Morse Theory in Random Topology

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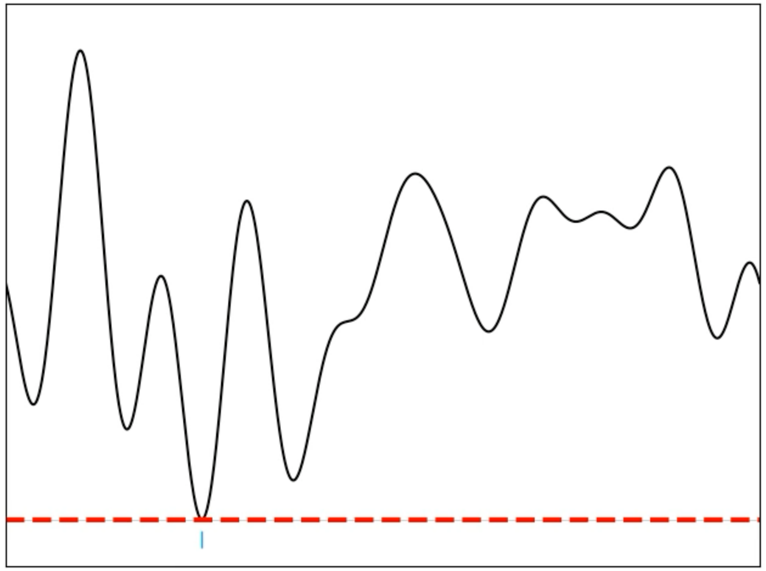
The Geometric Realization of AATRN

August 21, 2025

Morse Theory

Morse Theory

- $\mathcal{M}$ – compact smooth manifold
- $f : \mathcal{M} \to \mathbb{R}$ – a **Morse** function:
  - Smooth
  - At most one critical point at each level
  - Non-degenerate



- Sublevel sets:

$$\mathcal{M}_\alpha = f^{-1}((-\infty, \alpha])$$

- Main question:

What happens to the topology of $\mathcal{M}_\alpha$ as we increase α?

Morse Theory

$$\mathcal{M}_\alpha = f^{-1}((-\infty, \alpha])$$

1. If no critical points in the range $[\alpha - \varepsilon, \alpha + \varepsilon]$:

$$H_k(\mathcal{M}_{\alpha+\varepsilon}) \cong H_k(\mathcal{M}_{\alpha-\varepsilon})$$

2. If c critical point, $f(c) = \alpha$, index= k, only one of two options:

$$\beta_k(\mathcal{M}_{\alpha+\varepsilon}) = \beta_k(\mathcal{M}_{\alpha-\varepsilon}) + 1$$ k-birth

$$\beta_{k-1}(\mathcal{M}_{\alpha+\varepsilon}) = \beta_{k-1}(\mathcal{M}_{\alpha-\varepsilon}) - 1$$ (k-1)-death

Useful Conclusions

- $N_k(\alpha)$ = #critical points of index k in $\mathcal{M}_\alpha$

- Morse inequalities:

$$(1)\ \beta_m(\mathcal{M}_\alpha) \leq N_m(\alpha) \qquad (2)\ \sum_{k=0}^{m}(-1)^{m-k}\beta_k(\mathcal{M}_\alpha) \leq \sum_{k=0}^{m}(-1)^{m-k}N_k(\alpha)$$

- Euler characteteristic:

$$\chi(\mathcal{M}_\alpha) := \sum_{k=0}^{d}(-1)^k \underbrace{\beta_k(\mathcal{M}_\alpha)}_{\text{global}} = \sum_{k=0}^{d}(-1)^k \underbrace{N_k(\alpha)}_{\text{local}}$$

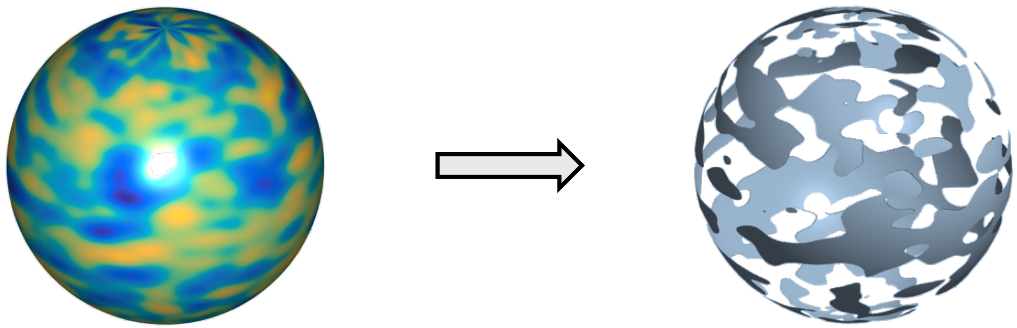
- Extremes:

$$N_k(\alpha) = 0 \quad \Longleftrightarrow \quad H_k(\mathcal{M}_t) \cong 0,\ \forall t \leq \alpha$$

$$\left.\begin{aligned} N_k(\alpha) &= N_k(\infty) \\ N_{k+1}(\alpha) &= N_{k+1}(\infty) \end{aligned}\right\} \quad \Longleftrightarrow \quad H_k(\mathcal{M}_t) \cong H_k(\mathcal{M}),\ \forall t \geq \alpha$$

Gaussian Random Fields

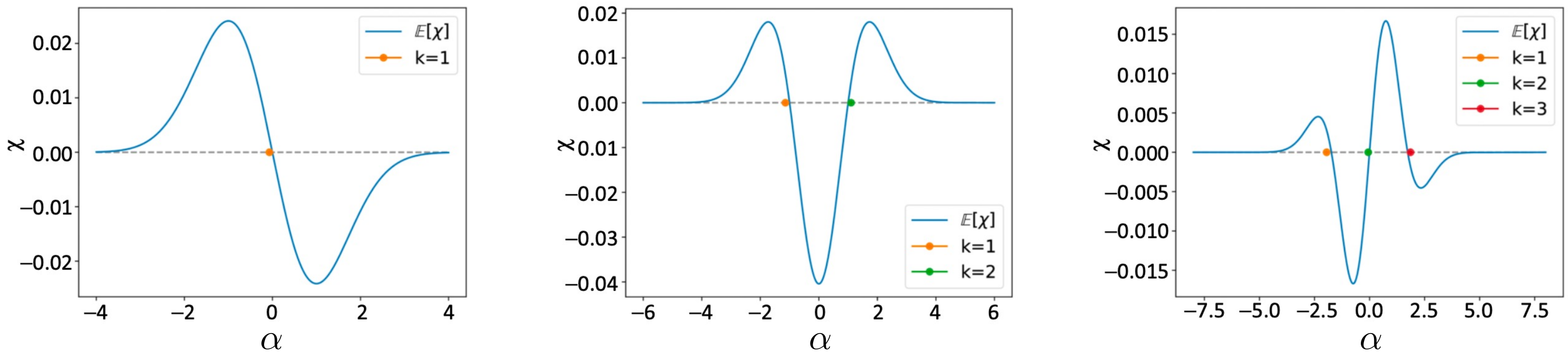
- Smooth random functions $f : \mathcal{M} \to \mathbb{R}$



$f : S^2 \to \mathbb{R}$ $\qquad\qquad$ $\mathcal{M}_\alpha = f^{-1}((-\infty, \alpha])$

- Gaussian Kinematic Formula [Adler & Taylor, 2003]

$$\mathbb{E}\{\chi(\mathcal{M}_\alpha)\} = \sum_{k=0}^{d} (-1)^k \mathbb{E}\{N_k(\alpha)\} = \cdots = \sum_{j=0}^{d} \mathcal{L}_j(\mathcal{M}) \rho_j(\alpha)$$



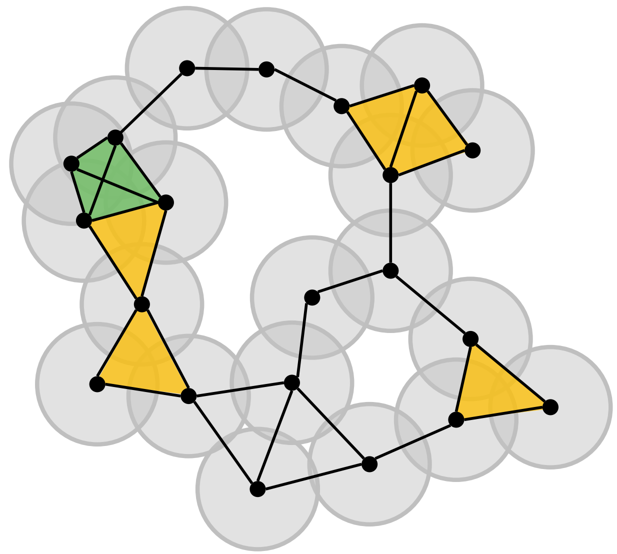
Random Čech Complex

Geometric Complexes

- $\mathcal{X} = \{x_1, \ldots, x_n\}$ = point cloud
- **Čech complex:**

  $S \subset \mathcal{X}$ is a simplex if:

