Spectral Ranking Inferences Based on General Multiway Comparisons

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Zhipeng Lou, **Weichen Wang**, **Mengxin Yu**

Outlines

★Chen, E. Y., Xia, D., Cai, C. and Fan, J. (2024). Semiparametric tensor factor analysis by iteratively projected SVD. *Journal of Royal Statistical Society, B*, **86** (3), 793–823.

★Chen, E.Y. and Fan, J. (2023). Statistical inference for high-dimensional matrix-variate factor model. *Journal of American Statistical Association*, **118**, 1038-1055

1. Introduction
2. A Discrete Choice Model
3. Estimation and Uncertainty Quantification
4. Spectral Ranking and Inferences
5. Theorectical Justifications
6. Numerical studies and Conclusion

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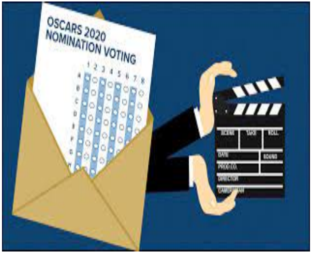
Introduction

Ranking Examples

★Ranking plays an important role in many applications: web search, voting, movie/music/book rating, recommendation systems, product designs, sports competitions, **refereeing**, **LLM** ...



Refereeing in Conference

- ★ In 2024, NeurIPS had 16,671 submissions. In 2023, ICML received 6,538 submissions from 18,535 authors.
- ★ Burden on the system, quality of reviews, indivudual noises.
- ★ $\approx$ half of the accepted papers in NeurIPS 2021 would be rejected upon a second round of reviews. *(Su, et al 2025+)*
- ★ Many referees reviewed multiple papers and therefore have a complete ranking among reviewed papers. ★allow partial ranking
- ★ Estimate the quality scores and its rank of each paper ★widson of crowd

Data: $\{(c_\ell, A_\ell)\}$ —top choice c_ℓ in the set A_ℓ

Comparison graph: Draw edge when compared.

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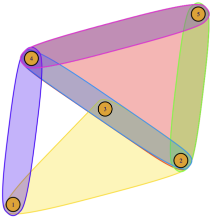

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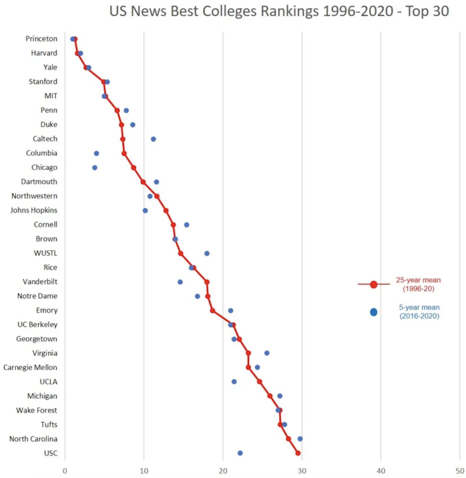
Preference scores and Intrisinc Ability

★Most current practical usage of ranks only involves estimating preference scores and displaying the estimated ranks.



Uncertainty of displayed ranks?

- ★ Is school A indeed better than school B?
- ★ Is school C indeed among top-20 rankings?
- ★ How many schools to apply to ensure the top 5 are selected?

Challenge: ★Limitted comparisons ★discrete parameters

★Involve all unknown scores

contribute to high-dim inference

Open Problems

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Related Literature

Ranking Estimation: Pairwise Comparison

- Rank centrality for Top-K recovery (Negahban et al., 2016).
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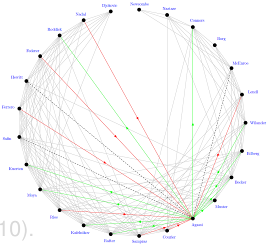

Ranking Estimation: M-way comparison

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- Fast estimation of PL models (Maystre et al., 2015).
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Ranking Inference: (Han et al, 2020; Liu et al,. 2022; Gao et al,.2022; Fan, et al., 2025+)

★Pairwise comparisons ★Not general enough ★Suboptimal for ranking infer.

