Möbius Homology

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The Geometric Realization of AATRN

August 18-22, 2025

Möbius inversion is a well known counting technique in algebraic combinatorics.

Given a locally finite poset P and a function $P \xrightarrow{f} \mathbb{Z}$ there is a unique function $P \xrightarrow{\partial f} \mathbb{Z}$ such that

$$f(b) = \sum_{a : a \leq b} \partial f(a) \quad \forall b \in P$$

Möbius inversion Formula

Note the resemblance to the fundamental theorem of calculus.

We call ∂f the <u>Möbius inverse</u> or <u>inversion</u> of f.

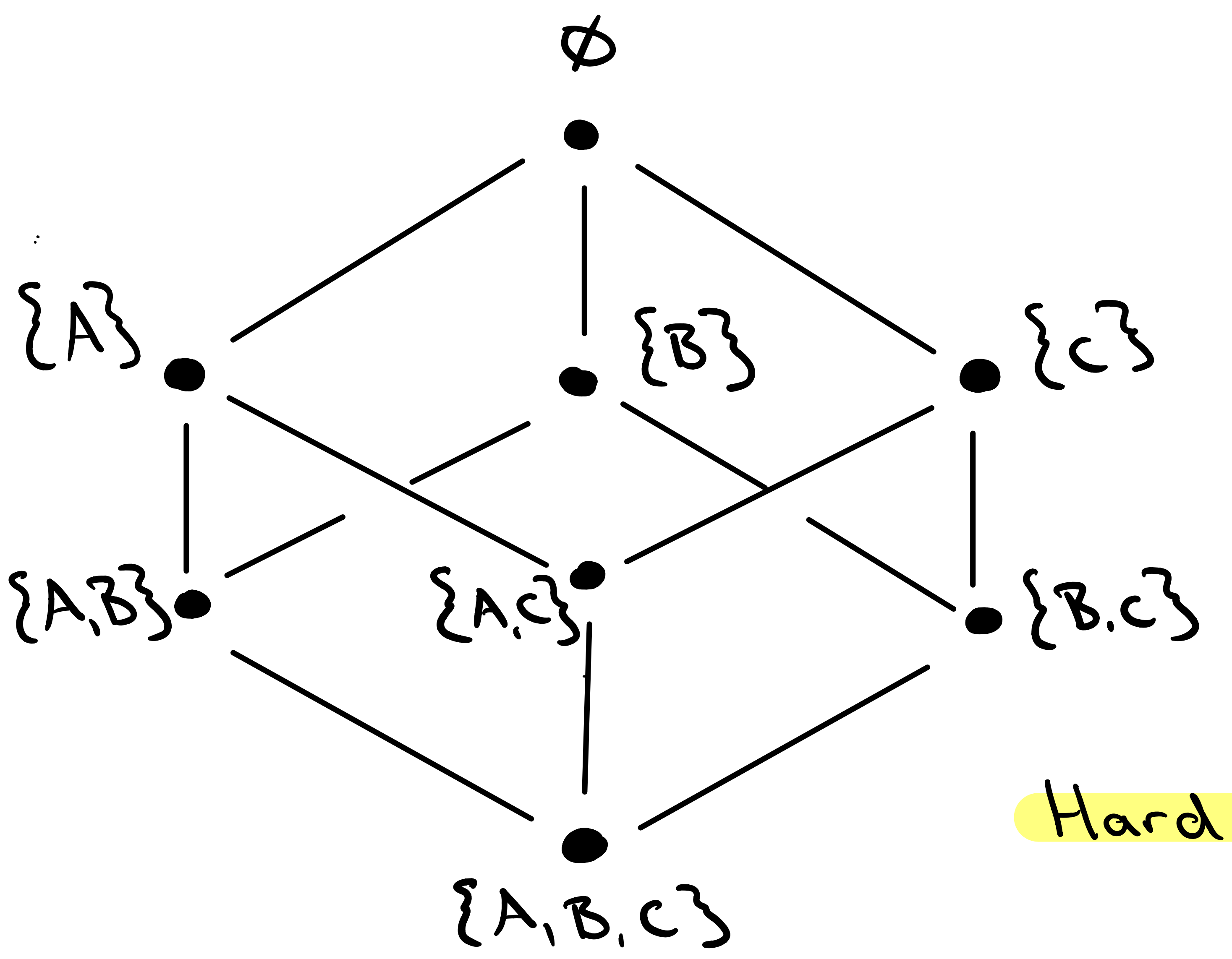
Möbius inversion has many applications.

For us, the main application is the persistence diagram aka barcode.

Application 1: Inclusion-Exclusion

Consider three sets A, B, C

Let B_3 be the Boolean lattice of subsets of $\{A, B, C\}$ ordered by reverse-inclusion.



Easy function: $B_3 \xrightarrow{f} \mathbb{Z}$

For each $S \subseteq \{A,B,C\}$, let $f(S) = \#\{$elements in **all** sets indexed by $S\}$

For example, $f(\{A,B\}) = |A \cap B|$

Hard function: $B_3 \xrightarrow{\partial f} \mathbb{Z}$

For each $S \subseteq \{A,B,C\}$, $\partial f(S) = \#\{$elements in *exactly* the sets indexed by $S\}$

For example, $\partial f(\{A,B\})$ counts elements in $A \cap B$ but not in C.

$$\boxed{f(T) = \sum_{S : S \leq T} \partial f(S) \quad \forall T \subseteq \{A,B,C\}}$$

Application 2: Chromatic number

Fix a finite graph $G = (V, E)$ and a number $n \in \mathbb{N}$.

Poset : $P = (2^E, \leq)$ with reverse inclusion. ($A \leq A' \iff A \supseteq A'$)

Given $A \subseteq E$, let $c(A)$ be the number of components of (V, A).

Easy function: $P \xrightarrow{f_n} \mathbb{Z}$

$f_n(A) = n^{c(A)} =$ # colorings that make all edges in (V, A) monochromatic.

Hard function: $P \xrightarrow{\partial f_n} \mathbb{Z}$

For each $A \subseteq E$, $\partial f_n(A) =$ # colorings of (V, A) whose set of monochromatic edges is exactly A

$$\boxed{f_n(B) = \sum_{A : A \leq B} \partial f_n(A) \quad \forall A \subset E}$$

Proper n-coloring of G is $\partial f_n(\emptyset)$.

Application 3: Euler's Totient Function

Poset: $\mathbb{N} = \{1, 2, 3, \dots\}$ ordered by divisibility $(d \le e \iff d \mid e)$

Easy function: $\mathbb{N} \xrightarrow{f} \mathbb{Z}$

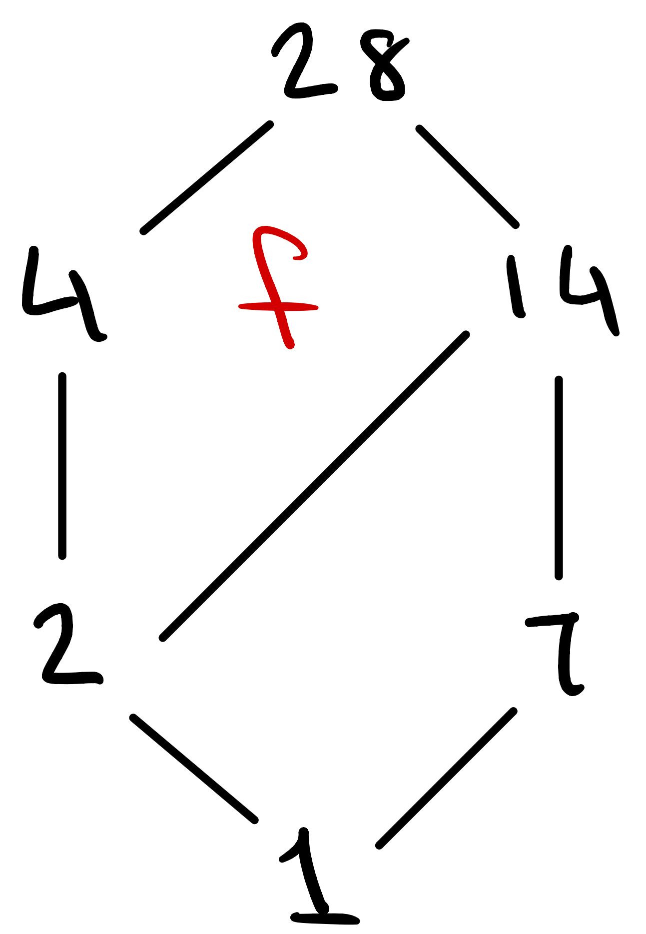
$f(n) = n$ (count all integers $1, \dots, n$)