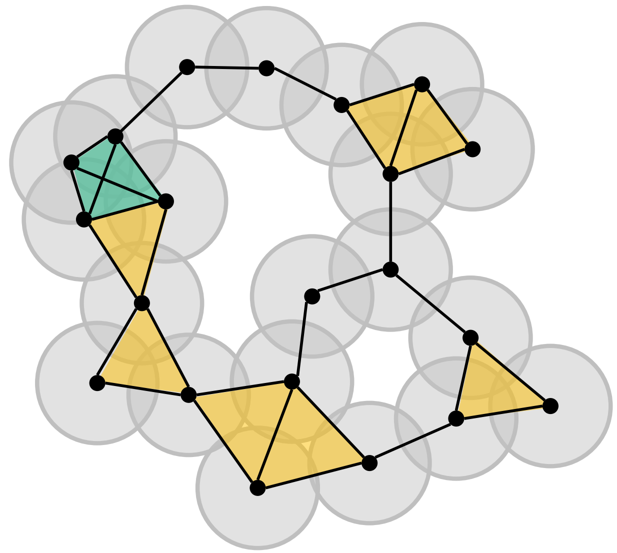
  $$\boxed{\bigcap_{x \in S} B_r(x) \neq \emptyset}$$



- **Vietoris-Rips complex:**

  $S \subset \mathcal{X}$ is a simplex if:

  $$\boxed{B_r(x) \cap B_r(x') \neq \emptyset} \quad \forall x, x' \in S$$



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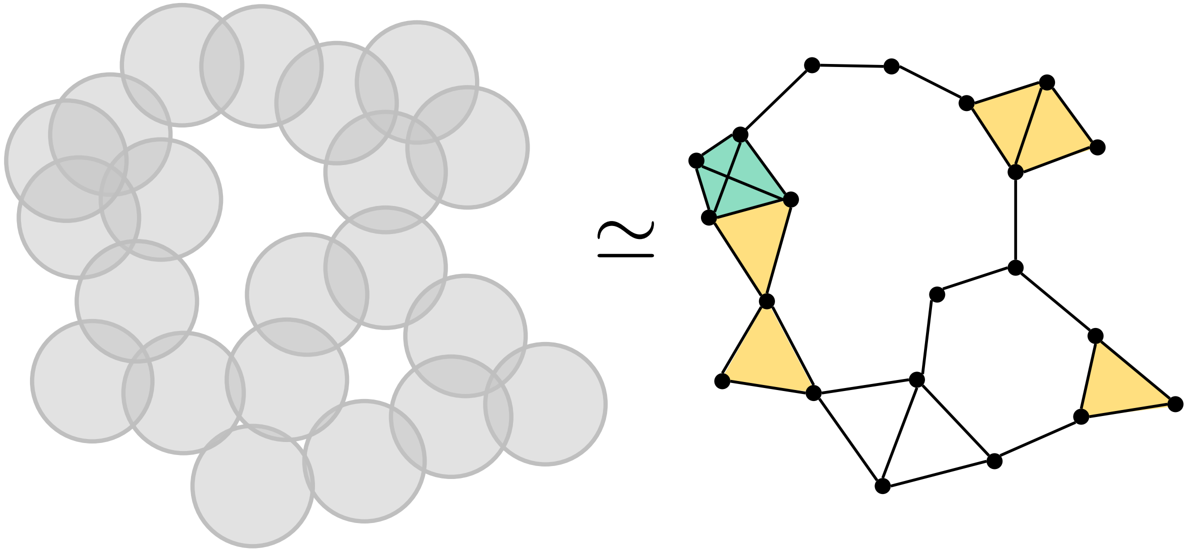
Nerve Lemma



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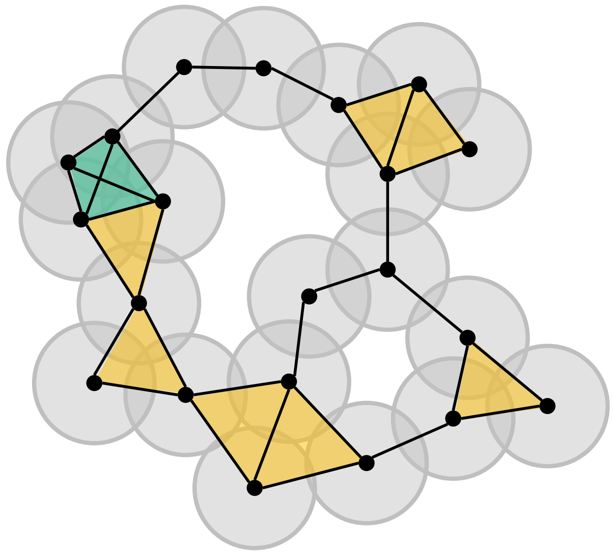


Random Point Clouds

- $f : \mathcal{M} \to \mathbb{R}$ – probability density function (pdf)
- **Binomial process:**