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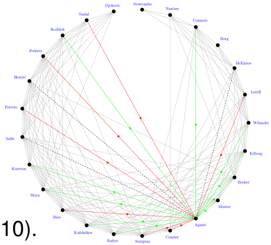

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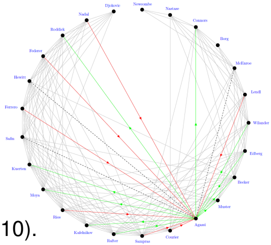

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A discrete choice model

under a general comparison graph

Model Settings

Preference scores: *n* items are associated preference scores

$$\theta^* = [\theta_1^*, \cdots, \theta_n^*]^\top, \qquad \theta_i^* \in [\theta_L, \theta_U], \; \forall i \in [n].$$

Data: $\{(c_\ell, A_\ell)\}$ —top choice c_ℓ in the set A_ℓ.

Multinomial outcomes: For each $(i_1, \cdots, i_M) \in A_\ell$, observe L indep. comparisons and obtain outcomes $\{(y_{i_1}^{(\ell)}, \cdots, y_{i_M}^{(\ell)})\}_{\ell=1}^{L}$ with winning prob (Luce's (1959) Choice Axiom)

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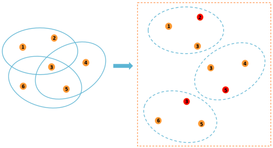
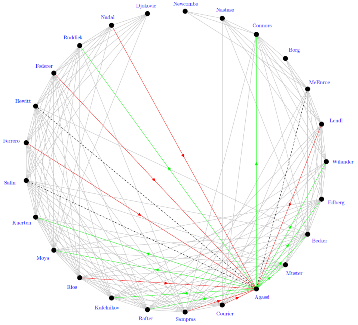


Learning Objectives

★Provide **estimation** and **uncertainty quantification** of $\{\theta_i^*\}_{i=1}^n$ via heterogeneous number of comparisons.



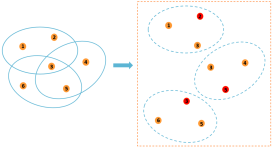
★Give ranking inferences

★$M = 2 \Longrightarrow$ Bradley-Terry-Luce (BTL) model

Versality of Models

Top-choice: General $M \geq 2$ and observe the top-choice *(Fan, et al, 25+)* $\Longrightarrow$ BTL

$$P(i_1 \succ \{i_2, \ldots, i_M\}) = \exp(\theta^*_{i_1}) / (\textstyle\sum_{k=1}^{M} \exp(\theta^*_{i_k}))\,.$$



Plakett-Luce: If compared, observe the full ranking for L independent times.

$$P(i_1 \succ i_2 \succ \cdots \succ i_M) = \prod_{j=1}^{M-1} \frac{\exp(\theta^*_{i_j})}{\sum_{k=j}^{M} \exp(\theta^*_{i_k})}$$

Take $A_1 = \{i_1, \cdots, i_M\}, \quad A_2 = \{i_2, \cdots, i_M\}, \cdots$

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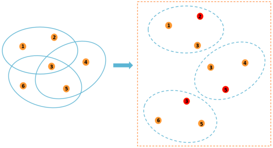

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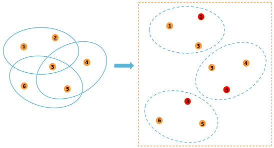

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Estimation and Uncertainty Quantification

Creation of A Transition Matrix

Markov chain (S, P): S contains n items with transition probability $\mathbf{P}$

$$P_{ij} = \begin{cases} \boxed{\frac{1}{d} \sum_{\ell \in \mathcal{W}_j \cap \mathcal{L}_i} \frac{1}{f(A_\ell)}}, & \text{if } i \neq j, \qquad \mathcal{W}_j = \{\ell \in \mathcal{D} | j \in A_\ell, c_\ell = j\}, \\ 1 - \sum_{k: k \neq i} P_{ik}, & \text{if } i = j. \qquad \mathcal{L}_i = \{\ell \in \mathcal{D} | i \in A_\ell, c_\ell \neq i\}. \end{cases}$$

- ★ $\mathcal{W}_j =$ winning instances for item j — $\mathcal{L}_i$ = losing instances for item i
- ★ $\mathcal{W}_j \cap \mathcal{L}_i =$ instances that j wins when i, j are compared.
- ★ $f(A_\ell) > 0$ is a weight, e.g. $f(A_\ell) = |A_\ell|$, **efficiency**.
- ★ d is chosen large enough to make the diagonals of P nonnegative.