Hard function: $\mathbb{N} \xrightarrow{\partial f} \mathbb{Z}$

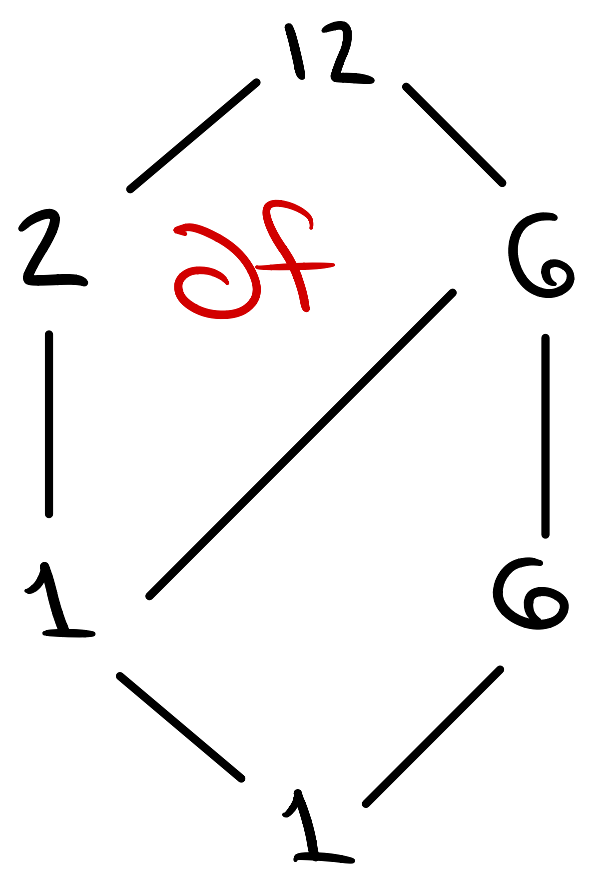
$\partial f(n) = \varphi(n) = \#\{m \in \mathbb{N} : 1 \le m \le n \text{ and } \gcd(m,n) = 1\}$ (i.e., exactly coprime to n)

Möbius 1832

Euler's totient function (1763)



$\mathbb{N}_{\le 28}$



$\mathbb{N}_{\le 28}$

Application 4: Persistence Diagram aka Barcodes

Let $P = \{1 < 2 < \cdots < n < \infty\}$ and $P \xrightarrow{M} \mathrm{vec}_{\mathbb{F}}$ a persistence module

Poset: $\mathrm{Int}\, P := \{[a,b) : a \leq b\}$ where $[a,b) \leq [c,d) \iff a \leq c$ and $b \leq d$

Easy function: $\mathrm{Int}\, P \xrightarrow{f_M} \mathbb{Z}$

$$f_M[a,b) := \begin{cases} \dim \ker M(a \leq b) & \text{for } b < \infty \\ \dim M(a) & \text{for } b = \infty \end{cases}$$

Counts # generators born before a and dies before b

Hard function: $\mathrm{Int}\, P \xrightarrow{\partial f_M} \mathbb{Z}$

$\partial f_M[a,b)$ counts the number of generators born exactly at a and dead exactly at b.

∂f_M is the persistence diagram of M

Calculating Möbius Inversion

P is a locally finite poset

Möbius Function $\mu : P \times P \longrightarrow \mathbb{Z}$

$$\mu(a,b) := \begin{cases} 0 & \text{if } a \nleq b \\ 1 & \text{if } a = b \\ -\sum\limits_{a < x \leq b} \mu(x,b) & \text{if } a < b \end{cases}$$

For any $P \xrightarrow{f} \mathbb{Z}$, its Möbius inversion $P \xrightarrow{\partial f} \mathbb{Z}$ is:

$$\partial f(b) = \sum_{a : a \leq b} f(a) \cdot \mu(a,b) \qquad \forall b \in P$$

Persistence over Posets

P be any finite (locally finite is enough) perhaps with a top ∞.

$P \xrightarrow{M} \mathrm{vec}_{\mathbb{F}}$ a persistence module.

Poset: $\mathrm{Int}\, P := \{[a,b) : a \leq b\}$ where $[a,b) \leq [c,d) \iff a \leq c$ and $b \leq d$

Easy function: $\mathrm{Int}\, P \xrightarrow{f_M} \mathbb{Z}$

$$f_M[a,b) := \begin{cases} \dim \ker M(a \leq b) & \text{for } b < \infty \\ \dim M(a) & \text{for } b = \infty \end{cases}$$

Counts # generators born before a and dies before b

Hard function: $\mathrm{Int}\, P \xrightarrow{\partial f_M} \mathbb{Z}$

$\partial f_M[a,b)$ counts the number of generators born exactly at a and dead exactly at b.

∂f_M is the "persistence diagram" of M?

Example

$$
\begin{tikzcd}
b \arrow[r] & \infty \\
a \arrow[u] \arrow[r] \arrow[ur, phantom, "P"] & c \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \\
\mathbb{F}^2 \arrow[u, "(0\ 0)"] \arrow[r, "(1\ 0)"] & \mathbb{F} \arrow[u]
\end{tikzcd}
$$

M

$$
\begin{tikzcd}
 & \overset{0}{[\infty,\infty)} & \\
1\ [b,\infty) \arrow[ur] & & [c,\infty)\ 1 \arrow[ul] \\
0\ [b,b) \arrow[u] & \overset{2}{[a,\infty)} \arrow[ul] \arrow[ur] & [c,c)\ 0 \arrow[u] \\
2\ [a,b) \arrow[u] \arrow[ur] & & [a,c)\ 1 \arrow[u] \arrow[ul] \\
 & \overset{0}{[a,a)} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

f_M

$$
\begin{tikzcd}
 & \overset{0}{[\infty,\infty)} & \\
1\ [b,\infty) \arrow[ur] & & [c,\infty)\ 0 \arrow[ul] \\
-2\ [b,b) \arrow[u] & \overset{-1}{[a,\infty)} \arrow[ul] \arrow[ur] & [c,c)\ -1 \arrow[u] \\
2\ [a,b) \arrow[u] \arrow[ur] & & [a,c)\ 1 \arrow[u] \arrow[ul] \\
 & \overset{0}{[a,a)} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

∂f_M

In the previous example, $\partial f_M < 0$ for a non-diagonal interval

How do we understand these negative values?

Is the Möbius inversion stable?

Möbius Homology

categorifies Möbius Inversion

joint work with Primoz Skraba to appear in Transactions of the AMS

Main Theorem

Given $P \xrightarrow{M} \mathrm{vec}_{\mathbb{F}}$, let $P \xrightarrow{m} \mathbb{Z}$ be its dimension function.

That is, $m(a) = \dim M(a) \quad \forall a \in P$

Möbius Homology

Then, there is a homology theory $H^{\downarrow}_{*}(-;M) : P \longrightarrow \mathrm{vec}_{\mathbb{F}}$ such that

$$\partial m(a) = \sum_{i=0} (-1)^i \dim H^{\downarrow}_i(a;M) \quad \forall a \in P$$

