Detecting Toricness via Lie Algebras

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based on work with **Arpan Pal** (University of Idaho)
arxiv: *Symmetry Lie algebras of varieties with applications to algebraic statistics*
and
Thomas Kahle and Julian Vill, arxiv: *Efficiently deciding if an ideal is toric after a linear coordinate change*

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toric structures

$\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ a polynomial ring.

Ideal $I \subseteq \mathbb{C}[x]$ is toric if
(a) I is prime and binomial ideal, or equivalently,
(b) I is the kernel of a monomial map $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[\theta_1^{\pm 1}, \ldots, \theta_m^{\pm 1}]$.

$$\psi : \mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[\theta_1, \theta_2], \; x_1 \mapsto \theta_1^2, \; x_2 \mapsto \theta_1\theta_2, \; x_3 \mapsto \theta_2^2$$
$$\ker \psi = \langle x_1 x_3 - x_2^2 \rangle$$

$V(I) = \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for } f \in I\}$ the variety of I.

Toric varieties are isomorphic to solutions sets of toric ideals.

An ideal I may not be toric and its $V(I)$ may be toric. A linear change of variables is useful for reveling the toric structure of $V(I)$.

Take $I = \langle x_1 x_3 - x_2^2 - x_1 x_2 \rangle$ and $x_1 = p_1, \; x_2 = p_2 - p_1, \; x_3 = p_3 - p_2$. Then,

$$I = \langle p_1(p_3 - p_2) - (p_2 - p_1)^2 - p_1(p_2 - p_1) \rangle = \langle p_1 p_3 - p_2^2 \rangle \text{ is toric in } \mathbb{C}[p_1, \ldots, p_3]$$

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$$I = \langle p_1(p_3 - p_2) - (p_2 - p_1)^2 - p_1(p_2 - p_1) \rangle = \langle p_1p_3 - p_2^2 \rangle \text{ is toric in } \mathbb{C}[p_1, \ldots, p_3]$$

PROBLEM: Suppose I is not a binomial ideal. Determine when there is no invertible linear transformation that turns I into a toric ideal.

In fact, the converse to this proposition also holds, with a slightly more inclusive definition of toric variety. Informally speaking, toric varieties are precisely those varieties that become linear spaces under taking logarithms.

8.3. The World is Toric

The occurrence of toric structures in an application can be either obvious or hidden. A typical example for the former is log-linear models in statistics. These are obviously toric, as seen around Example 8.26. In this section we discuss some scenarios where the toric structure is hidden, and it needs to be unearthed, often by a non-trivial chance of coordinates. Our style in this section is extremely informal. We briefly visit four fields where toric varieties arise. Under each header we focus on one concrete instance of a toric variety $X_A \subset \mathbb{P}^{p-1}$. The broader context is discussed alongside that example.

Mateusz Michałek, and Bernd Sturmfels. "Invitation to nonlinear algebra." American Mathematical Soc., (2021).

Toric Ideals of Phylogenetic Invariants

Bernd Sturmfels and Seth Sullivant
Department of Mathematics, University of California, Berkeley

Abstract

Statistical models of evolution are algebraic varieties in the space of joint probability distributions on the leaf colorations of a phylogenetic tree. The phylogenetic invariants of a model are the polynomials which vanish on the variety. Several widely used models for biological sequences have transition matrices that can be diagonalized by means of the Fourier transform of an abelian group. Their phylogenetic invariants form a toric ideal in the Fourier coordinates. We determine generators and Gröbner bases for these toric ideals. For the Jukes-Cantor and Kimura models on a binary tree, our Gröbner bases consist of certain explicitly constructed polynomials of degree at most four.