★Faster than MLE

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Spectral Method

Spectral Estimation: Let $\widehat{\pi}$ be the stationary distribution, i.e. $\widehat{\pi}^\top P = \widehat{\pi}^\top$. Set

$$\tilde{\theta}_i := \log \widehat{\pi}_i - \frac{1}{n} \sum_{k=1}^n \log \widehat{\pi}_k \,.$$

Rationale: Conditioning on $\mathcal{G} = \{A_\ell | \ell \in \mathcal{D}\}$, the population transition matrix is

$$P^*_{ij} = E[P_{ij} | \mathcal{G}] = \frac{1}{d} \sum_{l \in \mathcal{D}} 1(i, j \in A_\ell) \frac{\exp(\theta^*_j)}{\sum_{u \in A_\ell} \exp(\theta^*_u)} \frac{1}{f(A_\ell)}, \qquad \text{if } i \neq j,$$

Then $\pi^* = (e^{\theta^*_1}, \ldots, e^{\theta^*_n}) / \sum_{k=1}^n e^{\theta^*_k}$ is stationary distribution of $\mathbf{P}^*$, since
Detailed balance: $P^*_{ij}\pi^*_i = P^*_{ji}\pi^*_j$.

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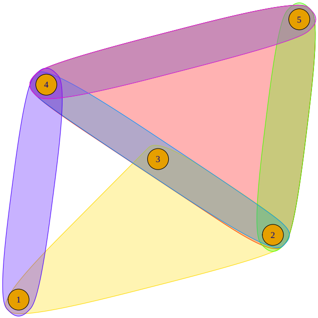
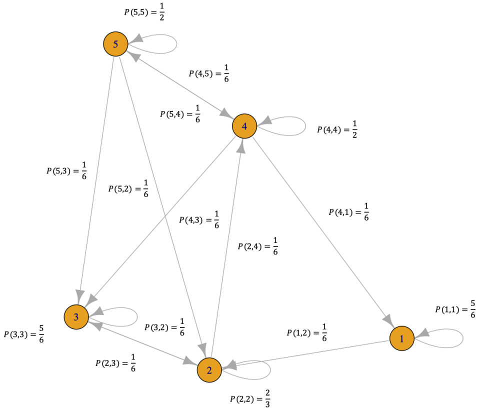
An illustration

Data: $(c_1, A_1) = (3, \{2,3,4,5\})$, $(c_2, A_2) = (2, \{1,2,3\})$, $(c_3, A_3) = (2, \{2,5\})$, $(c_4, A_4) = (4, \{4,5\})$, $(c_5, A_5) = (4, \{2,4\})$, $(c_6, A_6) = (1, \{1,4\})$, $(c_7, A_7) = (5, \{4,5\})$.





★A directed edge from i to j exists if i, j are compared and j wins

★$d = 6$ and $\widehat{\pi} = (0.199, 0.531, 0.796, 0.199, 0.066)^\top$. $3 \succ 2 \succ 1 = 4 \succ 5$.

Uncertainty Quantification in Estimation

1. Show $\widehat{\pi}_i = \frac{\sum_{j:j\neq i} P_{ji}\widehat{\pi}_j}{\sum_{j:j\neq i} P_{ij}} \approx \frac{\sum_{j:j\neq i} P_{ji}\pi_j^*}{\sum_{j:j\neq i} P_{ij}}$ leave-one-out

2. Show $\frac{\widehat{\pi}_i - \pi_i^*}{\pi_i^*} \approx \frac{\sum_{j:j\neq i}(P_{ji}\pi_j^* - P_{ij}\pi_i^*)}{\pi_i^* \sum_{j:j\neq i} P_{ij}} \approx \frac{\sum_{j:j\neq i}(P_{ji}e^{\theta_j^*} - P_{ij}e^{\theta_i^*})}{\sum_{j:j\neq i} E[P_{ij}|\mathcal{G}]e^{\theta_i^*}} =: J_i^*$ uniformly

3. Note that J_i^*

$$= \frac{\tau_i(\theta^*)}{d} \sum_{l\in\mathcal{D}} \frac{1(i\in A_l)}{f(A_l)} \{1(c_l = i) \sum_{u\in A_l, u\neq i} e^{\theta_u^*} - e^{\theta_i^*} 1(c_l \neq i)\}$$