$$\mathcal{X}_n = \{X_1, \ldots, X_n\}$$

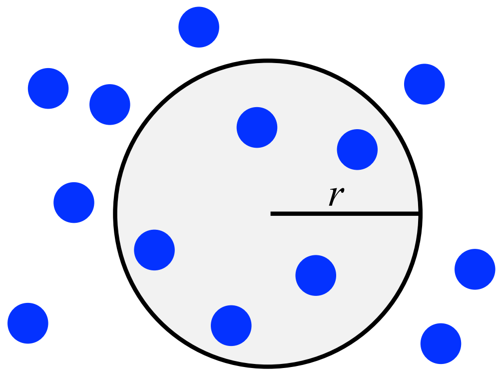
independent points, generated by f

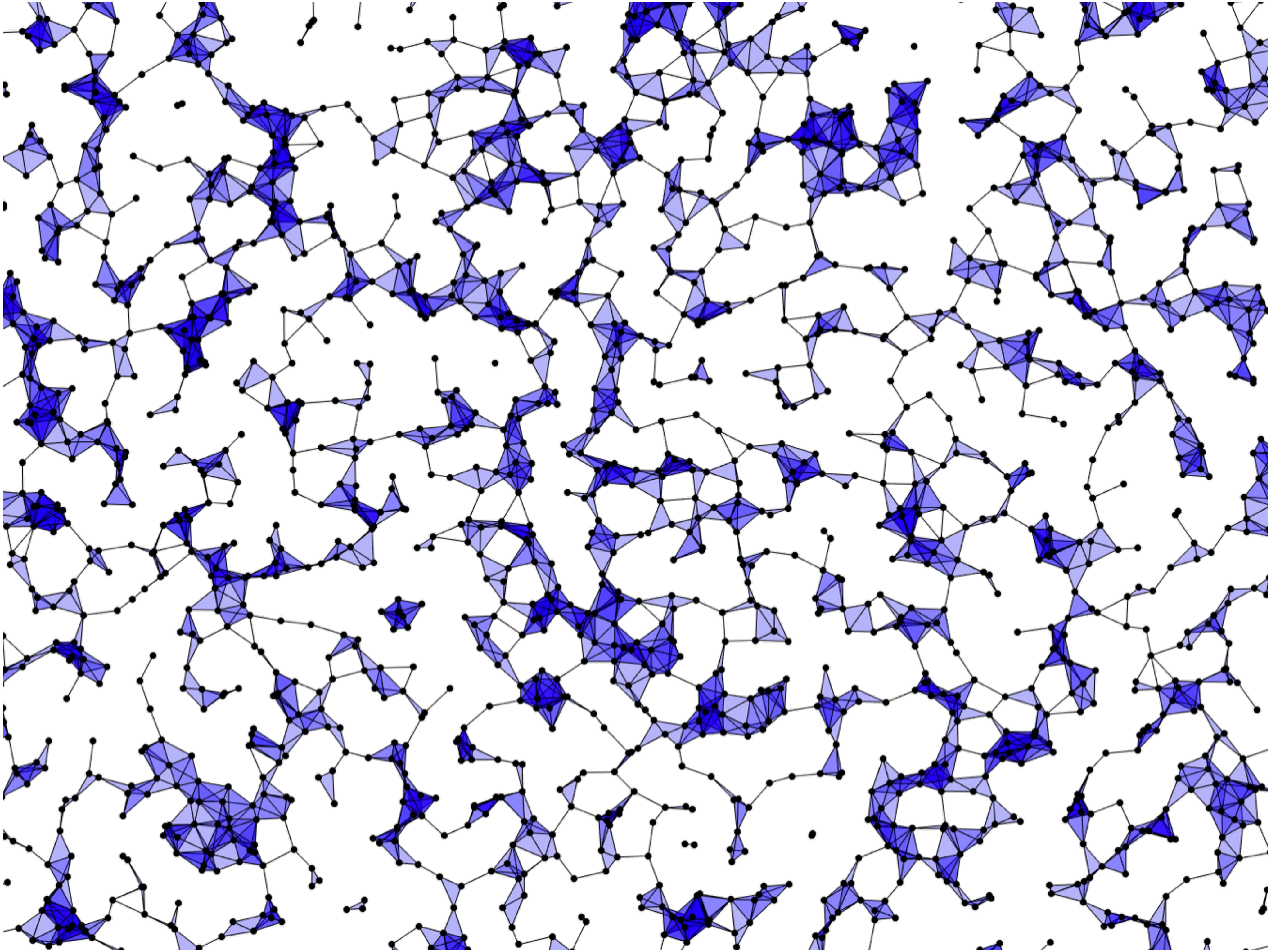
- **Poisson process:**

$$\mathcal{P}_n = \mathcal{X}_N \qquad N \sim \text{Poisson}\,(n)$$

random number of points, n on average

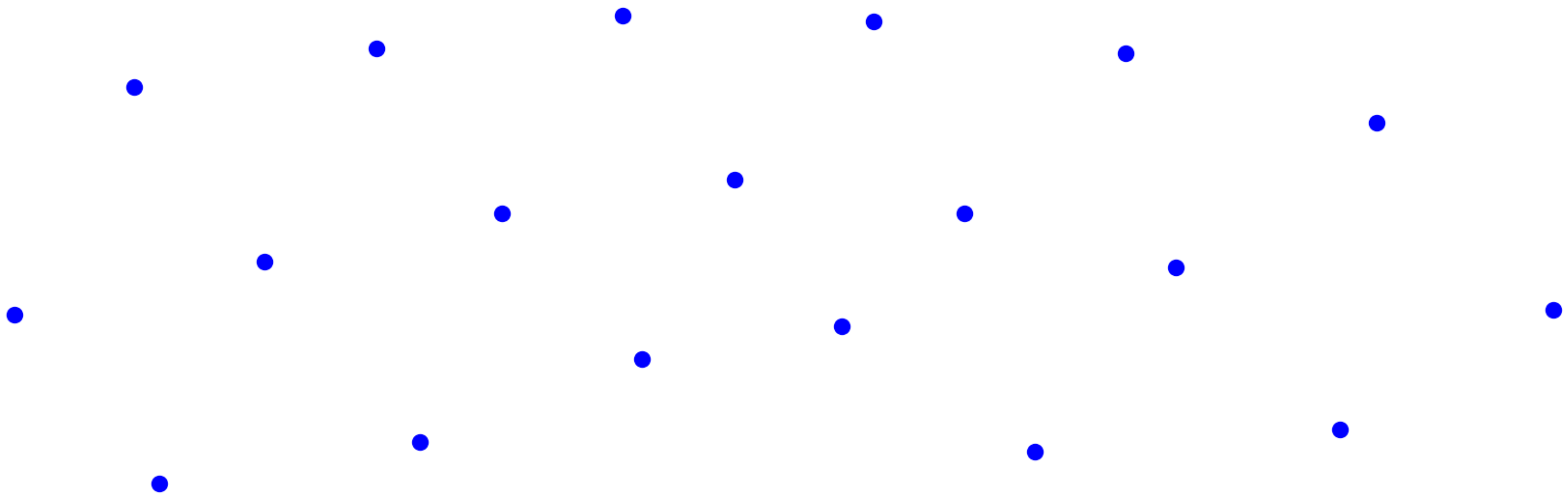
- Random Čech/Rips complexes: $C_r(n), R_r(n)$
- Limits: $n \to \infty,\ r = r(n) \to 0$
- Average points in a ball $\propto n \cdot r^d$





The Distance Function

- $\mathcal{P}$ = point cloud in a manifold $\mathcal{M}$



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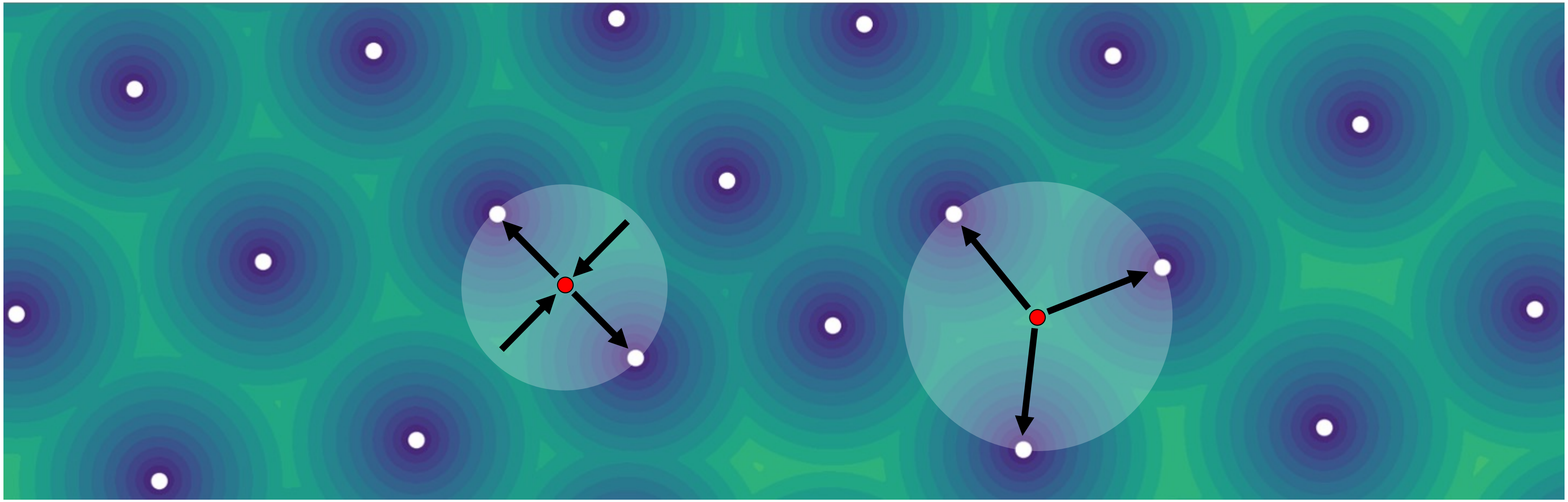

$$d_{\mathcal{P}}(x) := \min_{p \in \mathcal{P}} \operatorname{dist}(x, p)$$

- Sublevel sets: $\mathcal{M}_\alpha = \bigcup_{\mathcal{P}} B_\alpha(p) \simeq C_\alpha(\mathcal{P})$
- Nondifferentiable function
- Has critical points

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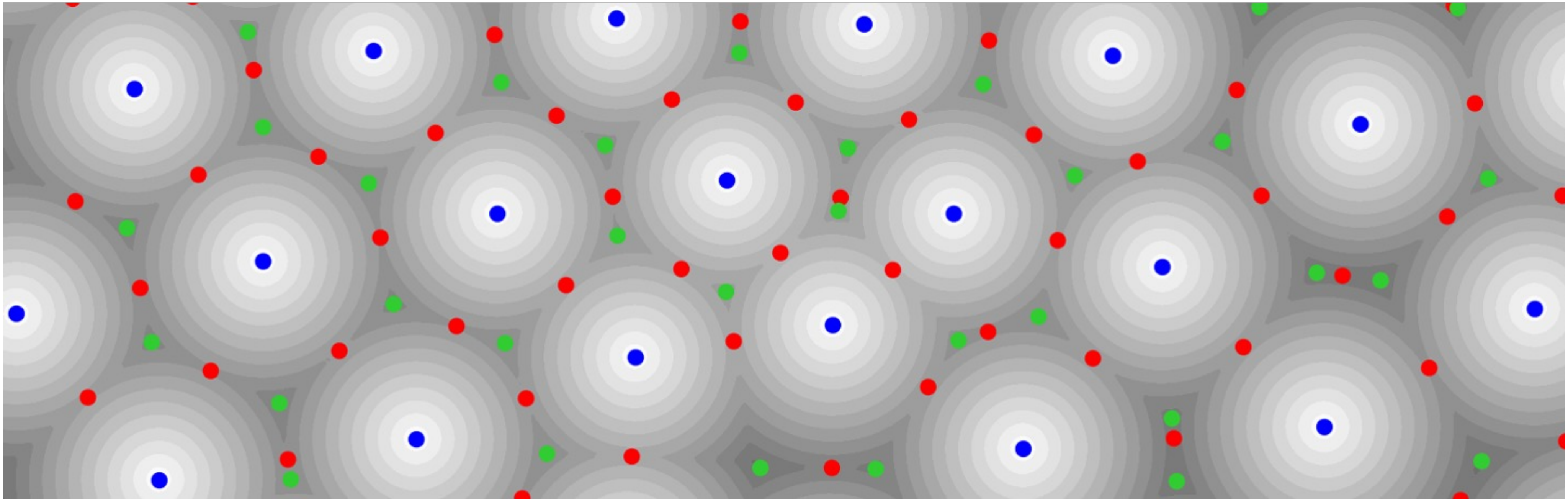


$$d_{\mathcal{P}}(x) := \min_{p \in \mathcal{P}} \mathrm{dist}(x, p)$$

Index 0 (minimum)
Index 1 (saddle)
Index 2(maximum)