Review of Simplicial Cosheaf Homology

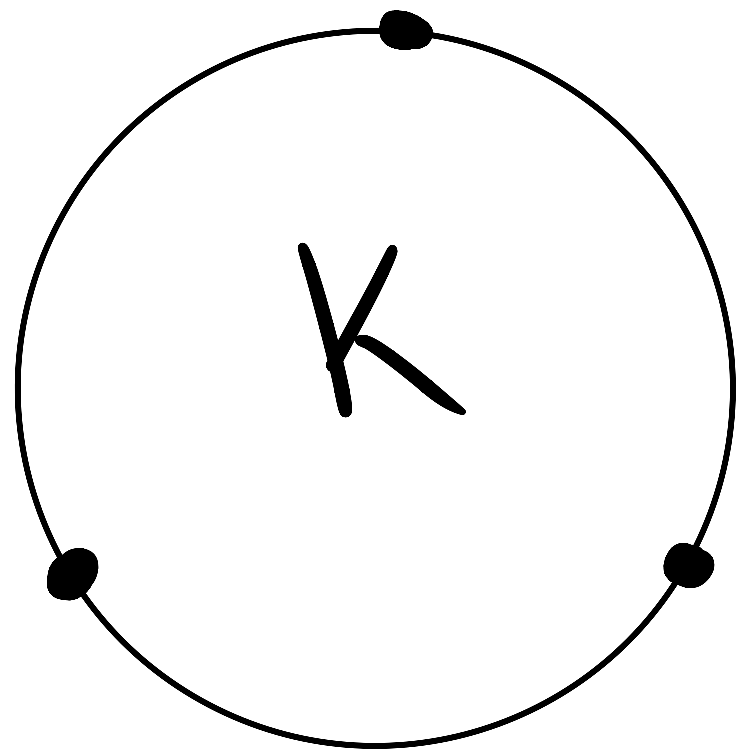
K a locally finite simplicial complex

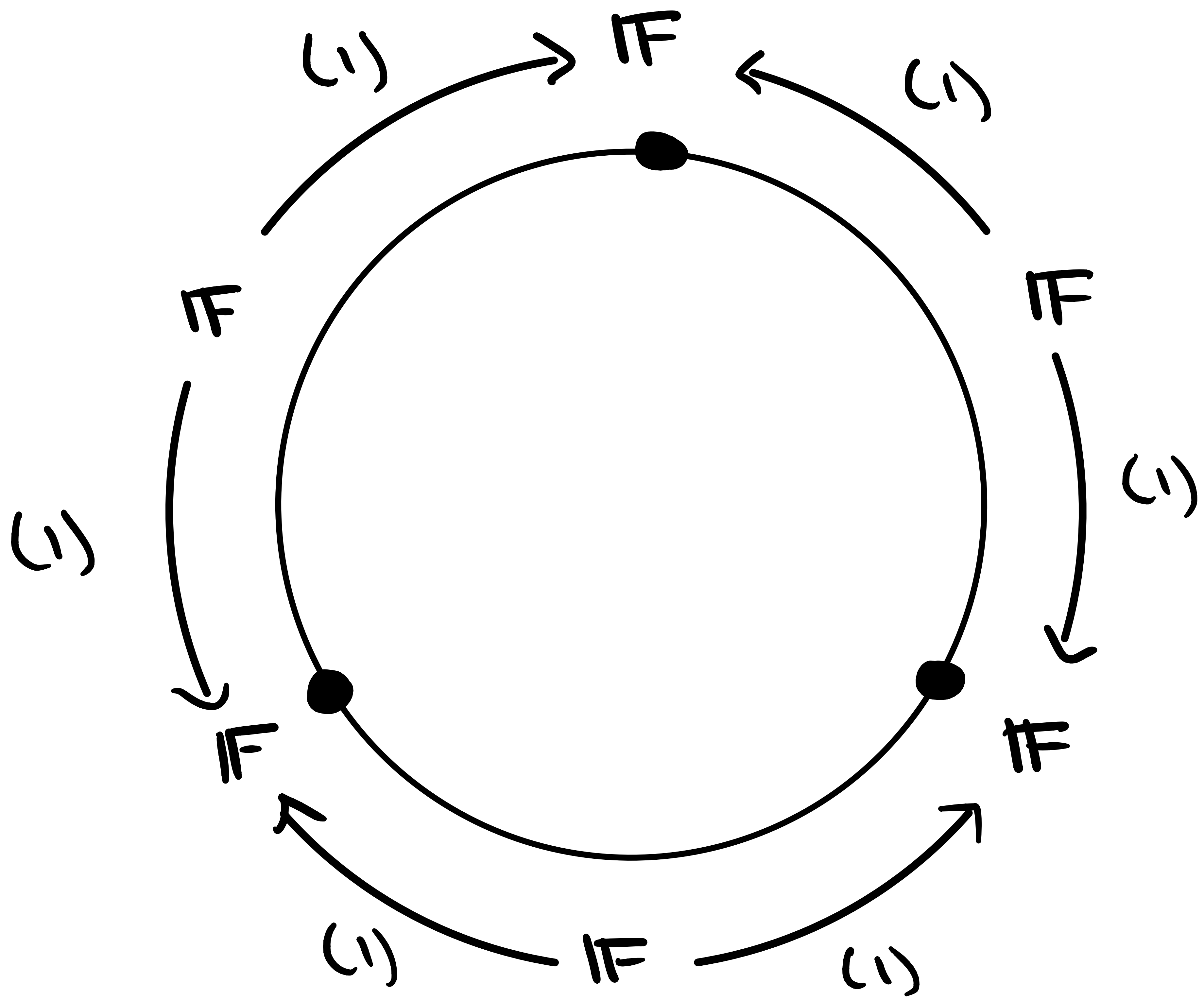
Face(K) - poset of face relations

$\tau \leq \sigma$ if σ is a face of τ

A <u>simplicial cosheaf</u> over K is a functor $\mathrm{Face}(K) \xrightarrow{F} \mathrm{vec}_{\mathbb{F}}$.

Examples


K


$\mathbb{F}$
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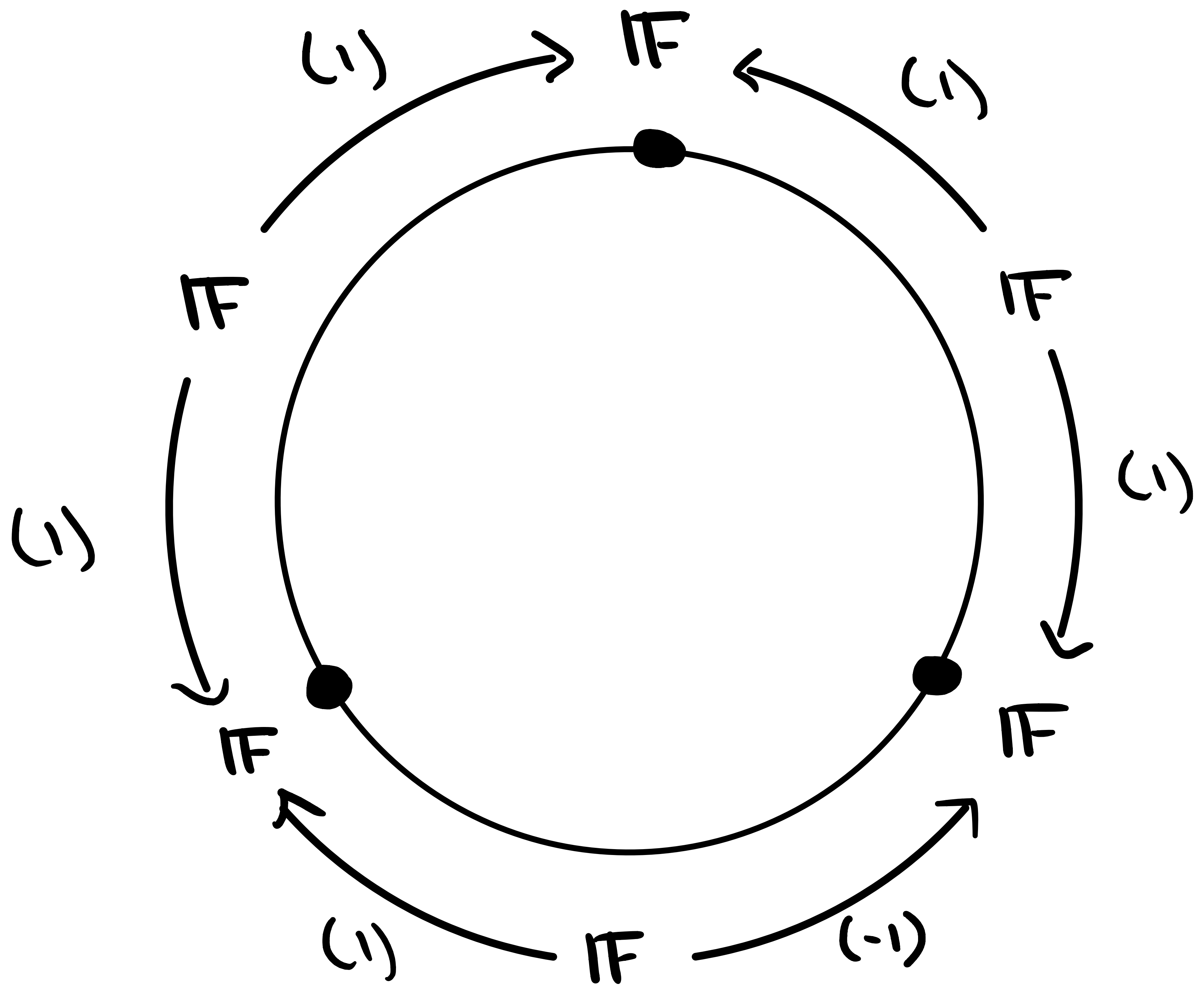
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Simplicial Cosheaf Homology

$$\mathrm{Face}(K) \xrightarrow{F} \mathcal{Z}$$

$$\cdots \longrightarrow \bigoplus_{\substack{\sigma \in K \\ \dim\sigma = d+1}} F(\sigma) \xrightarrow{\delta_{d+1}} \bigoplus_{\substack{\sigma \in K \\ \dim\sigma = d}} F(\sigma) \xrightarrow{\delta_d} \bigoplus_{\substack{\sigma \in K \\ \dim\sigma = d-1}} F(\sigma) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{\sigma \in K \\ \dim\sigma = 1}} F(\sigma) \xrightarrow{\delta_1} \bigoplus_{\substack{\sigma \in K \\ \dim\sigma = 0}} F(\sigma) \longrightarrow 0$$

Orient simplices of K arbitrarily.

Suppose $\tau, \sigma \in K$, $\dim \tau = d$, $\dim \sigma = d-1$, and $\tau \leq \sigma$.