4. variance can be analytically computed with optimal $f(A_l) \propto \sum_{u\in A_l} e^{\theta_u^*}$

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3. Note that $J_i^* =: \frac{1}{d}\sum_{l\in\mathcal{D}} J_{il}(\theta^*)$ ← sum of indep var.

$$= \frac{\tau_i(\theta^*)}{d} \sum_{l\in\mathcal{D}} \frac{1(i\in A_l)}{f(A_l)} \left\{ 1(c_l = i) \sum_{u\in A_l, u\neq i} e^{\theta_u^*} - e^{\theta_i^*} 1(c_l \neq i) \right\}$$

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Uncertainty Quantification in Estimation

1. Show $\widehat{\pi}_i = \frac{\sum_{j:j\neq i} P_{ji}\widehat{\pi}_j}{\sum_{j:j\neq i} P_{ij}} \approx \frac{\sum_{j:j\neq i} P_{ji}\pi_j^*}{\sum_{j:j\neq i} P_{ij}}$ — leave-one-out

2. Show $\frac{\widehat{\pi}_i - \pi_i^*}{\pi_i^*} \approx \frac{\sum_{j:j\neq i}(P_{ji}\pi_j^* - P_{ij}\pi_i^*)}{\pi_i^* \sum_{j:j\neq i} P_{ij}} \approx \frac{\sum_{j:j\neq i}(P_{ji}e^{\theta_j^*} - P_{ij}e^{\theta_i^*})}{\sum_{j:j\neq i} E[P_{ij}|\mathcal{G}]e^{\theta_i^*}} =: J_i^*$ — uniformly

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Spectral Ranking

under General Comparison Graphs

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★Are the top 2 ranked movies really statistically different?

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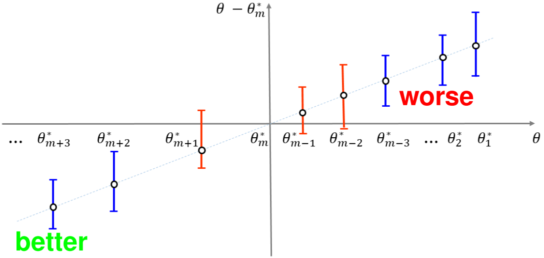
Basic Idea of Rank Inference

■ Let $\mathcal{M} = \{m\}$ be the item of interest and we have **simultaneous CI**

$$P\{\theta_k^* - \theta_m^* \in [C_L(k,m), C_U(k,m)], \forall k \neq m\} \geq 1 - \alpha$$



$(1-\alpha)$ **CI for r_m**: $\left[1 + \sum_{k \neq m} \mathbb{I}\{C_L(k,m) > 0\}, n - \sum_{k \neq m} \mathbb{I}\{C_U(k,m) < 0\}\right]$

Simultaneous pairwise comparisons

★ With $\{\tilde\sigma_{mk}\}$ is a uniform consistent SD, define

$$\mathcal{T}_{\mathcal{M}} = \max_{k\neq m}\left|\frac{\tilde\theta_k - \tilde\theta_m - (\theta_k^* - \theta_m^*)}{\tilde\sigma_{mk}}\right|.$$

★ Show that $\tilde\theta_k - \theta_k^* \approx J_i(\theta^*)$ so that $\tilde\sigma_{mk}$ can be computed.

★ Using $J_i(\tilde\theta) = \frac{1}{d}\sum_{l\in\mathcal{D}} J_{il}(\theta^*)$, Gaussian multiplier bootstrap of $\mathcal{T}_{\mathcal{M}}$ is

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High-confidence selection of top K items

Aim: To find $\widehat{I}_K$ such that $\mathbb{P}\left(\mathcal{K} \subset \widehat{I}_K\right) \geq 1-\alpha$.

