Tropical Toric Maximum Likelihood Estimation

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Joint Work

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Karel Devriendt (MPI CBG Dresden)

arXiv:2404.10567

Motivation:

▻ Two independent binary random variables X_1 and X_2

▻ Data from 100/1000 observations

$$u = \begin{pmatrix} 70 & 9 \\ 20 & 1 \end{pmatrix} \quad \begin{pmatrix} 919 & 16 \\ 63 & 2 \end{pmatrix}$$

▻ Maximum likelihood estimate $\hat{p}$ maximizes $\prod p_i^{u_i}$ among all rank-one joint distribution matrices p

$$\hat{p} = \begin{pmatrix} 71.1 & 7.9 \\ 18.9 & 2.1 \end{pmatrix} \quad \begin{pmatrix} 918.17 & 16.83 \\ 63.83 & 1.17 \end{pmatrix}$$

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$$u = \begin{pmatrix} 70 & 9 \\ 20 & 1 \end{pmatrix} \quad \begin{pmatrix} 919 & 16 \\ 63 & 2 \end{pmatrix} \quad \begin{pmatrix} 9834 & 38 \\ 123 & 5 \end{pmatrix}$$

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$$u = \begin{pmatrix} 70 & 9 \\ 20 & 1 \end{pmatrix} \quad \begin{pmatrix} 919 & 16 \\ 63 & 2 \end{pmatrix} \quad \begin{pmatrix} 9834 & 38 \\ 123 & 5 \end{pmatrix} \quad \approx N \begin{pmatrix} 1 & t^2 \\ t & t^4 \end{pmatrix} \quad (\text{small } t)$$

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History:

2022 Agostini–BSKFT: *tropical ML for linear models*

2024 Ardila–Eur–Penaguiao: *tropical ML for matroids*

2024 Boniface–Devriendt–H.: *tropical ML for toric models*

2025 Friedman–Sturmfels–Wiesmann: *tropical ML for squared lin. models*

Problem formulation

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1. Matrix $A \in \mathbb{Z}^{d \times n}$ full rank with $\mathrm{row}(A) \ni (1, \ldots, 1)$

$\rightarrow$ Toric variety X_A

$$X_A = V\left(\left\langle x^\alpha - x^\beta : \alpha, \beta \in \mathbb{N}^n \text{ s.t. } A(\alpha - \beta) = 0 \right\rangle\right) \subset (\mathbb{C}^*)^n$$

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$\rightarrow$ Affine subspace $Y_{A,u}$

$$Y_{A,u} \;=\; u \,+\, \ker(A)$$

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Definition (Maximum likelihood estimate (MLE))

The maximum likelihood estimate (if $\exists$) for the data pair (A, u) is the unique positive real point in $cX_A \cap Y_{A,u}$.

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▷ $cX_A \cap \Delta_{n-1}$ = **log-affine statistical model**
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$$Ap = Au \quad \text{and} \quad p \in cX_A$$

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▷ c generic $\implies$ $\deg(X_A)$-many critical points

Running example:

▻ Two independent binary random variables

$$\begin{array}{c} X_1 \in \{1,2\} \\ \text{w.p.} \quad \theta_1, \theta_2 \end{array} \qquad \text{and} \qquad \begin{array}{c} X_2 \in \{1,2\} \\ \text{w.p.} \quad \phi_1, \phi_2 \end{array}$$

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$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} \theta_1\phi_1 & \theta_1\phi_2 \\ \theta_2\phi_1 & \theta_2\phi_2 \end{pmatrix} \longrightarrow A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

with $\ker(A) = \text{span}(1, -1, -1, 1)$

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$\rightsquigarrow$ 2 critical points for generic u, c

Puiseux Series and Valuations

1. Field of Puiseux series $K := \mathbb{C}\{\{t\}\}$

$$K \ni u(t) = \sum_{k=0}^{\infty} c_k t^{\alpha_k},$$

with $c_k \in \mathbb{C}$ and $\alpha_k \in \mathbb{Q}$ with bounded denominator.

$$\text{E.g. } \begin{pmatrix} 1 & t^2 \\ t & t^4 \end{pmatrix} \in K^{2\times 2}$$

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2. Valuation $\mathrm{val} : K^* \to \mathbb{R}$

$$u(t) = \sum_{k=0}^{\infty} c_k t^{\alpha_k} \longmapsto \min(\alpha_k : c_k \neq 0)$$

$$\text{E.g. } \mathrm{val}\begin{pmatrix} 1 & t^2 \\ t & t^4 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$$

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Definition (Tropical critical points)

The **tropical critical points** for the pair $(A, u(t))$ are the valuations $\mathrm{val}(cX_A \cap Y_{A,u})$ of the critical points, counted with multiplicity.