ON THE TORIC ALGEBRA OF GRAPHICAL MODELS

BY DAN GEIGER, CHRISTOPHER MEEK AND BERND STURMFELS[1]
Technion–Haifa, Microsoft Research and University of California, Berkeley

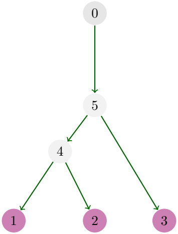
We formulate necessary and sufficient conditions for an arbitrary discrete probability distribution to factor according to an undirected graphical model, or a log-linear model, or other more general exponential models. For decomposable graphical models these conditions are equivalent to a set of conditional independence statements similar to the Hammersley–Clifford theorem; however, we show that for nondecomposable graphical models they are not. We also show that nondecomposable models can have nonrational maximum likelihood estimates. These results are used to give several novel characterizations of decomposable graphical models.

good story at IMSI last year and follow up

BROWNIAN MOTION TREE MODELS ARE TORIC

BERND STURMFELS, CAROLINE UHLER, AND PIOTR ZWIERNIK

Abstract. Felsenstein's classical model for Gaussian distributions on a phylogenetic tree is shown to be a toric variety in the space of concentration matrices. We present an exact semialgebraic characterization of this model, and we demonstrate how the toric structure leads to exact methods for maximum likelihood estimation. Our results also give new insights into the geometry of ultrametric matrices.



$$\mathcal{L}_T = \{\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{13} \\ \sigma_{13} & \sigma_{13} & \sigma_{33} \end{bmatrix} \mid \sigma_{11}, \sigma_{12} \cdots, \sigma_{33} \in \mathbb{R}\}$$

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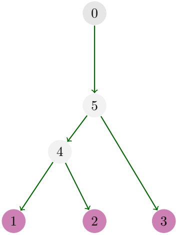


$$\mathcal{L}_T = \{\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{13} \\ \sigma_{13} & \sigma_{13} & \sigma_{33} \end{bmatrix} \mid \sigma_{11}, \sigma_{12} \cdots, \sigma_{33} \in \mathbb{R}\}$$

$$\mathcal{L}_T^{-1} = \overline{\{K \in \mathrm{Sym}_n | K^{-1} \in \mathcal{L}_T\}} = \{\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3} \mid k_{12}k_{23} - k_{13}k_{22} = -(k_{11}k_{23} - k_{12}k_{13})\}$$

BMT models are under the reduced graph Laplacian transformation with monomial parametrization given by shortest paths between leaves in the tree.

1. Tobias Boege, Jane Ivy Coons, Chris Eur, Aida Maraj, Frank Röttger, **Reciprocal Maximum Likelihood Degrees of Brownian Motion Tree Models**, Le Matematiche 76 (2), 383-398 (2021)

2. Jane Ivy Coons, Shelby Cox, Aida Maraj, Ikenna Nometa, **Maximum Likelihood Degrees of Brownian Motion Tree Models: Star Tree and Root Invariance**, arxiv:2402.10322 (2024)

4. Amer Goel, Aida Maraj, Alvaro Ribot, **Halfspace Representations of Path Polytopes of Trees**, arxiv:2309.10741 (2025)

3. Emma Cardwell, Aida Maraj, Alvaro Ribot, **Toric Multivariate Gaussian Models from Symmetries in a Tree**, arxiv:2412.00895 (2024)

symmetry Lie groups of ideals

Determine the group action of $\mathrm{GL}_n(\mathbb{C})$ in $\mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$ by

$$\text{for } g = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \in \mathrm{GL}_n(\mathbb{C}), f \in \mathbb{C}[x], \quad g \cdot f(x) = f(g^{-1}x).$$

$G_I = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid g \cdot f \in I, \forall f \in I\}$ is the stabilizer of I.