Method: Let $\mathcal{M} = [n]$ and $\{[\mathcal{R}_m^\diamond, n]\}_{m\in[n]}$ be associated $(1-\alpha)$ simultaneous left-sided CIs. A natural and **valid** choice is

$$\widehat{I}_K = \{1 \leq m \leq n : \mathcal{R}_m^\diamond \leq K\}.$$

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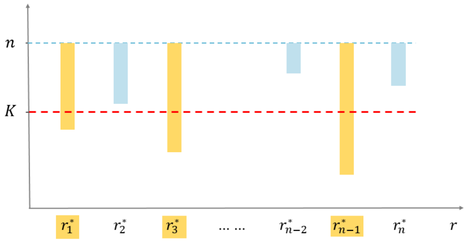

Two-Sample Rank Inference (I)

Two-sample rank change $H_0 : r_{1m} = r_{2m}$?

- ★ Rank changes of item m before and after a treatment or policy change.
- ★ Different communities e.g. males vs females have different preferences.
- ★ Preferences change in two time periods.

Test: Construct simul CI : $\mathbb{P}(r_{1m} \in [R_{1mL}, R_{1mU}] \text{ and } r_{2m} \in [R_{2mL}, R_{2mU}]) \geq 1 - \alpha$ and reject H_0 if $[R_{1mL}, R_{1mU}] \cap [R_{2mL}, R_{2mU}] = \emptyset$.

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Two-Sample Rank Inferences (II)

Two-sample top-K set change $H_0 : \mathcal{S}_{1K} = \mathcal{S}_{2K}$?

■ Test whether two top-K sets are identical or not, between two groups, two periods of time, or before and after a significant event or change.

Method: Construct $(1-\alpha)$ simultaneous confi. sets $\mathbb{P}\left(\mathcal{S}_{1K} \subset \widehat{\mathcal{I}}_{1K} \text{ and } \mathcal{S}_{2K} \subset \widehat{\mathcal{I}}_{2K}\right) \geq 1-\alpha$. Then the α-level test is

$$\phi_K = \mathbb{I}\{|\widehat{\mathcal{I}}_{1K} \cap \widehat{\mathcal{I}}_{2K}| < K\}.$$

Theorectical Justifications

Assumptions

Assumption 1: Graph is **connected** and $n^{\ddagger} n^{1/2} (\log n)^{1/2} = o(n^{\dagger})$.

$$n^{\dagger} := \max_i \sum_{\ell \in \mathcal{D}} 1(i \in A_\ell), \qquad n^{\ddagger} := \max_{i \neq j} \sum_{\ell \in \mathcal{D}} 1(i, j \in A_\ell).$$

Assumption 2: Define $\Omega = (\Omega_{ij})$ where $\Omega_{ij} = -P_{ji}\pi_j^*$ and $\Omega_{ii} = \sum_{j: j \neq i} P_{ij}\pi_i^*$.

$$C_1 \frac{n^{\dagger}}{dn} \leq \lambda_{\min,\perp}(E[\Omega | \mathcal{G}]) \leq \lambda_{\max}(E[\Omega | \mathcal{G}]) \leq C_2 \frac{n^{\dagger}}{dn},$$

$$\|\Omega - E[\Omega | \mathcal{G}]\| = o_P\left(\frac{n^{\dagger}}{dn}\right).$$

■ If each pair is compared for at least one time, then $n^{\ddagger} \asymp 1, n^{\dagger} \asymp n$ and both assumptions hold.

Theoretical Justification of Spectral Estimator

Theorem 1 (Uniform Approximation of Spectral Estimator)

It holds $\tilde{\theta}_i - \theta_i^* = J_i^* + o_P(1/\sqrt{n^\dagger})$, uniformly for all $i \in [n]$, where

$$J_i^* := \frac{\sum_{j:j\neq i}(P_{ji}e^{\theta_j^*} - P_{ij}e^{\theta_i^*})}{\sum_{j:j\neq i} E[P_{ij}|\mathcal{G}]e^{\theta_i^*}}.$$

This means

1. $\|\tilde{\theta} - \theta^*\|_\infty \asymp \|J^*\|_\infty \lesssim \sqrt{\frac{\log n}{n^\dagger}}$, with probability $1 - o(1)$.
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★ For the PL model w/ homogenous random graph, Assumption 1 can be relaxed to $n^{\dagger} \gtrsim n^{\ddagger} \log n$ and Assumption 2 holds.

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★ Two-step approach: •obtain an estimator $\tilde{\theta}^{(\text{init})}$ with $f(A_\ell) = |A_\ell|$. (**vanilla**)
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★ For the PL model, we break $i \succ j \succ k$ into indep. events $(c_\ell = i, A_\ell = \{i, j, k\})$ and $(c_\ell = j, A_\ell = \{j, k\})$ given $\mathcal{G}$. Run spectral ranking.