Then, the restriction of δ_d to $\tau \leq \sigma$ is:

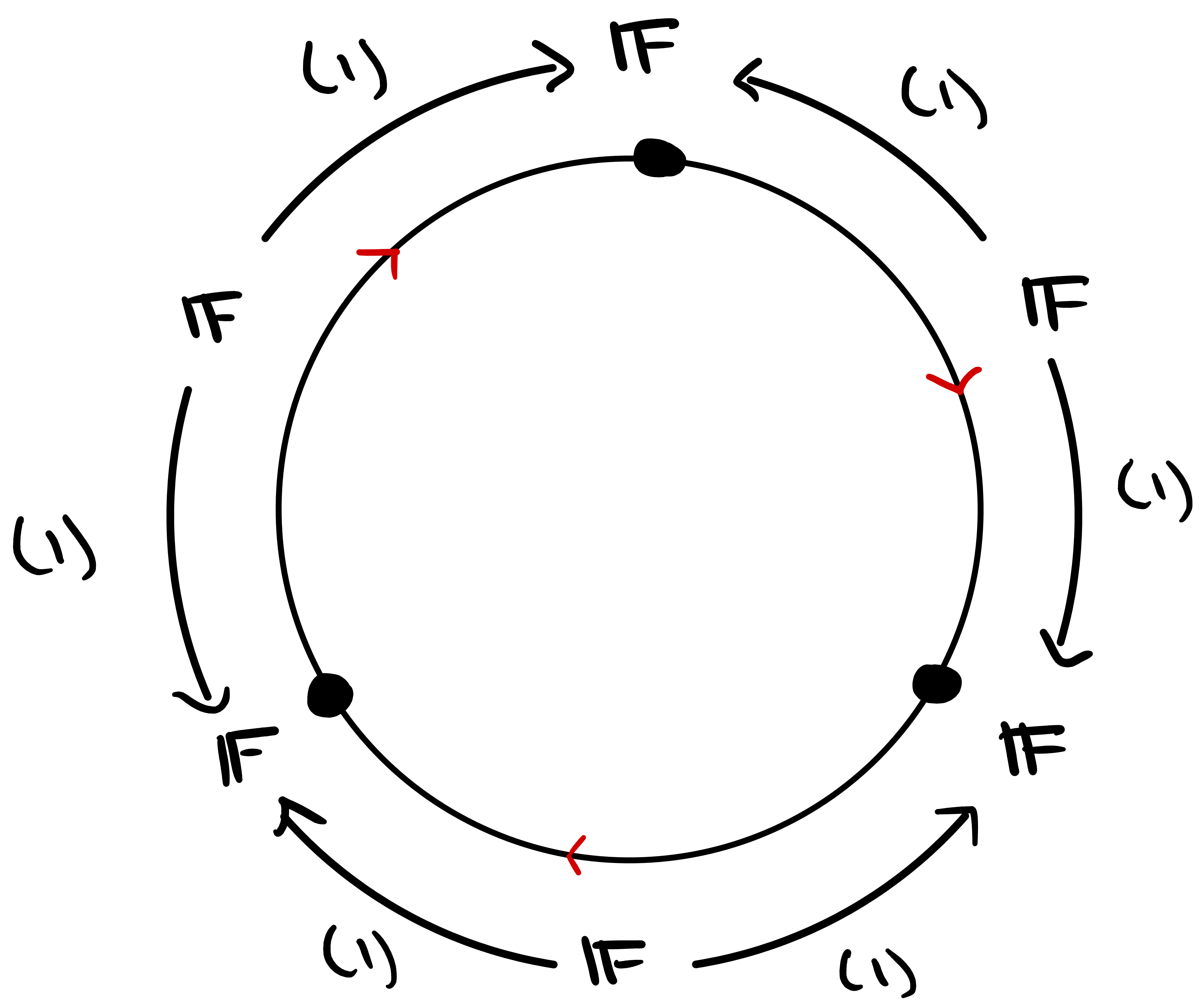
$$\delta_d\big|_{\tau \leq \sigma} : F(\tau) \longrightarrow F(\sigma)$$

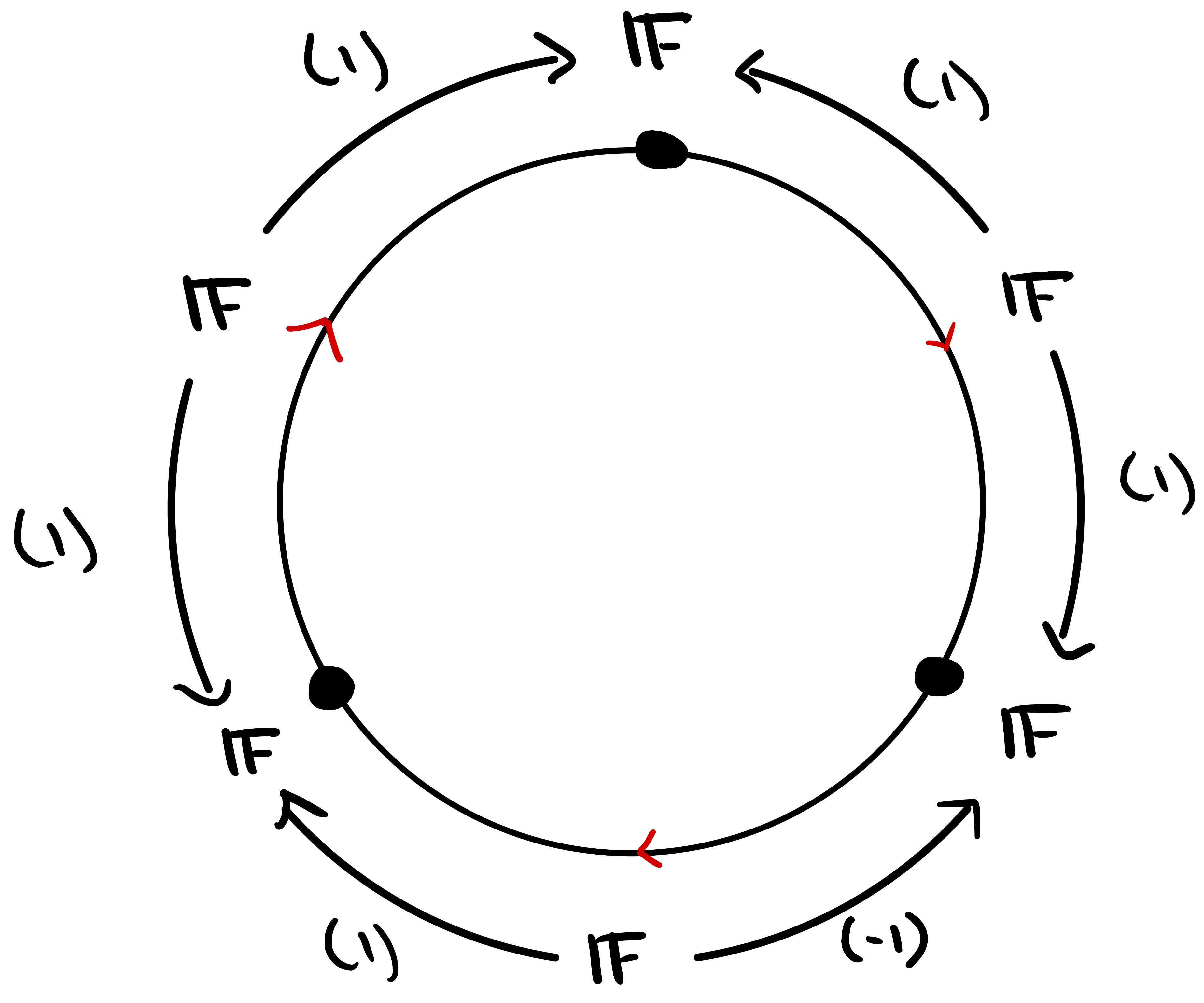
$$v \longmapsto [\tau : \sigma] F(\tau \leq \sigma)(v)$$

sign

$H_*(K; F)$ simplicial homology w/ coefficients in F.

Examples





$$0 \longrightarrow \underset{\color{blue}{1}}{\mathbb{F}^3} \xrightarrow{\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} \underset{\color{blue}{0}}{\mathbb{F}^3} \longrightarrow 0$$

$H_0 \cong \mathbb{F} \quad H_1 \cong \mathbb{F}$

$$0 \longrightarrow \underset{\color{blue}{1}}{\mathbb{F}^3} \xrightarrow{\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}} \underset{\color{blue}{0}}{\mathbb{F}^3} \longrightarrow 0$$

$H_0 = 0 \quad H_1 = 0$

Relative Simplicial Cosheaf Homology

$$\mathrm{Face}(K) \xrightarrow{F} Z$$

$L \subseteq K$ subcomplex

$$\cdots \longrightarrow \frac{\displaystyle\bigoplus_{\substack{\sigma \in K \\ \dim\sigma = d}} F(\sigma)}{\displaystyle\bigoplus_{\substack{\sigma \in L \\ \dim\sigma = d}} F(\sigma)} \xrightarrow{\partial_d} \frac{\displaystyle\bigoplus_{\substack{\sigma \in K \\ \dim\sigma = d-1}} F(\sigma)}{\displaystyle\bigoplus_{\substack{\sigma \in L \\ \dim\sigma = d+1}} F(\sigma)} \longrightarrow \cdots \longrightarrow \frac{\displaystyle\bigoplus_{\substack{\sigma \in K \\ \dim\sigma = 1}} F(\sigma)}{\displaystyle\bigoplus_{\substack{\sigma \in L \\ \dim\sigma = d+1}} F(\sigma)} \xrightarrow{\partial_1} \frac{\displaystyle\bigoplus_{\substack{\sigma \in K \\ \dim\sigma = 0}} F(\sigma)}{\displaystyle\bigoplus_{\substack{\sigma \in L \\ \dim\sigma = d+1}} F(\sigma)} \longrightarrow 0$$

$H_*(K, L; F)$ relative simplicial homology w/ coefficients in F.