Example:

$I = \langle f \rangle \subset \mathbb{C}[x_1, x_2]$ where $f = x_1^2 + x_2^2 + x_1x_2$. Take $g^{-1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$.

$$\begin{aligned} g \cdot f =& (g_{11}x_1 + g_{12}x_2)^2 - (g_{21}x_1 + g_{22}x_2)^2 - (g_{11}x_1 + g_{12}x_2)(g_{11}x_1 + g_{12}x_2) \\ =&(g_{11}^2 + g_{21}^2 + g_{11}g_{21})x_1^2 + (g_{12}^2 + g_{22}^2 + g_{12}g_{22})x_2^2 \\ &+ (2g_{11}g_{12} + 2g_{21}g_{22} + g_{11}g_{22} + g_{12}g_{21})x_1x_2 \end{aligned}$$

$G_I = \langle g \in GL_n(\mathbb{C}) \mid g_{11}^2 + g_{21}^2 + g_{11}g_{21} = g_{12}^2 + g_{22}^2 + g_{12}g_{22} = 2g_{11}g_{12} + 2g_{21}g_{22} + g_{11}g_{22} + g_{12}g_{21} \rangle$

$\dim(G_I) = 2$

detecting non-toricness via Lie groups

$G_I = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid g \cdot f \in I, \forall f \in I\}$ is the stabilizer of I.

> Theorem: Let $I \subseteq \mathbb{C}[x]$ be a prime homogeneous ideal with $\dim(V(I)) = r$. If $\dim(G_I) < r$ then there is **no** linear change of variables that turns I into a toric ideal.

Proof idea:

1. G_I is a Lie group as a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ (Cartan's theorem)

2. If I is toric, then the torus $\mathbb{T}^r$ acting on $V(I)$ is embedded in G_I ($\mathrm{diag}(\mathbb{T}^r) \subset G_I$). So, $r \leq \dim(G_I)$

2′ (contrapositive) If $\dim(G_I) < \dim(V(I))$, then I cannot be turned toric under any linear change of variables

some symmetry Lie groups are investigated in: Fulvio Gesmundo, Young Han, Benjamin Lovitz, *Linear Preservers of Secant Varieties and Other Varieties of Tensors*, 2024

Project initiated in conversations with JM Landsberg at 2022 Texas Algebraic Geometry Symposium at Texas A&M

symmetry Lie algebras of ideals

—where there is a Lie group, there is a Lie algebra (with same dimension)

The Lie algebra for G_I is

$$\mathfrak{g}_I = \{g \in M_n(\mathbb{C}) \mid e^{tg} \in G_I, \forall t \in \mathbb{R}\} = \{g \in M_n(\mathbb{C}) \mid g * f(x) \in I, \ \forall f(x) \in I\},$$

where $g * f(x) := \frac{d}{dt}(e^{gt} \cdot f(x))|_{t=0}$.

Proposition: The $*$ operation on $\mathbb{C}[x]$ is fully determined by the rules:

1. $g * c = 0$ for any constant $c \in \mathbb{C}[x]$,
2. $g * x_i = -\sum_{j=1}^{n} g_{ij} \cdot x_j$ for any variable $x_i \in \mathbb{C}[x]$,
3. $g * (p_1 p_2) = (g * p_1)p_2 + p_1(g * p_2)$, for any $p_1, p_2 \in \mathbb{C}[x]$,

extended linearly to $\mathbb{C}[x]$.

$$\begin{aligned} g * (x_1^2 + x_2^2 + x_1x_2) =& g * x_1^2 + g * x_2^2 + g * x_1x_2 \\ =& -2x_1(g_{11}x_1 + g_{12}x_2) - 2x_2(g_{21}x_1 + g_{22}x_2) \\ & - (g_{11}x_1 + g_{12}x_2)x_2 - x_1(g_{21}x_1 + g_{22}x_2) \\ =& -(2g_{11} + g_{21})x_1^2 - (2g_{12} + 2g_{21} + g_{11} + g_{22})x_1x_2 - (2g_{22} + g_{12})x_2^2 \end{aligned}$$

$$\mathfrak{g}_I = \{g \in M_2(\mathbb{C}) \mid 2g_{11} + g_{21} = 2g_{12} + 2g_{21} + g_{11} + g_{22} = 2g_{22} + g_{12}\}$$

Theorem: Let $I = \langle f_1, \ldots, f_k \rangle \subseteq \mathbb{C}[x]$ be a homogeneous prime ideal. Then,

$$\mathfrak{g}_I = \{g \in M_n(\mathbb{C}) \mid g * f_i \in I, \ \text{for } i = 1, \ldots, k\}.$$

an algorithm for computing symmetry Lie algebras

Observation: Suppose $f(x)$ is of degree d. Then both $g \cdot f(x)$ and $g * f(x)$ are also polynomials of degree d.