★ Assumption 2 holds with probability $1 - o(1)$ under the PL model.

★ The graph is connected iff $\binom{n-1}{M-1} p \gtrsim \log n$ (*Cooley et al., 16*). Under the PL model, when $L \asymp 1$, we can prove

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Under the PL model with $p \gtrsim \mathrm{poly}(\log n)/\binom{n-1}{M-1}$, the spectral estimator $\tilde{\theta}_i$ has the uniform approximation: $\tilde{\theta}_i - \theta_i^* = J_i^* + o_P(1/\sqrt{n^\dagger})$, uniformly for all i. This implies

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Theorem 3 (Gaussian Multiplier Bootstrap). Let $Q_{1-\alpha}$ be $(1-\alpha)$-th quantile of $G_{\mathcal{M}}$.

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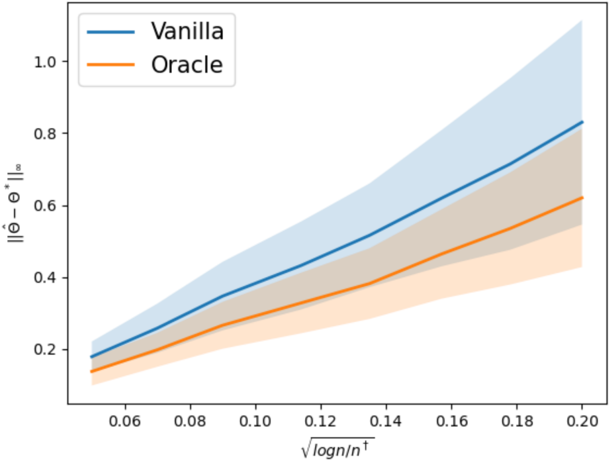
Simulations and Empirical Applications

Simulation models and Rates

- $n = 50$ with θ_i^* evenly distributed on $[-2, 2]$
- Heterogeneous comparisons among $\{2, 3, 4, 5\}$ items



Coverages of Confidence Intervals

		Vanilla Two-Sided CI			**Oracle** Two-Sided CI		
	$\|\mathcal{D}\|$	EC(θ)	EC(r)	Length	EC(θ)	EC(r)	Length
θ_8^*	$\|\mathcal{D}\| = 12000$	0.954	1.000	6.384	0.954	1.000	6.298
	$\|\mathcal{D}\| = 24000$	0.950	1.000	4.092	0.968	1.000	4.090
	$\|\mathcal{D}\| = 36000$	0.956	1.000	3.008	0.954	1.000	2.928
θ_{20}^*	$\|\mathcal{D}\| = 12000$	0.952	1.000	**11.602**	0.960	1.000	**10.082**
	$\|\mathcal{D}\| = 24000$	0.958	1.000	**7.450**	0.952	1.000	**6.524**
	$\|\mathcal{D}\| = 36000$	0.954	1.000	**5.788**	0.958	1.000	**5.068**
θ_{30}^*	$\|\mathcal{D}\| = 12000$	0.950	1.000	17.502	0.962	1.000	14.072
	$\|\mathcal{D}\| = 24000$	0.952	1.000	11.620	0.960	1.000	9.528
	$\|\mathcal{D}\| = 36000$	0.956	1.000	9.262	0.958	1.000	7.748