Ready for Möbius Homology

Given $P \xrightarrow{M} \mathrm{vec}_{\mathbb{F}}$

we want

$$H_*^{\downarrow}(- ; M) : P \longrightarrow \mathrm{vec}_{\mathbb{F}}$$

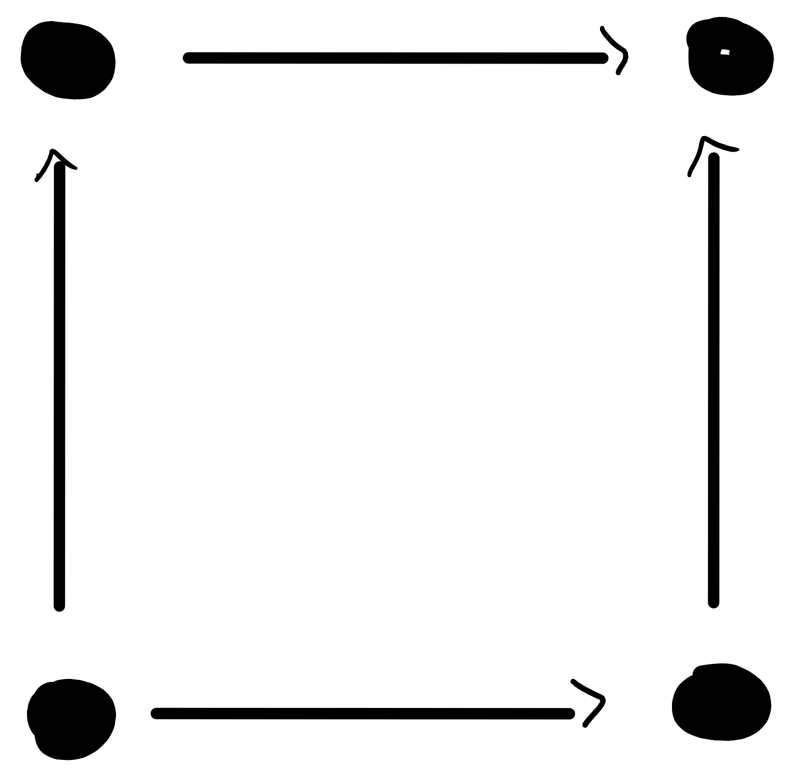
Order Complex

A <u>chain</u> in P is a strictly increasing sequence $a_0 < \cdots < a_n$.

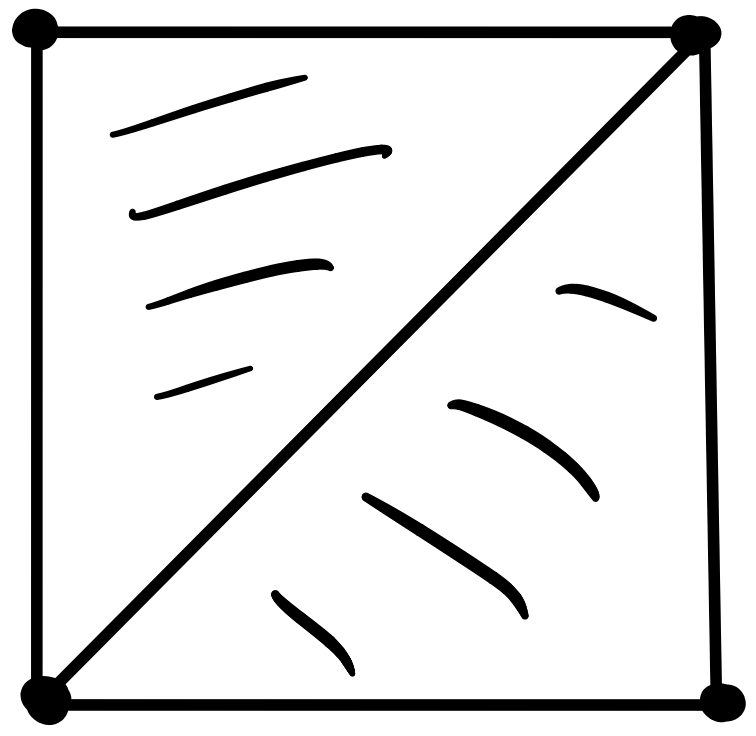
The <u>length</u> of a chain is the number of strict relations.

The <u>order complex</u> of P, denoted ΔP, is the simplicial complex consisting of chains (i.e., an n-simplex is a length n chain)

<u>Ex</u>



P



ΔP

Order Complex Cosheaf

order preserving map $\quad \mathrm{Face}(\Delta P) \xrightarrow{\min} P$

$$\tau \leq \sigma \longmapsto \min \tau \leq \min \sigma$$

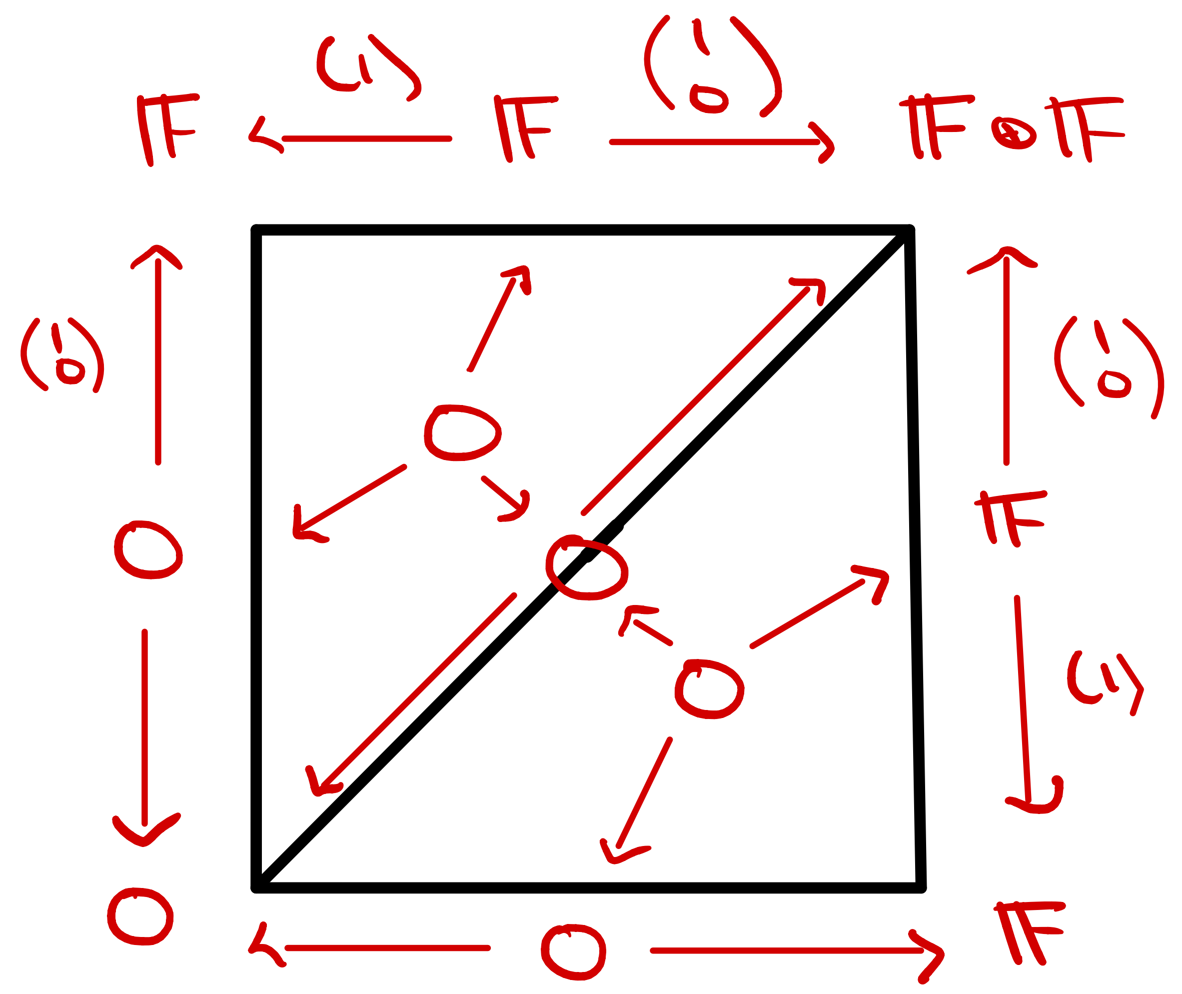
Given $P \xrightarrow{M} \mathrm{vec}_{\mathbb{F}}$, define $\mathrm{Face}(\Delta P) \xrightarrow{\underline{M}} \mathrm{vec}_{\mathbb{F}}$ as the composition:

$$\begin{tikzcd} \mathrm{Face}(\Delta P) \arrow[r, "\min"] \arrow[rr, bend left, dashed, "\underline{M}"] & P \arrow[r, "M"] & \mathrm{vec}_{\mathbb{F}} \end{tikzcd}$$

Example

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "\binom{1}{0}"] & \mathbb{F}\oplus\mathbb{F} \\
0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "\binom{1}{0}"']
\end{tikzcd}$$