- Sublevel sets: $$\mathcal{M}_\alpha = \bigcup_{\mathcal{P}} B_\alpha(p) \simeq C_\alpha(\mathcal{P})$$

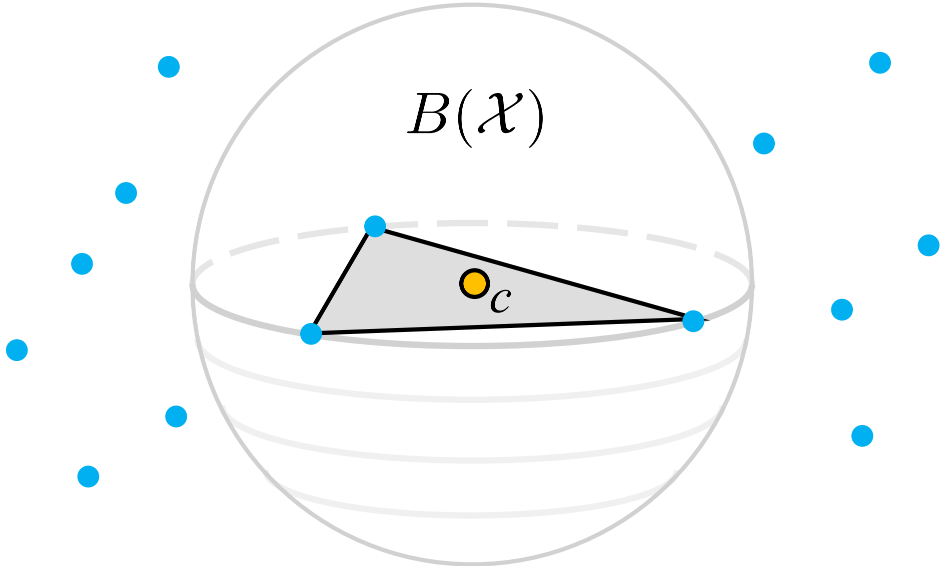
- Nondifferentiable function

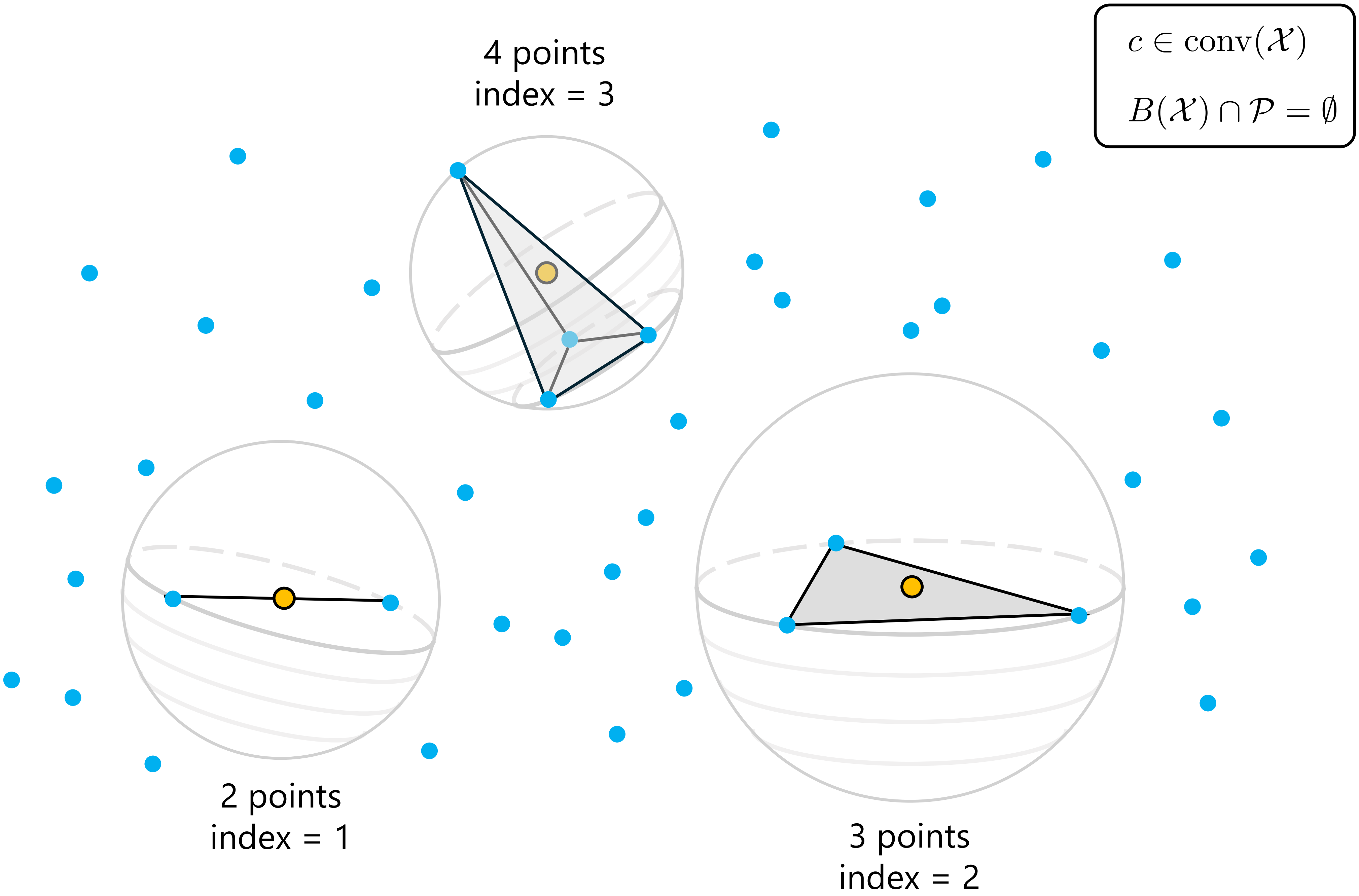
- Has critical points

Critical Points for the Distance Fn.

$$d_{\mathcal{P}}(x) := \min_{p \in \mathcal{P}} \mathrm{dist}(x, p)$$

- Morse Theory for min-type functions [Gershkovich & Rubinstein, 1997]
- Distance function as a special case
- A critical point c is defined via a subset $\mathcal{X} \subset \mathcal{P}$ where:
  - $c \in \mathrm{conv}(\mathcal{X})$
  - $B(\mathcal{X}) \cap \mathcal{P} = \emptyset$ ($B(\mathcal{X})$ = smallest open ball with $\mathcal{X}$ on its boundary)
  - $\mu(c) = |\mathcal{X}| - 1$ (index)




$c \in \mathrm{conv}(\mathcal{X})$
$B(\mathcal{X}) \cap \mathcal{P} = \emptyset$
4 points
index = 3
2 points
index = 1
3 points
index = 2

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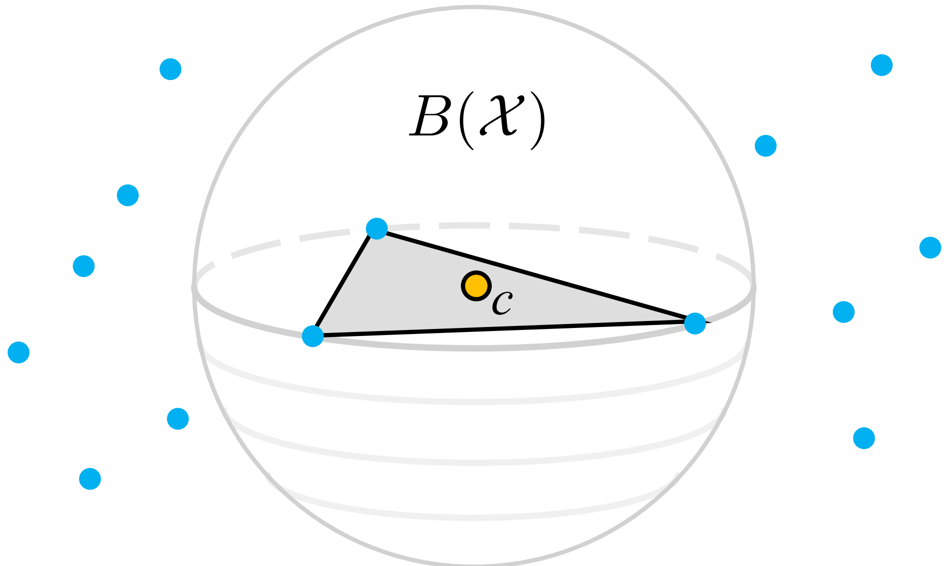
Analysis → Combinatorics