★also verified •one-side CI •top-K CS •two-sample inferences

Two-step versus MLE

	Estimator	$p = 0.02$	$p = 0.05$	$p = 0.08$	$p = 0.11$	$p = 0.14$
ℓ_2	**Vanilla**	1.092 (0.140)	0.688 (0.086)	0.543 (0.061)	0.301 (0.052)	0.181 (0.047)
	Oracle	0.902 (0.102)	0.561 (0.061)	0.447 (0.043)	0.248 (0.040)	0.150 (0.037)
	Two Step	0.906 (0.103)	0.562 (0.061)	0.447 (0.043)	0.248 (0.040)	0.150 (0.037)
	MLE	0.902 (1.102)	0.562 (0.061)	0.447 (0.043)	0.248 (0.040)	0.150 (0.037)
	Two Step − MLE	0.046 (0.012)	0.018 (0.004)	0.011 (0.002)	0.008 (0.002)	0.006 (0.001)
ℓ_∞	**Vanilla**	0.427 (0.081)	0.259 (0.059)	0.206 (0.041)	0.116 (0.039)	0.070 (0.037)
	Oracle	0.338 (0.063)	0.204 (0.034)	0.162 (0.030)	0.091 (0.027)	0.054 (0.022)
	Two Step	0.337 (0.063)	0.204 (0.034)	0.162 (0.030)	0.091 (0.027)	0.054 (0.022)
	MLE	0.337 (0.063)	0.204 (0.034)	0.162 (0.030)	0.091 (0.027)	0.054 (0.022)
	Two Step − MLE	0.021 (0.007)	0.008 (0.002)	0.005 (0.002)	0.003 (0.001)	0.002 (0.001)

★Two-step spectral method and MLE have **very similar performance** in terms of ℓ_2-norm and ℓ_∞-norm.

Ranking of Statistics Journals

Data: Multi-Attribute Dataset on Statisticians (MADStat) containing citation information from 83,331 papers published in 36 journals during 1975-2015 (Ji et al., 23).

Comparisons: Journal A ranks higher than Journal B by a paper in year Y $\Longleftrightarrow$ a paper published in Journal B in year Y cited another paper published in Journal A between the years $Y - 10$ and Y.

Two-sample testing: We compare journal rankings using papers published in 2006-2010 vs 2011-2015.

Ranking of Statistics Journals

	2006 – 2010						2011 – 2015					
Journal	$\widetilde{\theta}$	$\widetilde{r}$	TCI	OCI	UOCI	Count	$\widetilde{\theta}$	$\widetilde{r}$	TCI	OCI	UOCI	Count
JRSSB	1.654	1	$[1,1]$	$[1,n]$	$[1,n]$	5282	1.553	1	$[1,2]$	$[1,n]$	$[1,n]$	5513
AoS	1.206	3	$[2,4]$	$[2,n]$	$[2,n]$	7674	1.522	2	$[1,2]$	$[1,n]$	$[1,n]$	11316
Bka	1.316	2	$[2,3]$	$[2,n]$	$[2,n]$	5579	1.202	3	$[3,3]$	$[3,n]$	$[3,n]$	6399
JASA	1.165	4	$[3,4]$	$[3,n]$	$[3,n]$	9652	1.064	4	$[4,4]$	$[4,n]$	$[4,n]$	10862
JMLR	−0.053	20	$[14,25]$	$[15,n]$	$[13,n]$	1100	0.721	5	$[5,7]$	$[5,n]$	$[5,n]$	2551
Biost	0.288	13	$[10,18]$	$[10,n]$	$[9,n]$	2175	0.591	6	$[5,9]$	$[5,n]$	$[5,n]$	2727
Bcs	0.820	5	$[5,7]$	$[5,n]$	$[5,n]$	6614	0.571	7	$[5,9]$	$[6,n]$	$[5,n]$	6450
StSci	0.668	7	$[5,9]$	$[5,n]$	$[5,n]$	1796	0.437	8	$[6,13]$	$[6,n]$	$[6,n]$	2461
Sini	0.416	10	$[9,14]$	$[9,n]$	$[8,n]$	3701	0.374	9	$[8,13]$	$[8,n]$	$[8,n]$	4915
JRSSA	0.239	14	$[10,20]$	$[10,n]$	$[9,n]$	893	0.370	10	$[6,13]$	$[8,n]$	$[6,n]$	865
JCGS	0.605	8	$[6,9]$	$[6,n]$	$[6,n]$	2493	0.338	11	$[8,13]$	$[8,n]$	$[8,n]$	3105
Bern	0.793	6	$[5,8]$	$[5,n]$	$[5,n]$	1575	0.336	12	$[8,13]$	$[8,n]$	$[8,n]$	2613
ScaJS	0.528	9	$[7,12]$	$[7,n]$	$[6,n]$	2442	0.258	13	$[8,13]$	$[9,n]$	$[8,n]$	2573
JRSSC	0.113	15	$[11,22]$	$[11,n]$	$[11,n]$	1401	0.020	14	$[14,19]$	$[14,n]$	$[12,n]$	1492
AoAS	−1.463	30	$[30,33]$	$[30,n]$	$[30,n]$	1258	−0.017	15	$[14,20]$	$[14,n]$	$[14,n]$	3768
CanJS	0.101	17	$[11,22]$	$[11,n]$	$[11,n]$	1694	−0.033	16	$[14,20]$	$[14,n]$	$[14,n]$	1702

- Results are based on two-step spectral estimator.