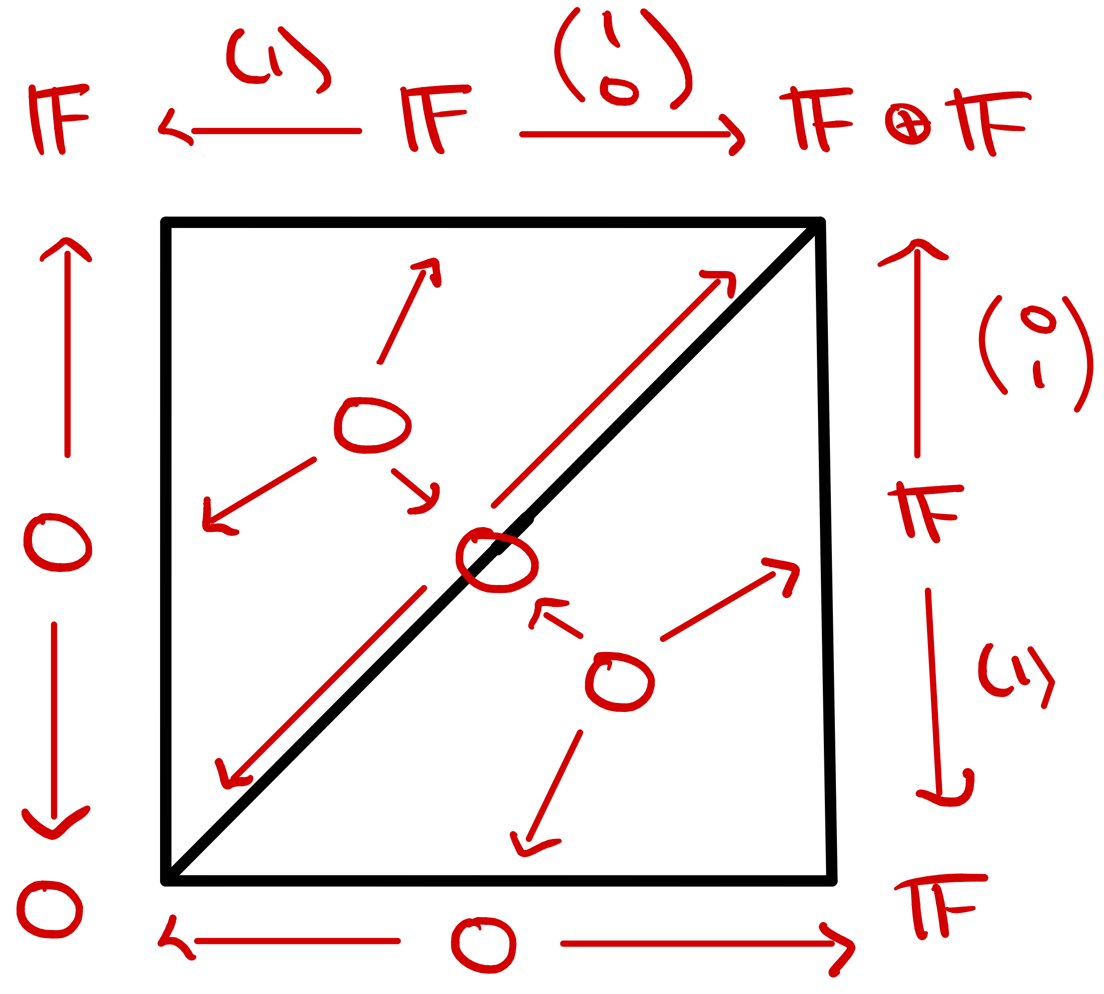
M_1



$\underline{M}_1$

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "\binom{1}{0}"] & \mathbb{F}\oplus\mathbb{F} \\
0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "\binom{0}{1}"']
\end{tikzcd}$$

M_2



$\underline{M}_2$

Möbius Homology

Given $P \xrightarrow{M} \mathrm{vec}_{\mathbb{F}}$, we have a cosheaf $\mathrm{Face}(\Delta P) \xrightarrow{\underline{M}} \mathrm{vec}_{\mathbb{F}}$.

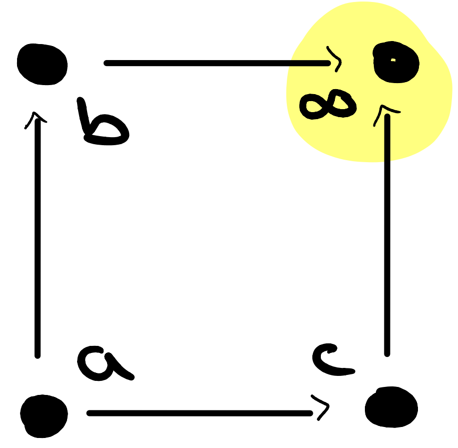
For $b \in P$, let $\Delta P_{\leq b} := \{\sigma \in \Delta P : \max \sigma \leq b\}$

$\Delta P_{< b} := \{\sigma \in \Delta P : \max < b\}$

Note $\Delta P_{< b} \subseteq \Delta P_{\leq b}$

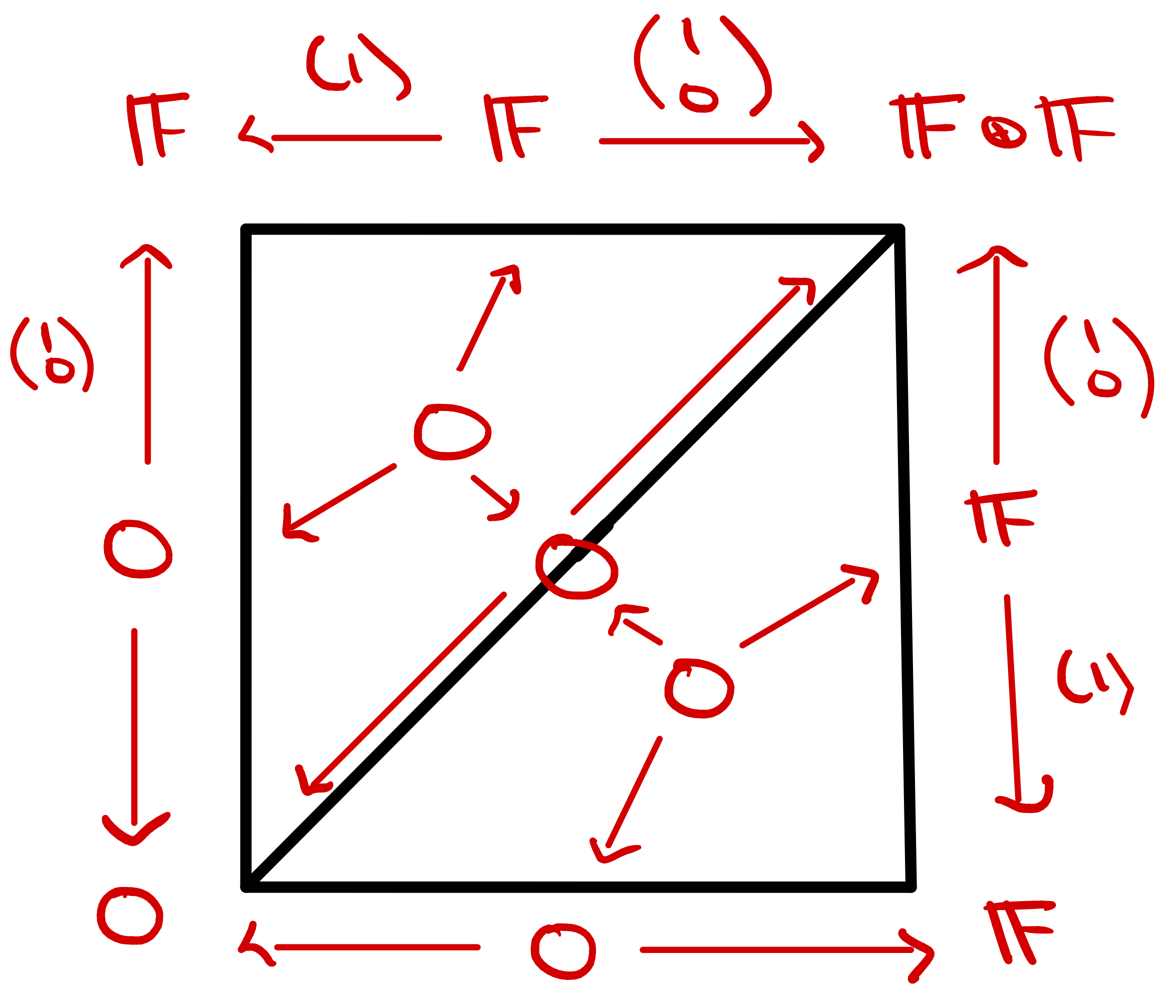
$$\boxed{\text{Define } H_*^{\downarrow}(b; M) := H_*(\Delta P_{\leq b}, \Delta P_{< b}; \tilde{\underline{M}})}$$