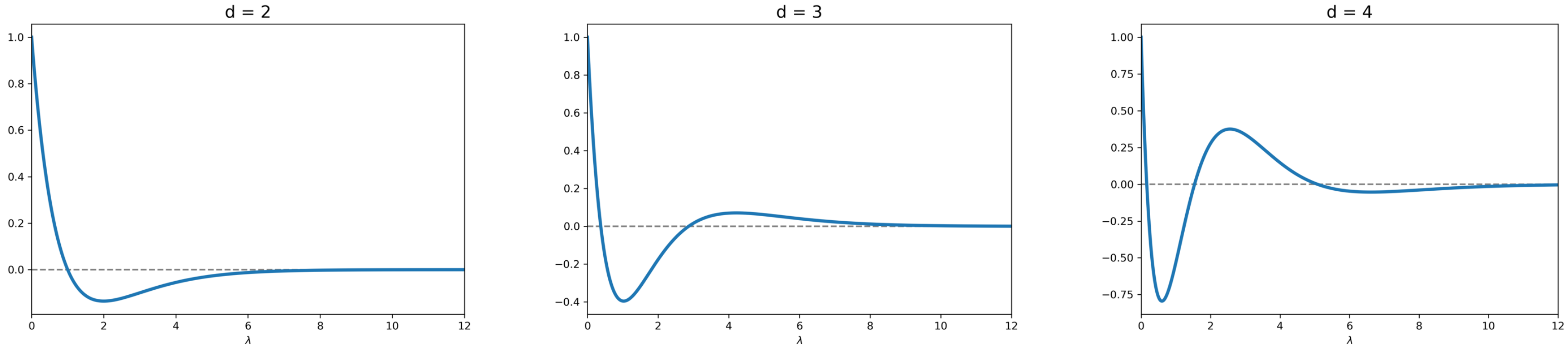


Applications of Morse Theory

1– Expected Euler Characteristic

- For $r = (\lambda/n)^{1/d}$ [B. & Weinberger, 2017]:

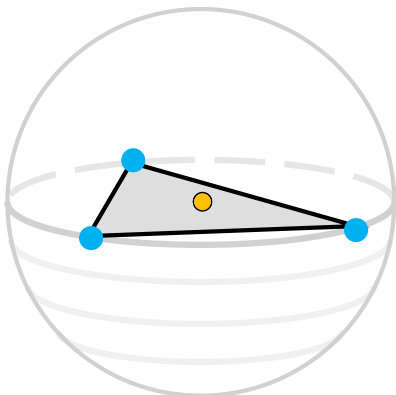
$$\mathbb{E}\left\{\mathcal{X}(C_r)\right\} = \sum_{k=0}^{d}(-1)^k \mathbb{E}\left\{N_k(r)\right\} = \cdots = ne^{-\lambda}\sum_{j=0}^{d-1} A_j \lambda^j$$



- A very useful tool to evaluate $\mathbb{E}\left\{N_k(r)\right\}$:

  Blaschke-Petkantschin formulae [Edelsrbunner, Nikitenko, Reitzner, 2017]

  Spherical coordinates around critical points



$$\mathbb{E}\left\{N_k(r)\right\} = n^{k+1}\int_c\int_\rho\int_\gamma\int_{\boldsymbol{\theta}} \mathbb{1}\left\{c \text{ is a critical point}\right\}$$

1– Expected Euler Characteristic

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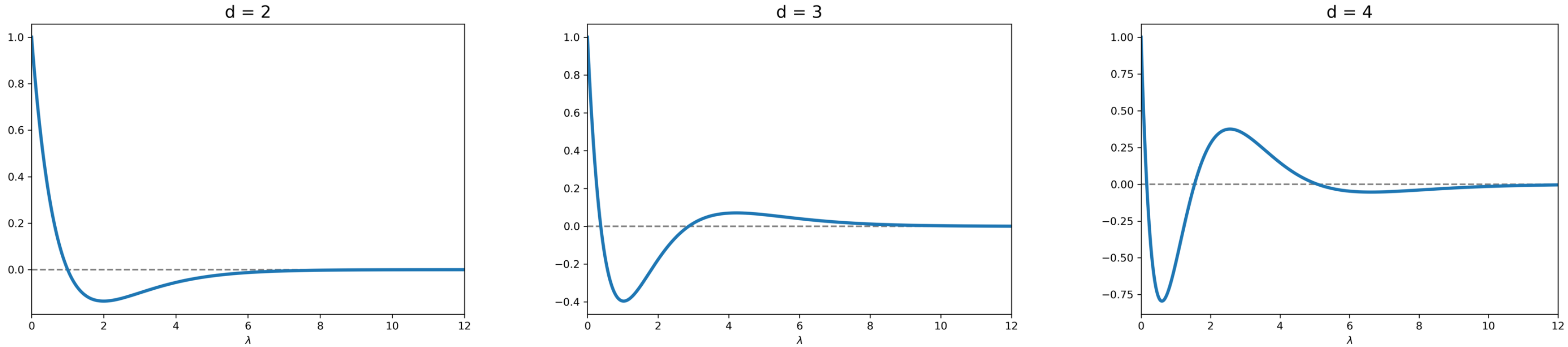
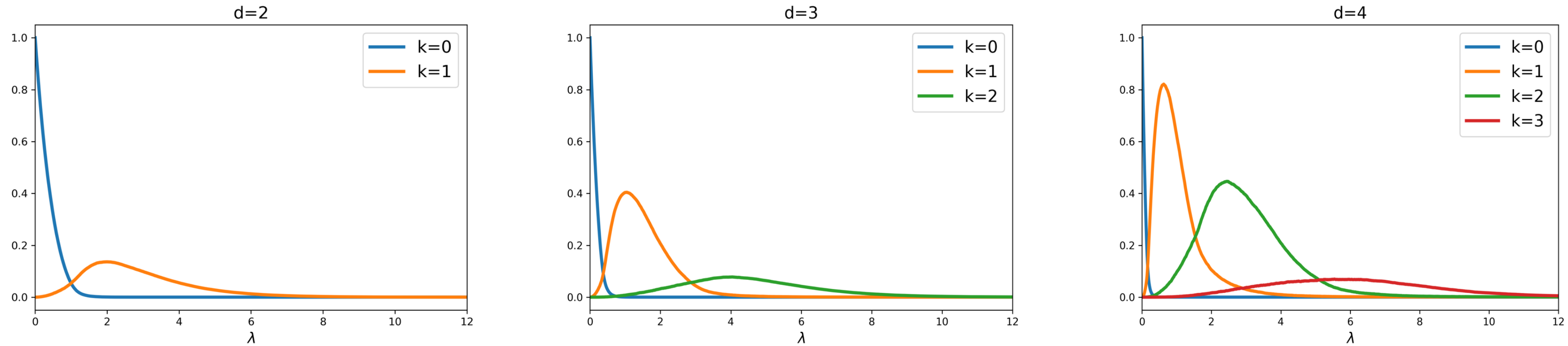
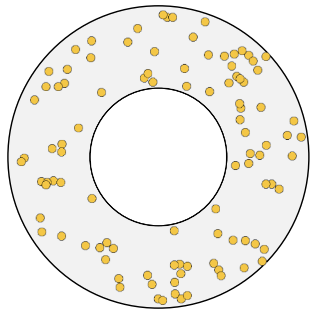


- Betti Numbers (experimental):



2 – Homological Connectivity



- $\mathcal{M}$ = smooth closed manifold
- What's the smallest r needed to recover $H_k(\mathcal{M})$?
- Connectivity [Penrose, 1997]:

$$\lim_{n\to\infty} \mathbb{P}\left(H_0(C_r) \cong H_0(\mathcal{M})\right) = \begin{cases} 1 & nr^d = \frac{1}{c}\log n + w(n), \\ 0 & nr^d = \frac{1}{c}\log n - w(n). \end{cases}$$