- $[\mathbb{C}[x]]_d$ = homogeneous polynomials of degree d in $\mathbb{C}[x]$
- $\mathrm{Mon}([\mathbb{C}[x]]_d)$ = monomials of degree d span $[\mathbb{C}[x]]_d$
- $\vec{f}$ = the vector representation of f with respect to $\mathrm{Mon}([\mathbb{C}[x]]_d)$

$$f = x_1^2 - 3x_1x_2 + x_3^2 \in [I]_2 = \mathrm{span}(x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2), \qquad \vec{f} = \begin{bmatrix} 1 & -3 & 0 & 0 & 1 \end{bmatrix}^T$$

Theorem: Let $I \subseteq \mathbb{C}[x]$ be a homogeneous prime ideal generated by polynomials of degree at most d. Let $\mathscr{B}([I]_d)$ be a basis for $[I]_d$. For each $f_i \in \mathscr{B}([I]_d)$ consider the matrix

$$M_i(g) := \begin{bmatrix} \overrightarrow{f_1} & \overrightarrow{f_2} & \dots & \overrightarrow{f_k} & \overrightarrow{g * f_i} \end{bmatrix}.$$

Then $\mathfrak{g}_I = \{g \in M_n(\mathbb{C}) \mid \mathrm{rank}(M_i(g)) = k \text{ for } f_i \in \mathscr{B}([I]_d)\}$.

examples

$$I = \langle x_1^2 + x_2^2 + x_1 x_2 \rangle$$

$$\begin{aligned} g * (x_1^2 + x_2^2 + x_1x_2) =& g * x_1^2 + g * x_2^2 + g * x_1x_2 \\ =& -2x_1(g_{11}x_1 + g_{12}x_2) - 2x_2(g_{21}x_1 + g_{22}x_2) \\ & -(g_{11}x_1 + g_{12}x_2)x_2 - x_1(g_{21}x_1 + g_{22}x_2) \\ =& -(2g_{11} + g_{21})x_1^2 - (2g_{12} + 2g_{21} + g_{11} + g_{22})x_1x_2 - (2g_{22} + g_{12})x_2^2 \end{aligned}$$

$$M_1(g) = \begin{bmatrix} 1 & 2g_{11} + g_{21} \\ 1 & 2g_{12} + 2g_{21} + g_{11} + g_{22} \\ 1 & 2g_{22} + g_{12} \end{bmatrix}$$

In [ ]:

```
R = PolynomialRing(QQ,['x', 'y','z'])
R.inject_variables()
LieI = symmalg([x^2+y^2+z^2,x+y],3)
```

3

Out[ ]:

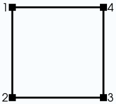
```
Defining x, y, z
Defining x, y, z
Defining x, y, z, g11, g12, g13, g21, g22, g23, g31, g32, g33

A basis of the Lie algebra consists of the following matrices:
```

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \right]$$

non-toric structures in algebraic statistics

The following is the first Gaussian graphical model proven to be non-toric.