Example



$$\begin{tikzcd}
\mathbb{F} \arrow[r, "{\binom{1}{0}}"] & \mathbb{F}\oplus\mathbb{F} \\
0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "{\binom{1}{0}}"']
\end{tikzcd}$$

M_1



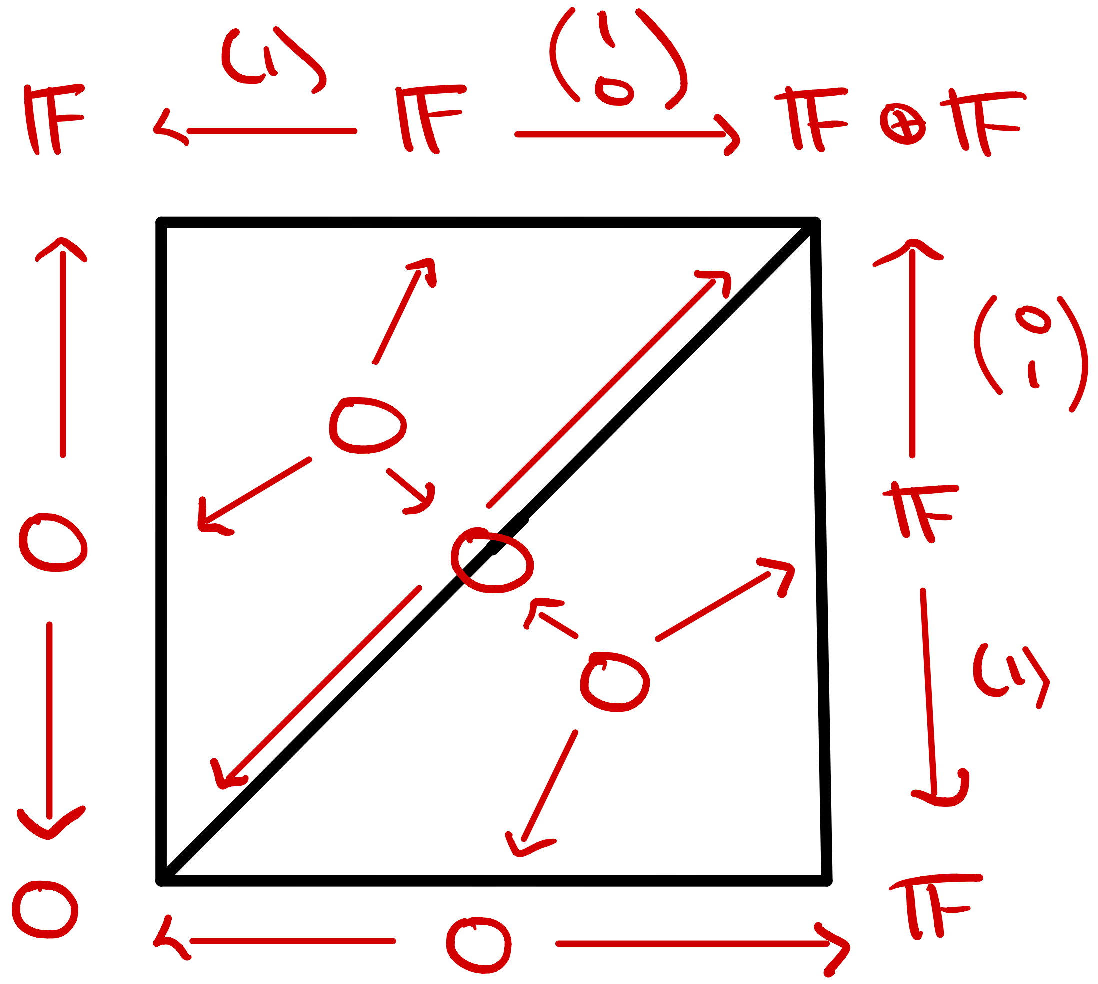
$\underline{M}_1$

$$0 \longrightarrow \overset{1}{\mathbb{F}\oplus\mathbb{F}} \xrightarrow{\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}} \overset{0}{\mathbb{F}\oplus\mathbb{F}} \longrightarrow 0$$

$$H_0^{\downarrow}(\infty; M_1) \cong \mathbb{F} \qquad H_1^{\downarrow}(\infty; M_1) \cong \mathbb{F}$$

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "{\binom{1}{0}}"] & \mathbb{F}\oplus\mathbb{F} \\
0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "{\binom{0}{1}}"']
\end{tikzcd}$$

M_2



$\underline{M}_2$

$$0 \longrightarrow \overset{1}{\mathbb{F}\oplus\mathbb{F}} \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} \overset{0}{\mathbb{F}\oplus\mathbb{F}} \longrightarrow 0$$

$$H_0^{\downarrow}(\infty; M_2) = 0 \qquad H_1^{\downarrow}(\infty; M_2) = 0$$

Decategorify

$$\begin{tikzcd} b \arrow[r] & \infty \\ a \arrow[u] \arrow[r] & c \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{F} \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] & \mathbb{F}\oplus\mathbb{F} \\ 0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "\begin{pmatrix}1\\0\end{pmatrix}"'] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & 2 \\ 0 \arrow[u] \arrow[r] & 1 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 1 \arrow[u] \end{tikzcd}$$

$M_1 \qquad m_1 \qquad \partial m_1$

$H_0^{\downarrow}(\infty; M_1) \cong \mathbb{F} \qquad H_1^{\downarrow}(\infty; M_1) \cong \mathbb{F}$

$\chi(\infty; M_1) = 0 = \partial m_1(\infty)$

$$\begin{tikzcd} \mathbb{F} \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] & \mathbb{F}\oplus\mathbb{F} \\ 0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "\begin{pmatrix}0\\1\end{pmatrix}"'] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] & 2 \\ 0 \arrow[u] \arrow[r] & 1 \arrow[u, "\begin{pmatrix}0\\1\end{pmatrix}"'] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] & 0 \\ 0 \arrow[u] \arrow[r] & 1 \arrow[u] \end{tikzcd}$$

$M_2 \qquad m_2 \qquad \partial m_2$

$H_0^{\downarrow}(\infty; M_2) = 0 \qquad H_1^{\downarrow}(\infty; M_2) = 0$

$\chi(\infty; M_2) = 0 = \partial m_2(\infty)$

Möbius Homology

and Persistence

$$\begin{tikzcd}
b \arrow[r] & \infty \\
a \arrow[u] \arrow[r] & c \arrow[u]
\arrow[from=2-1, to=1-2, phantom, "P"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \\
\mathbb{F}^2 \arrow[u, "(0\ 0)"] \arrow[r, "(1\ 0)"'] & \mathbb{F} \arrow[u]
\end{tikzcd}$$

M

$$\begin{tikzcd}
& {\overset{0}{[\infty,\infty)}} & \\
{1\ [b,\infty)} \arrow[ur] & & {[c,\infty)\ 1} \arrow[ul] \\
{0\ [b,b)} \arrow[u] & {\overset{2}{[a,\infty)}} \arrow[ul] \arrow[ur] & {[c,c)\ 0} \arrow[u] \\
{2\ [a,b)} \arrow[u] \arrow[ur] & & {[a,c)\ 1} \arrow[u] \arrow[ul] \\
& {\overset{0}{[a,a)}} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

f_M

$$\begin{tikzcd}
& {\overset{0}{[\infty,\infty)}} & \\
{1\ [b,\infty)} \arrow[ur] & & {[c,\infty)\ 0} \arrow[ul] \\
{-2\ [b,b)} \arrow[u] & {\overset{-1}{[a,\infty)}} \arrow[ul] \arrow[ur] & {[c,c)\ -1} \arrow[u] \\
{2\ [a,b)} \arrow[u] \arrow[ur] & & {[a,c)\ 1} \arrow[u] \arrow[ul] \\
& {\overset{0}{[a,a)}} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

∂f_M

Kernel Module

Given $P \xrightarrow{M} \mathrm{vec}_{\mathbb{F}}$

We want to lift the kernel function $\mathrm{Int}\, P \xrightarrow{f_M} \mathbb{Z}$ to $\mathrm{Int}\, P \longrightarrow \mathrm{vec}_{\mathbb{F}}$ so that we can apply Möbius Homology

For every $[a,b) \leq [c,d)$, consider

$$\begin{tikzcd}
M(a) \arrow[r, "M(a\leq b)"] \arrow[d, "M(a\leq c)"'] \arrow[dr, phantom, "\square"] & M(b) \arrow[d, "M(b\leq d)"] \\
M(c) \arrow[r, "M(c\leq d)"] & M(d)
\end{tikzcd}$$