$w(n) \to \infty$

- Homological connectivity [B., 2022]:

$$\lim_{n\to\infty} \mathbb{P}\left(H_k(C_r) \cong H_k(\mathcal{M})\right) = \begin{cases} 1 & nr^d = \frac{2}{c}\left(\log n + (k-1)\log\log n\right) + w(n), \\ 0 & nr^d = \frac{2}{c}\left(\log n + (k-1)\log\log n\right) - w(n). \end{cases}$$

- Proof main idea: count "future" critical points ($|N_k(\infty) - N_k(r)| \to 0$)
- Proof main challenge: show separate vanishing for deaths vs. births
- Manifolds with boundary [de Kergorlay, Tillmann, Vipond, 2022]

2 – Homological Connectivity

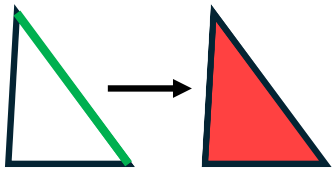
$$\lim_{n\to\infty} \mathbb{P}\left(H_k(\mathcal{C}_r) \cong H_k(\mathcal{M})\right) = \begin{cases} 1 & nr^d = \frac{2}{c}\left(\log n + (k-1)\log\log n\right) + w(n), \\ 0 & nr^d = \frac{2}{c}\left(\log n + (k-1)\log\log n\right) - w(n). \end{cases}$$

- Critical window:

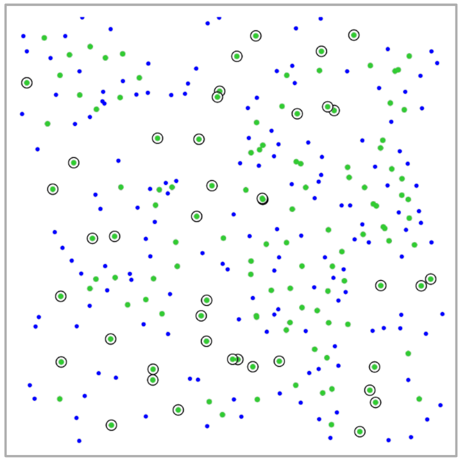
$$\boxed{nr^d = \frac{2}{c}\left(\log n + (k-1)\log\log n\right) + \text{const}}$$

- Theorem:

All cycles born in the critical window die instantly



- Poisson limit [B., Schulte, Yogeshwaran, 2022]:



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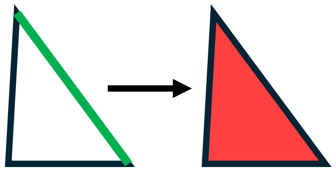

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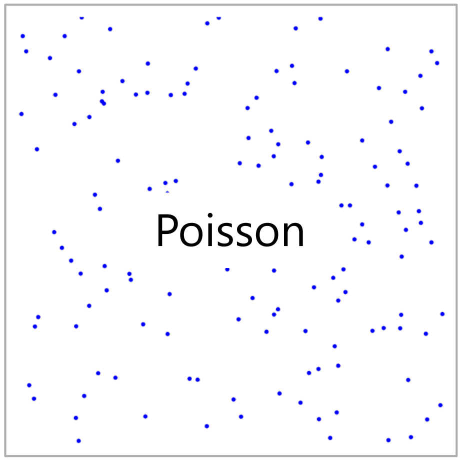
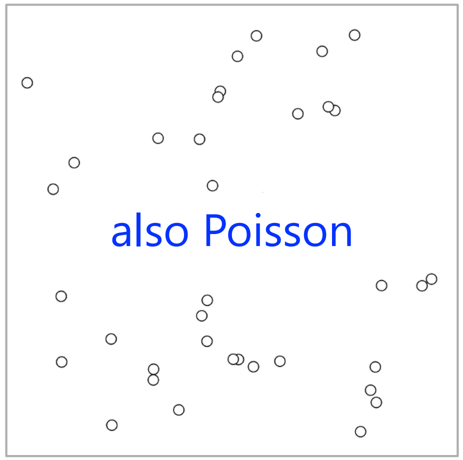
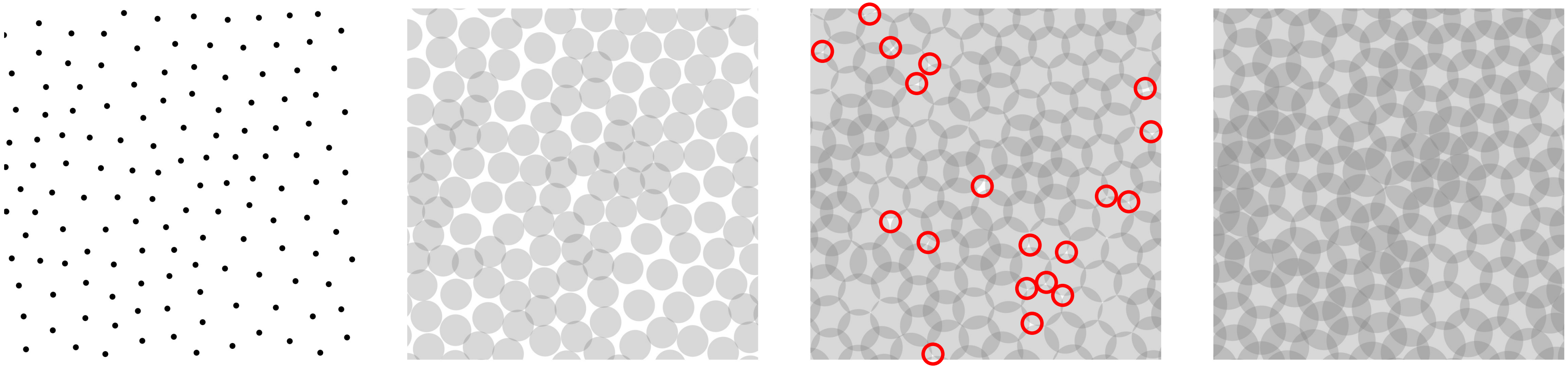


$$\xrightarrow[\text{= last cycles}]{\text{critical points}}$$



3 – Coverage

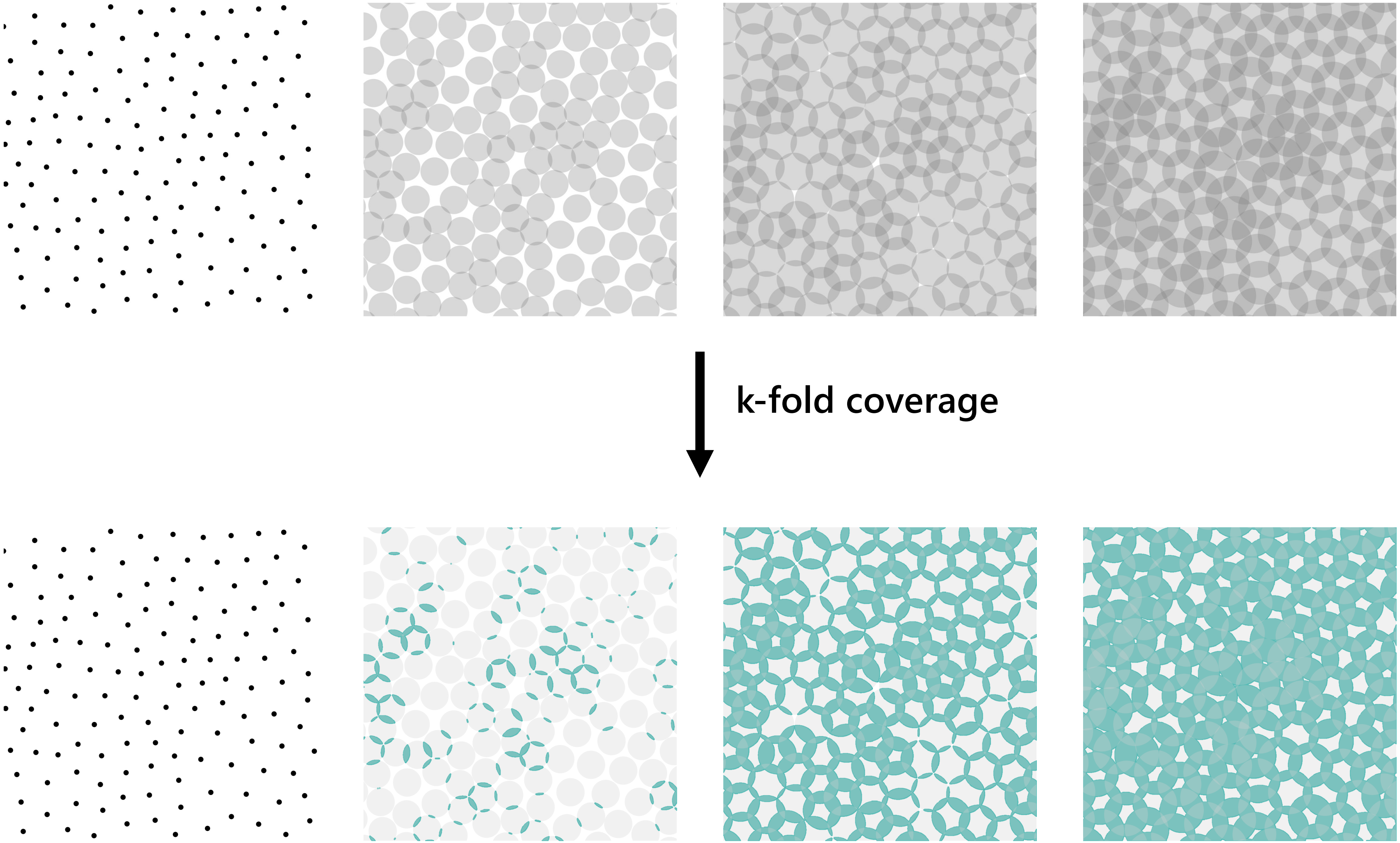


- When is $\mathcal{M}$ completely covered by the balls?
- What is the spatial distribution of last uncovered components?
- Classical methods are "brute force" and limited
- **Observation:** uncovered components $\longleftrightarrow$ maxima (index d)
- **Corollary:**

