times \mathcal{F}X$, where $\mathcal{F}S$ is the set of filters on S. If $(x, F) \in \lambda$, we say that F converges to x and write $F \to x$. The relation λ must satisfy

1) $[x] \to x \quad \forall x \in X$
2) $F \subseteq G$ and $F \to x$ then $G \to x$
3) If F is a filter s.t. every ultrafilter $G \supseteq F$ converges to x, then $F \to x$

Pseudotopological Spaces

Def A filter F on a set X is a collection of subsets s.t.

1) If $A \subset B$ and $A \in F \to B \in F$
2) $\exists A \subset X$ s.t. $A \in F$
3) If $A, B \in F$, then $\exists C \subset A \cap B$ s.t. $C \in F$

An ultrafilter is a maximal proper filter

Homotopy Theory

<u>Basic</u> <u>theory</u> (R '21, R '23): Since $I = [0,1]$ and the spheres S^n are closure (and pseudotopological) spaces, we may define homotopy groups as usual

$$\pi_1(S^1, c_r) = \begin{cases} \mathbb{Z} & 0 \le r < \frac{1}{3} \\ 1 & r \ge \frac{1}{2} \end{cases}$$

(Note similarity with Adamaszek and Adams '15)

Homotopy Theory

Model categories (R'23, Ebel-Kapulkin'25) A Quillen-type model category exists on pseudotopological spaces (R'23, Ebel-Kapulkin'25) and it is Quillen-equivalent to the Quillen model category on Top.

Homotopy Theory

Relationship to homotopy of Vietoris-Rips complexes:

If (X, c_G) is induced by an undirected graph $G = (X, E)$ (Treviño-Marroquín '24) or a finite directed graph (Milicevic + Scoville '25) then

$$\pi_*(X, c) \cong \pi_*(|VR(G)|)$$

pseudotopological homotopy groups

classical homotopy groups of the geometric realization of the clique complex of G.

Homotopy Theory

Discrete and exotic homotopy theories (Bubenik + Milicevic '24):
By using different intervals ($\bullet \to \bullet$, $\bullet - \bullet$, $[\!-\!-\!]$) and products to define homotopy, one recovers discrete homotopy theories in the literature (A-theory (Barcelo et al.)) as well as new ones.

Open problem: When are they isomorphic?

Čech Cohomology and Sheaf Theory

Čech cohomology (Palacios '19): Built using nerves of interior covers of a closure space

Sheaves (R '25): Grothendieck topology constructed using interior covers. Gives sheaf cohomology on possibly directed graphs. Graphs may have higher-dimensional sheaf cohomology.

Singular Homology

<u>Singular homology</u> (Bubenik and Milicevic '24): Constructed either in the standard way - using maps from $|\Delta^n|$ - or using maps from the complete graphs on $(n+1)$ vertices or the complete directed graphs on $(n+1)$ vertices.

Vietoris-Rips (Co)Homology

Vietoris-Rips Homology

1) (Bubenik + Milicevic '24) $\sigma \subset (X, c)$ is a simplex iff $\sigma \subset c(x)$ $\forall x \in \sigma$.
2) (Rieser '20) Defined through semi-uniform spaces. Also applies to topological spaces and directly generalizes Hausmann's metric cohomology

1) + 2) are isomorphic on unordered graphs

Simplifying Known Results

<u>Thm</u> (Virk '22) Let X be a compact metric space, $\mathbb{F}$ a field, $n \in \{0,1,2,\dots\}$. Suppose $f: X \to A$, $A \subset X$, is a 1-Lipschitz retract. Then

$$PD\left(H_*(VR(A,r))\right)_{r \geq 0} \subset PD\left(H_*(VR(X,r))\right)_{r \geq 0}$$

<u>Proof</u> (R '24): Consider $A \overset{i}{\hookrightarrow} X \xrightarrow{f} A$

i and f are continuous as functions $i: (A, c_r) \to (X, c_r)$

$f: (A, c_r) \to (X, c_r)$ $\forall\, r \geq 0$, since i is a topological inclusion and f is a 1-Lipschitz retraction.

Then

$$\mathrm{Id}\left\{\begin{tikzcd}
(A, c_0) \arrow[r, hook] \arrow[d, "i"'] & (A, c_1) \arrow[r, hook] \arrow[d, "i"'] & \cdots \arrow[r, hook] & (A, c_k) \arrow[r, hook] \arrow[d, "i"'] & \cdots \\
(X, c_0) \arrow[r, hook] \arrow[d, "f"'] & (X, c_1) \arrow[r, hook] \arrow[d, "f"'] & \cdots \arrow[r, hook] & (X, c_k) \arrow[r, hook] \arrow[d, "f"'] & \cdots \\
(A, c_0) \arrow[r, hook] & (A, c_1) \arrow[r, hook] & \cdots \arrow[r, hook] & (A, c_k) \arrow[r, hook] & \cdots
\end{tikzcd}\right.$$

commutes and

$$\mathrm{Id}\left\{\begin{tikzcd}
H_*^{VR}(A, c_0) \arrow[r, hook] \arrow[d, "H_*^{VR} i"'] & H_*^{VR}(A, c_1) \arrow[r, hook] \arrow[d, "H_*^{VR} i"'] & \cdots \arrow[r, hook] & H_*^{VR}(A, c_k) \arrow[r, hook] \arrow[d, "H_*^{VR} i"'] & \cdots \\
H_*^{VR}(X, c_0) \arrow[r, hook] \arrow[d, "H_*^{VR} f"'] & H_*^{VR}(X, c_1) \arrow[r, hook] \arrow[d, "H_*^{VR} f"'] & \cdots \arrow[r, hook] & H_*^{VR}(X, c_k) \arrow[r, hook] \arrow[d, "H_*^{VR} f"'] & \cdots \\
H_*^{VR}(A, c_0) \arrow[r, hook] & H_*^{VR}(A, c_1) \arrow[r, hook] & \cdots \arrow[r, hook] & H_*^{VR}(A, c_k) \arrow[r, hook] & \cdots
\end{tikzcd}\right.$$

Since $f \circ i = 1_A$, $H_*^{VR}(f) \circ H_*^{VR}(i) = 1_{Ab}$, and the result follows. $\square$

Persistence and Stability

<u>Thm</u> (Bubenik + Milićević '24): Suppose $F: \mathrm{Cl} \to \mathcal{C}$ a homotopy invariant functor, where $\mathcal{C}$ is a category. Let (X, d_X), (Y, d_Y) be metric spaces. Then the persistent version of F exists and

$$d_I(PF(X), PF(Y)) \leq 2 d_{GH}(X, Y)$$

Gives a new strategy for proving stability by proving homotopy invariance in Cl! Proof makes heavy use of functoriality in Cl.

¡¡¡Thank you!!!