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▷ Two independent binary random variables

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Find critical points $cX_A \cap Y_{A,u}$ by solving a quadratic equation

$$\widehat{p}_1 = (1+\alpha, -\alpha, -\alpha, \alpha) + \dots \qquad \text{and} \qquad \widehat{p}_2 = (1, t^2, t, \beta \cdot t^3) + \dots$$

with α, β some function of c

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▻ Tropical critical points:

$$\widehat{q}_1 = \mathrm{val}(\widehat{p}_1) = (0,0,0,0) \quad \text{and} \quad \widehat{q}_2 = \mathrm{val}(\widehat{p}_2) = (0,2,1,3)$$

Fundamental theorem of tropical geometry

▻ Let $f = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha x^\alpha \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.
Then the following two subsets of $\mathbb{R}^n$ coincide:

1. $\overline{\{\mathrm{val}(x) : x \in V(f) \subset (K^*)^n\}}$
2. $\{x \in \mathbb{R}^n : \min_{\alpha\in\mathbb{Z}^n}(\mathrm{val}(c_\alpha) + \alpha^T x)$ achieves min at least twice$\}$

This is the **tropical hypersurface** $\mathrm{trop}(V(f))$.

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Example:

$$\begin{aligned}
&\text{trop}(V(\langle p_{11}p_{22} - p_{12}p_{21} \rangle)) \\
&= \{x \in \mathbb{R}^4 : \min\{x_{11} + x_{22}, x_{12} + x_{21}\} \text{ achieves min at least twice}\} \\
&= \{x \in \mathbb{R}^4 : x_{11} + x_{22} = x_{12} + x_{21}\} \\
&= \text{row}(A)
\end{aligned}$$

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▷ Let $I \subseteq K[x_1^{\pm}, \ldots, x_n^{\pm}]$. Then the following two subsets of $\mathbb{R}^n$ coincide:

1. $\overline{\{\text{val}(x) : x \in V(I) \subset (K^*)^n\}}$
2. $\bigcap_{f \in I} \text{trop}(V(f))$.

This is the **tropical variety** $\text{trop}(V(I))$.

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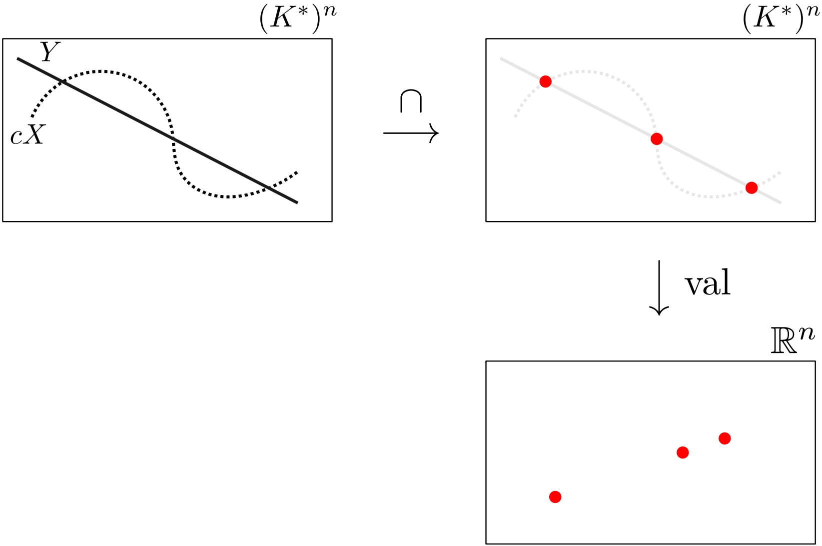
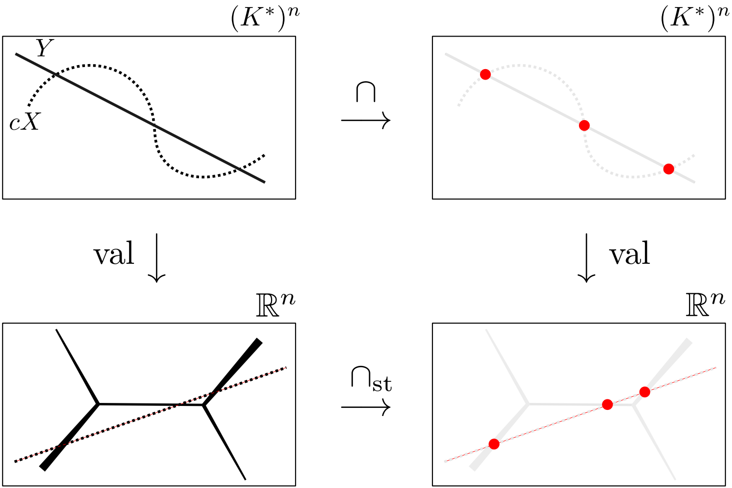
To remember: Tropical varieties are nice polyhedral complexes.