1. Coverage threshold: $nr^d \sim \log n + (d-1)\log\log n$
2. Poisson limit for last uncovered components

[Hall
Flatto-Newman
Janson
Penrose
Many more…]

3 – Coverage



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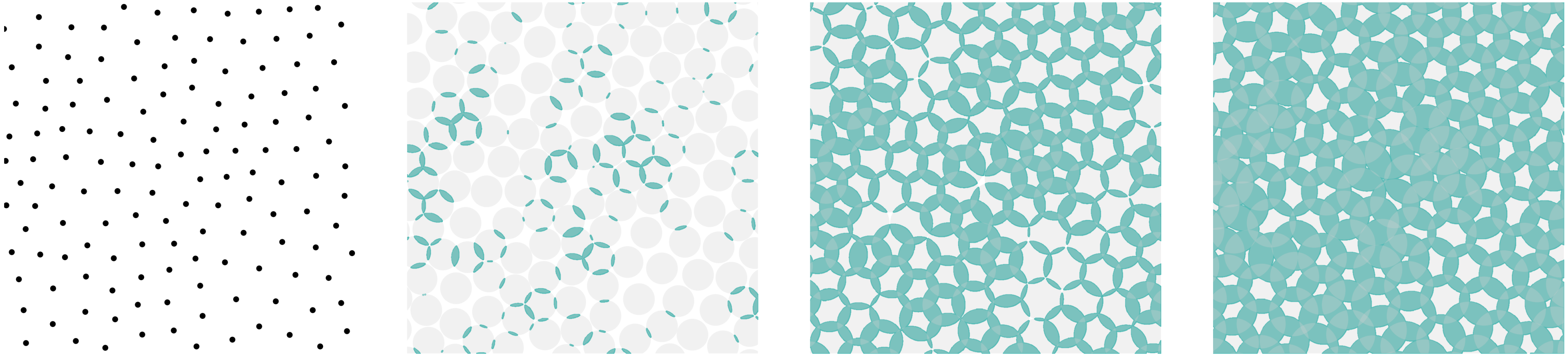


- When is $\mathcal{M}$ completely **k-covered**?
- What is the spatial distribution of last uncovered components?
- Requires new Morse theory for k-NN distance function
- Theorem [Reani & B., 2025]:

1. k-Coverage threshold: $nr^d \sim \log n + (d+k-2) \log \log n$
2. Poisson limit for last uncovered components

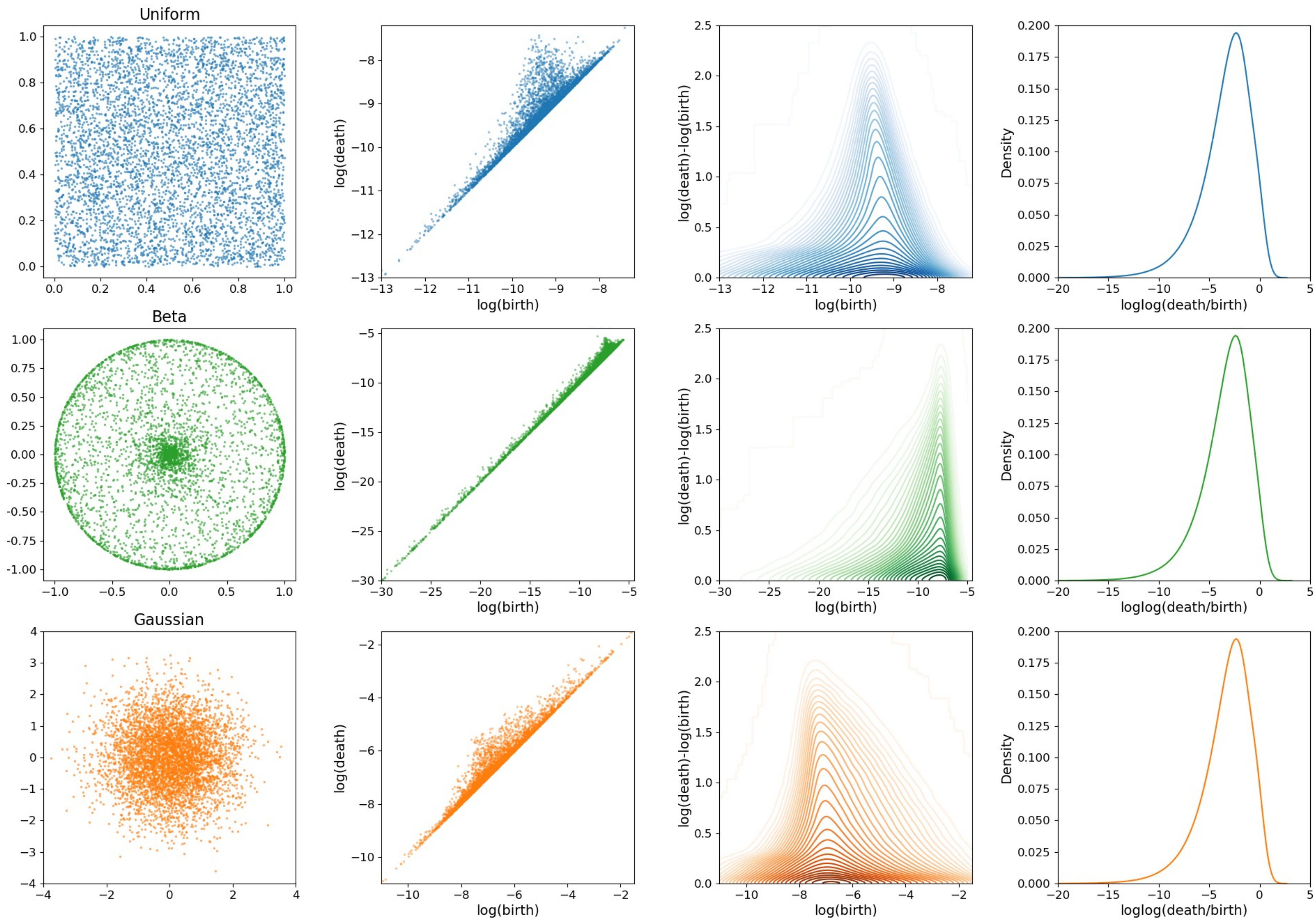
4 – Universality of Noise

$\mathcal{P}_n$	→	$\mathrm{dgm}_k(\mathcal{P}_n)$	→	$\Pi_{k,n}$
$\{X_1, \ldots, X_N\}$		$\{(b_1, d_1), \ldots, (b_M, d_M)\}$		$\{d_1/b_1, \ldots, d_M/b_M\}$
point-cloud		persistence diagram		persistence values

- $f : \mathbb{R}^d \to \mathbb{R}$ – "nice" probabiltiy density function
- $\mathcal{P}_n$ is Poisson/Binomial, with density f
- Consider $\Pi_{k,n}$ as a measure on $[1, \infty)$
- Theorem [B. & Skraba, 2024]:

$$\lim_{n\to\infty} \frac{1}{n} \Pi_{k,n} = \Pi^*_{k,d}$$

- $\Pi^*_{k,d}$ = non-random measure, supported on $[1, \infty)$ **independent of f = universal**
- Applications: hypothesis testing, clustering, dimensionality estimation…



Uniform
Beta
Gaussian
log(death)
log(birth)
log(death)-log(birth)
Density
loglog(death/birth)