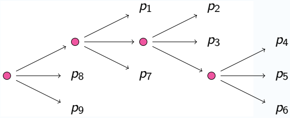
$$\sigma_{23}\sigma_{14}\sigma_{24} - \sigma_{13}\sigma_{24}^2 - \sigma_{22}\sigma_{14}\sigma_{34} + \sigma_{12}\sigma_{24}\sigma_{34} + \sigma_{22}\sigma_{13}\sigma_{44} - \sigma_{12}\sigma_{23}\sigma_{44},$$

$$\sigma_{13}\sigma_{23}\sigma_{14} - \sigma_{24}\sigma_{13}^2 - \sigma_{12}\sigma_{33}\sigma_{14} + \sigma_{11}\sigma_{33}\sigma_{24} + \sigma_{12}\sigma_{13}\sigma_{34} - \sigma_{11}\sigma_{23}\sigma_{34}$$

$$\dim(G_I) = 4 < 8 = \dim(V(I)).$$

compare with: Jane Coons, Aida Maraj, Pratik Misra, Stefana Sorea, *Symmetrically colored Gaussian graphical models with toric vanishing ideals*. SIAM Journal of Applied Algebra and Geometry (2023)

The following is the first staged tree model with one stage proven to be non-toric.



$$P = \begin{bmatrix} p_1 + \ldots + p_7 & p_1 & p_3 & p_4 \\ p_8 & p_2 + \ldots + p_6 & p_3 & p_5 \\ p_9 & p_7 & p_4 + p_5 + p_6 & p_6 \end{bmatrix}$$

$I = \langle 2 \times 2 \text{ minors of } P \rangle$. $\dim(G_I) = 2 < 3 = \dim(V(I))$.

compare with: Christiane Görgen, Aida Maraj, Lisa Nicklasson, *staged tree models with toric structures*. Journal of Symbolic Computation (2022)

principal ideals defined by cycles

work with student Joan Pascual Ribes.

$$I_{n,m} = \langle \sum_{i=1}^{n} \prod_{j=i}^{i+m-1} x_{j \bmod n} \rangle$$

$$I_{5,2} = \langle x_1x_2 + x_2x_3 + x_3x_4 + x_4x_4 + x_5x_1 \rangle$$
$$I_{5,3} = \langle x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 \rangle$$

$n\backslash m$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
2	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
3	4	3	1	1	3	1	1	3	1	1	3	1	1	3	1	1	3	1	1
4	10	1	4	1	2	1	4	1	2	1	4	1	2	1	4	1	2	1	4
5	11	1	1	5	1	1	1	1	5	1	1	1	1	5	1	1	1	1	5
6	16	5	2	1	6	1	2	3	2	1	6	1	2	3	2	1	6	1	2
7	22	1	1	1	1	7	1	1	1	1	1	1	7	1	1	1	1	1	1
8	32	1	5	1	2	1	8	1	2	1	4	1	2	1	8	1	2	1	4
9	37	4	1	1	3	1	1	9	1	1	3	1	1	3	1	1	9	1	1
10	46	1	2	5	2	1	2	1	10	1	2	1	2	5	2	1	2	1	10
11	56	1	1	1	1	1	1	1	1	11	1	1	1	1	1	1	1	1	1
12	70	3	4	1	6	1	4	3	2	1	12	1	2	3	4	1	6	1	4
13	79	1	1	1	1	1	1	1	1	1	1	13	1	1	1	1	1	1	1
14	92	1	2	1	2	7	2	1	2	1	2	1	14	1	2	1	2	1	2
15	106	3	1	5	3	1	1	3	5	1	3	1	1	15	1	1	3	1	5
16	124	1	4	1	2	1	8	1	2	1	4	1	2	1	16	1	2	1	4
17	137	1	1	1	1	1	1	1	1	1	1	1	1	1	1	17	1	1	1
18	154	3	2	1	6	1	2	9	2	1	6	1	2	3	2	1	18	1	2
19	172	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	19	1
20	194	1	4	5	2	1	4	1	10	1	4	1	2	5	4	1	2	1	20

Table: $\dim(\mathfrak{g}_{I_{n,m}})$

principal ideals defined by cycles

work with student Joan Pascual Ribes.