Ranking of Statistics Journals

★ Is each journal's rank changed significantly? At significance level 10%, the following journals demonstrate significant differences:

AISM, AoAS, Biost, CSTM, EJS, JMLR, JoAS, JSPI.

★ Big-Four journals (AoS, Bka, JASA, and JRSSB) maintain their positions strongly.

★ Are the top-7 ranked journals remain unchanged? We reject. For 2006-2010, the 95% confidence set for the top-7 journals includes:

AoS, Bern, Bcs, Bka, JASA, JCGS JRSSB, ScaJS, StSci.

However, for 2011-2015, the 95% confidence set for the top-7 items includes:

AoS, Bcs, Biost, Bka, JASA, JMLR, JRSSA, JRSSB, StSci.

They only intersect at 6 items < 7, so we reject at $\alpha = 0.1$.

Ranking of Movies

Data: 100 random 3 and 4 candidate elections drawn from the Netflix Prize dataset

Movie	$\tilde{\theta}$	$\tilde{r}$	TCI	OCI	UOCI	Count
The Silence of the Lambs	3.002	1	$[1,1]$	$[1,n]$	$[1,n]$	19589
The Green Mile	2.649	2	$[2,4]$	$[2,n]$	$[2,n]$	5391
Shrek (Full-screen)	2.626	3	$[2,4]$	$[2,n]$	$[2,n]$	19447
The X-Files: Season 2	2.524	4	$[2,7]$	$[2,n]$	$[2,n]$	1114
Ray	2.426	5	$[4,7]$	$[4,n]$	$[4,n]$	7905
The X-Files: Season 3	2.357	6	$[4,10]$	$[4,n]$	$[2,n]$	1442
The West Wing: Season 1	2.278	7	$[4,10]$	$[4,n]$	$[4,n]$	3263
National Lampoon's Animal House	2.196	8	$[6,10]$	$[6,n]$	$[5,n]$	10074
Aladdin: Platinum Edition	2.154	9	$[6,13]$	$[6,n]$	$[5,n]$	3355
Seven	2.143	10	$[6,11]$	$[7,n]$	$[6,n]$	16305
Back to the Future	2.030	11	$[9,15]$	$[9,n]$	$[8,n]$	6428
Blade Runner	1.968	12	$[10,16]$	$[10,n]$	$[9,n]$	5597
Harry Potter and the Sorcerer's Stone	1.842	13	$[12,22]$	$[12,n]$	$[11,n]$	7976
High Noon	1.821	14	$[11,25]$	$[11,n]$	$[10,n]$	1902
Sex and the City: Season 6: Part 2	1.770	15	$[11,30]$	$[11,n]$	$[8,n]$	532
Jaws	1.749	16	$[13,25]$	$[13,n]$	$[13,n]$	8383
The Ten Commandments	1.735	17	$[13,28]$	$[13,n]$	$[12,n]$	2186
Willy Wonka & the Chocolate Factory	1.714	18	$[13,26]$	$[13,n]$	$[13,n]$	9188
Stalag 17	1.697	19	$[12,34]$	$[12,n]$	$[11,n]$	806
Unforgiven	1.633	20	$[14,29]$	$[14,n]$	$[14,n]$	9422

Concluding Remarks

- ★ Propose a spectral method for a discrete choice model (axiom of choice).
- ★ Allow general fixed comp. graph with relaxed conditions (varying M and $L = 1$).
  - •BTL model •PL model •Top choice model
- ★ Establish ℓ_∞-rate and the asymptotic normality based on uniform approx. With the optimal weighting, spectral estimator $\approx$ MLE under PL model.
- ★ Propose a multipler bootsrap and demonstrate it validity.
- ★ Add two-sample inference tools to the ranking inference framework.
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The End

Thank

You

—Fan, J., Lou, Z., Wang, W., and Yu, M. (2025+). Spectral Ranking Inferences based on General Multiway Comparisons. *Operations Research*, to appear.