4 – Universality of Noise

- $\Pi_{k,n} = \sum_i \delta_{d_i/b_i}$ – persistence values measure on $[1,\infty)$
- Total number of points in a diagram (Morse Theory):

$$\Pi_{k,n}([1,\infty)) = \sum_{i=0}^{k}(-1)^{k-i} N_k(\infty)$$

- Central claim:

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left\{N_k(\infty)\right\} = \lim_{\lambda\to\infty}\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left\{N_k(r)\right\} = C^*_{k,d}$$

universal constant

$r = (\lambda/n)^{1/d}$

thermodynamic limit

- Conclusions:
  - Diagram size is **order** n
  - Diagram size is **universal** (independent of f)
  - Most persistent cycles are in the **thermodynamic limit** [Hiraoka, Shirai, Trinh, 2018]

5 – Maximal Cycles

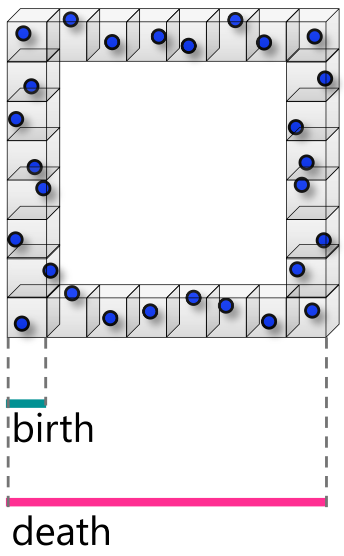
- How big "noisy" (random) cycles can be?

- Maximal size (persistence):

$$\Pi_{k,n}^{\max} := \max_{\gamma} \frac{\mathrm{death}(\gamma)}{\mathrm{birth}(\gamma)}$$

- Asymptotic rate [B., Kahle, Skraba, 2017]:

$$\Pi_{k,n}^{\max} = \Theta\left(\left(\frac{\log n}{\log\log n}\right)^{1/k}\right)$$

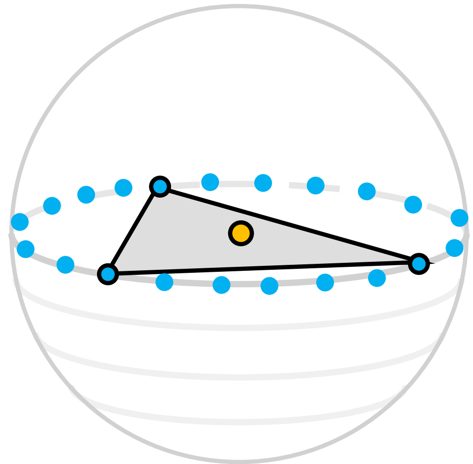


- Limitations:
    - No exact limits (rather than order of magnitude)
    - Uniform distribution in $[0,1]^d$

5 – Maximal Cycles

$$\Pi_{k,n}^{\max} = \Theta\left(\left(\frac{\log n}{\log\log n}\right)^{1/k}\right)$$

- New approach – "efficient" large cycles:
  - Look for critical points of index k+1 (**death**)
  - Check for a dense configuration near the equator



- New result [Duncan, B., Skraba, 2025]:

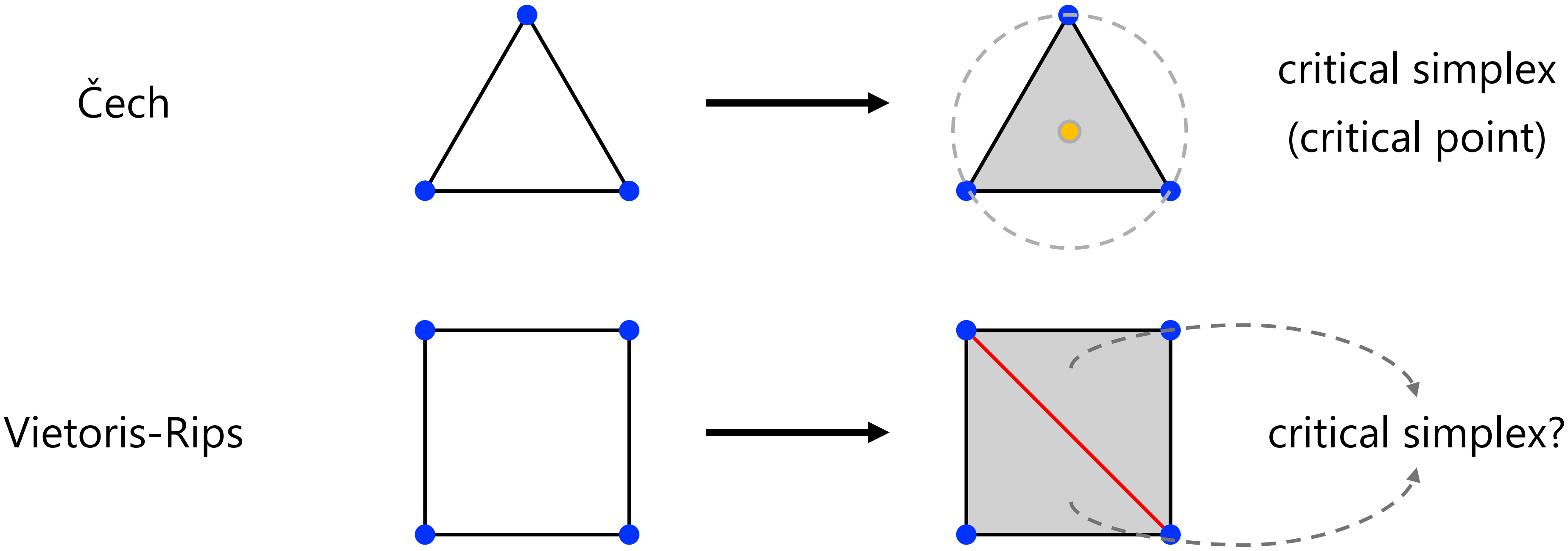
$$D_{\max}^{-1}\left(\frac{k}{d-k}\right)^{1/k} \leq \Pi_{k,n}^{\max}\left(\frac{\log n}{\log\log n}\right)^{-1/k} \leq D_{\min}^{-1}\left(\frac{k}{d-k}\right)^{1/k}$$

- Applies to any* distribution (i.e., **universal**)

Random Vietoris-Rips Complex

Challenges

- No direct relation to the distance function
- Not a Morse filtration (more than one change at a time)
- Homology is not bounded by dimension
- **No unique "critical objects":**



Two Possible Approaches

1. **Discrete Morse Theory [Kahle, 2011]:**
   - Order points by distance to origin
   - Create pairings (apparent pairs)
   - Count critical simplexes:

$$\mathbb{E}\left\{\beta_k(r)\right\} \leq \mathbb{E}\left\{N_k(r)\right\} = O(n(nr^d)^k e^{-cnr^d})$$

2. **Critical edges [B. & Skraba, 2024]:**
   - Claim:

$$\tilde{\beta}_{k-2}(\mathrm{lk}(e)) = \#k\text{-births} + \#(k-1)\text{-deaths}$$

   - Count changes = weighted sums of edges, with weights = $\tilde{\beta}_{k-2}(\mathrm{lk}(e))$

Conclusion

Summary

- Morse Theory provides a powerful “local-to-global” machinery
- Only provides partial information (nothing on positive/negative)
- Highly useful for random geometric complexes:
  - Euler characteristic
  - Homological connectivity
  - Coverage / k-Coverage
  - Universality
  - Maximal cycles
  - Quantitative CLTs for persistence diagrams
- Also useful in other areas:
  - Gaussian random fields [Adler & Taylor]
  - Configuration spaces [Baryshnikov, Bubenik, Kahle]

Random geometric complexes [Kahle, 2011]

Distance functions, critical points, and the topology of random Čech complexes [B. & Adler, 2014]

On the topology of random complexes built over stationary point processes [Yogeshwaran & Adler, 2015]

Expected sizes of Poisson-Delaunay mosaics and their discrete Morse functions [Edelsbrunner, Nikitenko, Reitzner, 2017]

On the vanishing of homology in random Čech complexes [B. & Weinberger, 2017]

Poisson process approximation under stabilization and Palm coupling [B., Schulte, Yogeshwaran, 2021]

Homological connectivity in random Čech complexes [B., 2022]

Random Čech complexes on manifolds with boundary [de Kergorlay, Tillmann, Vipond, 2022]

Large deviation principle for geometric and topological functionals [Hirsch & Owada, 2023]

Universality in random persistent homology and scale-invariant functionals [B. & Skraba, 2024]

Sharp phase transitions for k-fold coverage using Morse theory [Reani & B., in prep]

A universal limit for maximal cycles in random Čech complexes [Duncan, B., Skraba, in prep]

Quantitative central limit theorems for persistence diagrams [B., Skraba, Yukich, in prep]

Thank You.

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