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$n\backslash m$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
2	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
3	4	3	1	1	3	1	1	3	1	1	3	1	1	3	1	1	3	1	1
4	10	1	4	1	2	1	4	1	2	1	4	1	2	1	4	1	2	1	4
5	11	1	1	5	1	1	1	1	5	1	1	1	1	5	1	1	1	1	5
6	16	5	2	1	6	1	2	3	2	1	6	1	2	3	2	1	6	1	2
7	22	1	1	1	1	7	1	1	1	1	1	1	7	1	1	1	1	1	1
8	32	1	5	1	2	1	8	1	2	1	4	1	2	1	8	1	2	1	4
9	37	4	1	1	3	1	1	9	1	1	3	1	1	3	1	1	9	1	1
10	46	1	2	5	2	1	2	1	10	1	2	1	2	5	2	1	2	1	10
11	56	1	1	1	1	1	1	1	1	11	1	1	1	1	1	1	1	1	1
12	70	3	4	1	6	1	4	3	2	1	12	1	2	3	4	1	6	1	4
13	79	1	1	1	1	1	1	1	1	1	1	13	1	1	1	1	1	1	1
14	92	1	2	1	2	7	2	1	2	1	2	1	14	1	2	1	2	1	2
15	106	3	1	5	3	1	1	3	5	1	3	1	1	15	1	1	3	1	5
16	124	1	4	1	2	1	8	1	2	1	4	1	2	1	16	1	2	1	4
17	137	1	1	1	1	1	1	1	1	1	1	1	1	1	1	17	1	1	1
18	154	3	2	1	6	1	2	9	2	1	6	1	2	3	2	1	18	1	2
19	172	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	19	1
20	194	1	4	5	2	1	4	1	10	1	4	1	2	5	4	1	2	1	20

Table: $\mathbf{dim}(\mathfrak{g}_{I_{n,m}})$

almost all the time $\mathbf{dim}(I_{n,m}) = \gcd(m,n)$

what about positive answers?

$$I = \langle x_1^2 + x_2^2 + x_3^2 \rangle \subseteq \mathbb{C}[x_1, x_2, x_3]$$

Take $y_1 = x_1$, $y_2 = -x_2 + ix_3$ and $y_3 = x_2 + ix_3$. Then $I = \langle y_1^2 - y_2 y_3 \rangle \subseteq \mathbb{C}[y_1, y_2, y_3]$ is toric.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & i & i \end{pmatrix}$$ the matrix recording the linear change of variables

The symmetry Lie algebra of I is the 4-dimensional vector space with basis

$$\mathcal{L} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

Notice that B changes the basis $\mathcal{L}$ (so apply $B^{-1}AB$ to each element in the basis) to the list

$$B^{-1}\mathcal{L}B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ 0.5 & 0 & 0 \\ -0.5 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 1 \\ 0.5i & 0 & 0 \\ 0.5i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix} \right\},$$

which realizes the embedded 2-dimensional torus.

finding the torus $\mathbb{T}^r$ in G_I

...if we can simultaneously diagonalize r of the basis elements in g_I, then a torus lives in G_I.
algorithm by Thomas Kahle and Julian Vill:

1 Compute the Lie algebra $\mathfrak{g}$ of the group $G \subseteq \mathrm{GL}_n(\mathbb{C})$ fixing I, i.e. $G.I \subseteq I$.
2 Pick $x \in \mathfrak{g}$ at random and compute $\mathfrak{c} = \ker((\mathrm{ad}(x))^{\dim \mathfrak{g}})$.
3 Check if $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}$. If not go back to line 2.
4 Decompose $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{n}$.
5 Compute an $S \in \mathrm{GL}_n(\mathbb{C})$ that diagonalizes $\mathfrak{t}$.
6 Check if $S.I$ is a binomial ideal. If not, return `False`.
7 Check if the binomial ideal $S.I$ is prime. If not return `False`.

Cartan sub algebra = nilpotent and self-normalizing. Cartan subgroup is the centralizer of a maximal torus

and their application to Gaussian graphical models:

graph	dim model	dim Lie algebra	dim max tori	can be made toric
diamond	9	30	6	no
paw	8	52	8	yes
cycle	8	4	4	no
claw	7	37	7	yes
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Thank you!