Connectivity of random simplicial complexes

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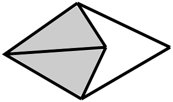

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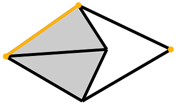

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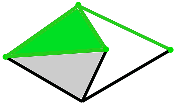

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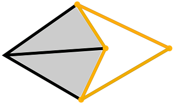

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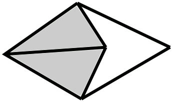

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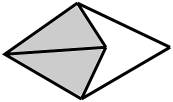

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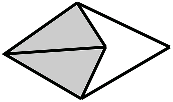

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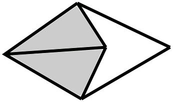

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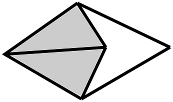

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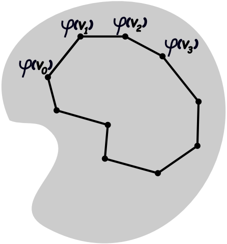

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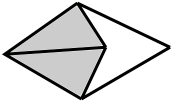

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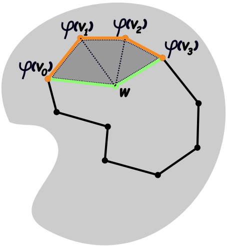

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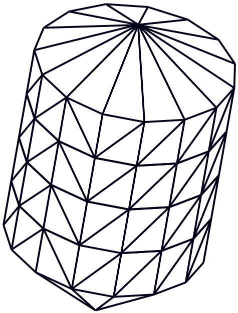

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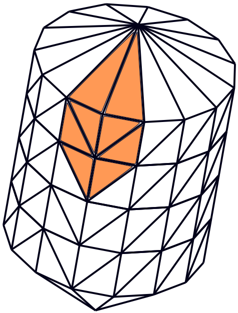

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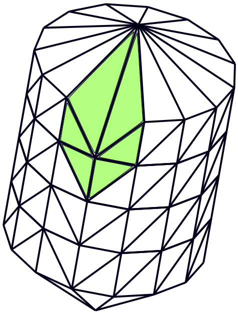

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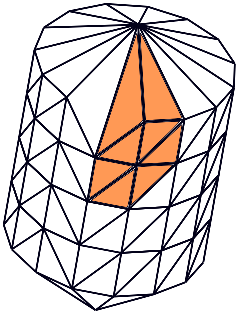

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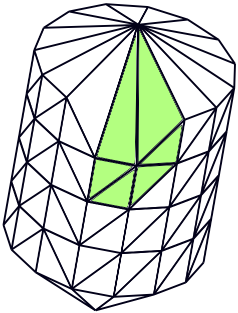

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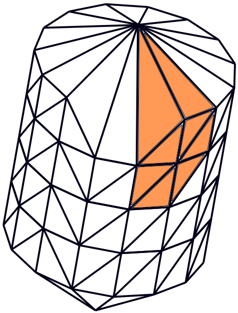

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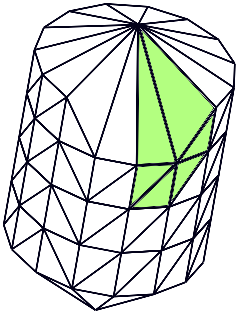

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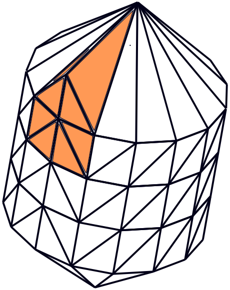

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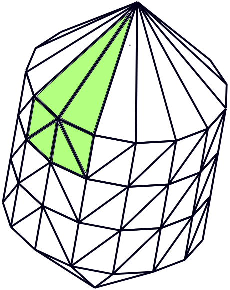

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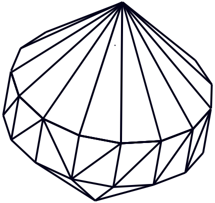

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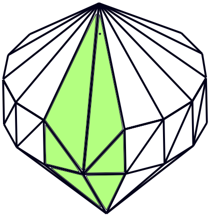

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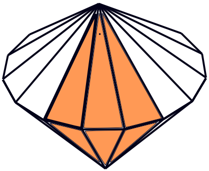

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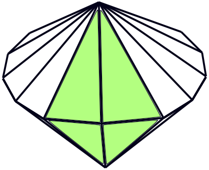

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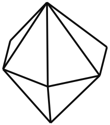

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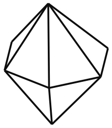

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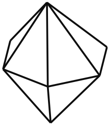

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However, there is a much simpler proof of this fact based on ideas by Meshulam and Kahle, using a version of the Nerve lemma.

Nerve lemma: Let K be a simplicial complex and $\{L_i\}_{i\in I}$ a family of subcomplexes covering K. Let $r \geq 0$. If each non-empty intersection $L_{i_1} \cap L_{i_2} \cap \ldots \cap L_{i_t}$ is $(r-t+1)$-connected for every $1 \leq t \leq r+1$, then K is r-connected if and only if the nerve $N(\{L_i\}_{i\in I})$ is r-connected.

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Proof of lemma. Let $L \leqslant S$ be a subcomplex of at most $r-t$ vertices. Then $\{v_1, v_2, \ldots, v_t\} \circledast L \leqslant K$ has at most r vertices and thus there exists $v \in K$ such that $\{v_1, v_2, \ldots, v_t\} \circledast L \in \mathrm{st}(v)$. This means that $v \circledast \{v_1, v_2, \ldots, v_t\} \circledast L \leqslant K$, so $v \circledast L \leqslant S$. Thus, $L \leqslant \mathrm{st}_S(v)$.

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During the last months we have been working to improve this bounds.

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Def: A complex K is (-1)-narrow if it is nonempty. For $r \geq 0$, K is r-narrow if for every $2 \leq t \leq r+2$ and $v_1, v_2, \ldots, v_t \in K$, $S = \bigcap \mathrm{st}(v_i) \leqslant K$ is $(r-t+1)$-narrow.

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Coro: A random complex is $\log_3 log_2(n)$-connected a.a.s.

Def: Let $r \geq -1$. We say that a complex K satisfies property T_r if for every $0 \leq k \leq r+1$, every $n_1, n_2, \ldots, n_k \geq 2$ such that $\sum_{i=1}^{k} n_i \leq r+k+1$ and 0-dimensional subcomplexes $Z_1, Z_2, \ldots, Z_k$ such that Z_i has n_i vertices and $\circledast_{i=1}^{k} Z_i \leqslant K$, we have that $\circledast_{i=1}^{k} Z_i$ is contained in the star of a vertex.

Thm: K is r-narrow if and only if it satisfies T_r.

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$$\mathrm{cat}(K) \leq \frac{\dim(K)}{\mathrm{conn}(K)+1} \leq \frac{\log_2 \log_2(n)}{\log_3 \log_2(n)} = \log_2(3) \sim 1.58$$

Thus, $\mathrm{cat}(K) \leq 1$ and $\mathrm{TC}(K) \leq 2$ a.a.s.

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Coro: K is homotopy equivalent to a suspension a.a.s.