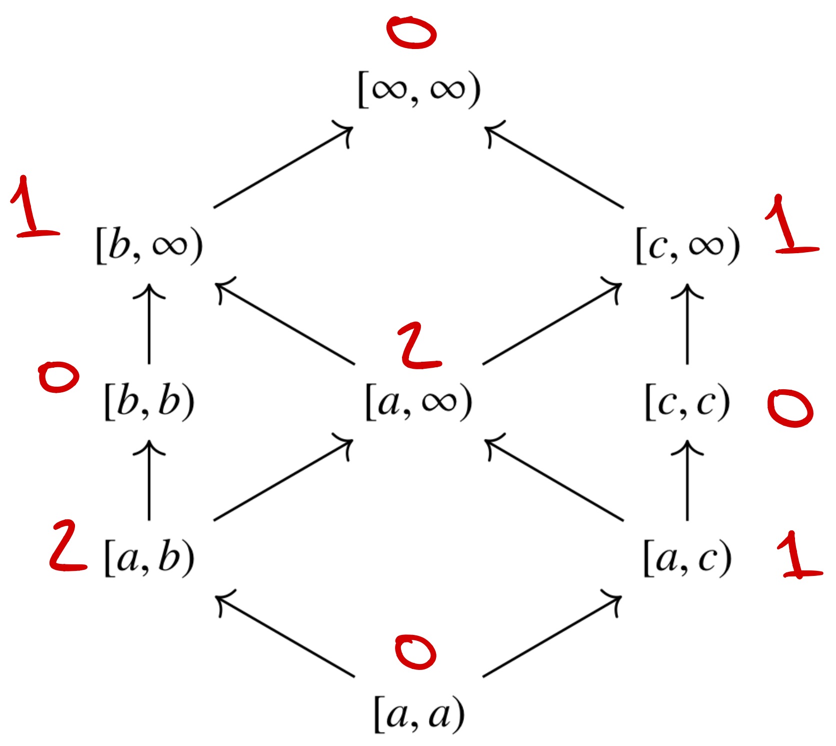
$M(a\leq c)$ restricts to a map $\ker M(a\leq b) \longrightarrow \ker M(c\leq d)$

Thus, we have $\mathrm{Int}\, P \xrightarrow{\ker M} \mathrm{vec}_{\mathbb{F}}$

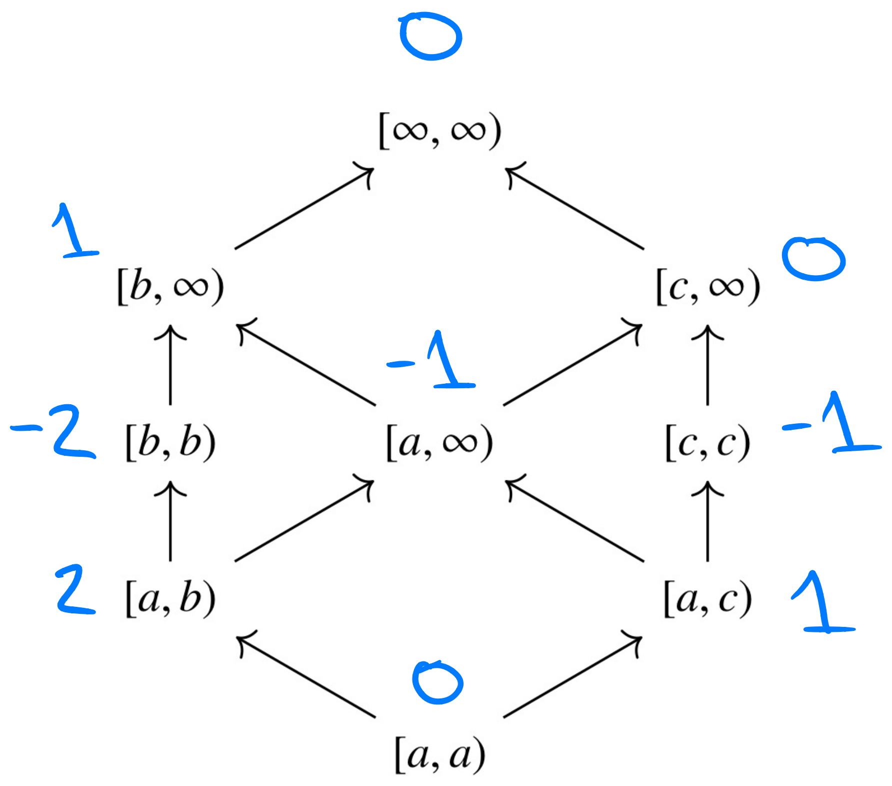
$$
\begin{tikzcd}
b \arrow[r] & \infty \\
a \arrow[u] \arrow[r] \arrow[ur, phantom, "P"] & c \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \\
\mathbb{F}^2 \arrow[u, "(0\ 0)"] \arrow[r, "(1\ 0)"] & \mathbb{F} \arrow[u]
\end{tikzcd}
$$

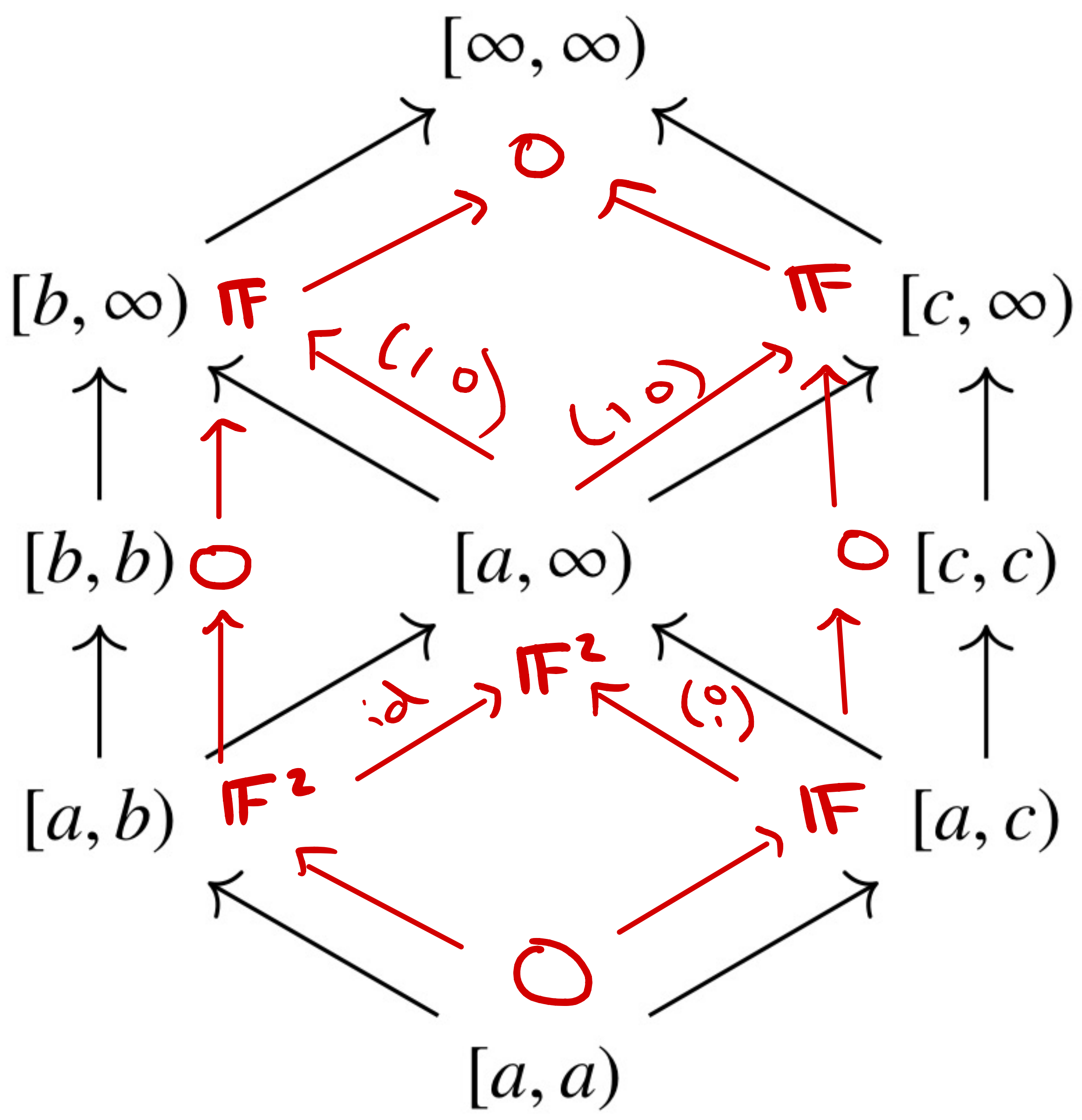
M



f_M



∂f_M



$\mathrm{Ker}\ M$

$$H_0^{!}\big([a,\infty);\ \ker M\big) = 0 \qquad H_1\big([a,\infty);\ \ker M\big) \cong \mathbb{F}$$

$$\chi\big([a,\infty);\ \ker M\big) = -1 = \partial f_M\big([a,\infty)\big)$$

Exciting Recent Development

Been searching for a Bottleneck/Interleaving-type stability theorem for Möbius inversions ~ 10 years

This summer, Hideto Asashiba (Prof Emeritus, Shizuoka U.) and I found such a theorem for Möbius Homology!

Interesting Problems

Can you figure out how to use Möbius homology to categorify well known examples of Möbius inversions?

- Chromatic Polynomial
- Tutte Polynomial
- Incidence Algebra of Subgroup Lattice
- Euler's Totient Function
- Intersection Lattice of Hyperplane Arrangement

If so, what does stability imply? Apply TDA ideas to combinatorics?

Thank You