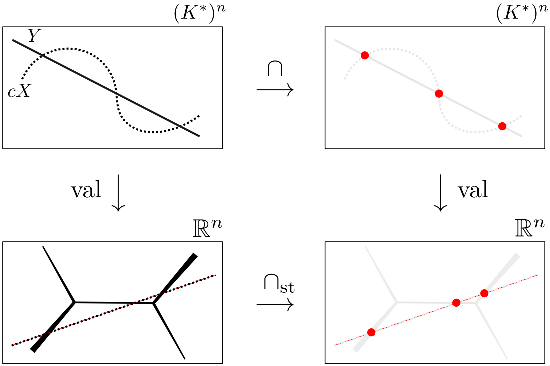
Tropical intersections



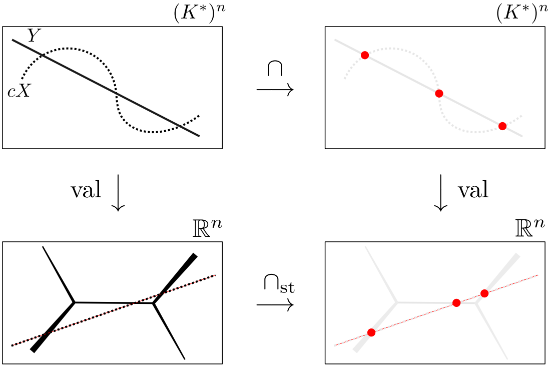
Tropical intersections



Maclagan–Sturmfels: for generic c, this diagram commutes



$(K^*)^n$
Y
cX
$(K^*)^n$
$\cap$
val
val
$\mathbb{R}^n$
$\mathbb{R}^n$
$\cap_{\mathrm{st}}$



Corollary

The tropical critical points for data $(A, u(t))$ are the points in the stable intersection $\text{trop}(cX_A) \cap_{\text{st}} \text{trop}(Y_{A,u})$*, counted with multiplicity.*

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▻ Tropical affine space $L_{A,w} := \text{trop}(Y_{A,u}) =$ polyhedral complex, with combinatorics governed by the matroid of A:

$$M(A) = \{\tau \subseteq [n] : \det(A_\tau) \neq 0\}$$

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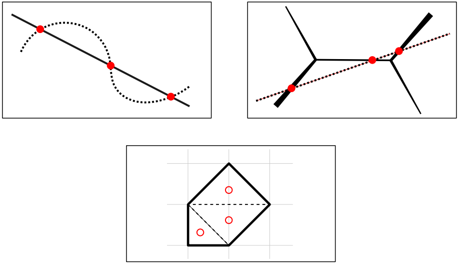

Main result

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$$x \longmapsto \begin{cases} x_j^{(\tau)} = \min(x_i \quad \text{s.t. } \tau - j + i \in M(A)) & \text{for } j \in \tau, \\ x_i^{(\tau)} = \max(x_j^{(\tau)} \text{ s.t. } \tau - j + i \in M(A)) & \text{for } i \in [n] \backslash \tau \end{cases}$$

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Running example: Let $w = \text{val}(1, t^2, t, t^4) = (0, 2, 1, 4)$ and $\tau = 123$

$$\begin{cases} w_1^{(\tau)} = \min(w_1, w_4) \\ w_2^{(\tau)} = \min(w_2, w_4) \\ w_3^{(\tau)} = \min(w_3, w_4) \\ w_4^{(\tau)} = \max(w_1^{(\tau)}, w_2^{(\tau)}, w_3^{(\tau)}) \end{cases} \longrightarrow \begin{cases} w_1^{(\tau)} = 0 \\ w_2^{(\tau)} = 2 \\ w_3^{(\tau)} = 1 \\ w_4^{(\tau)} = 2 \end{cases}$$

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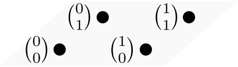
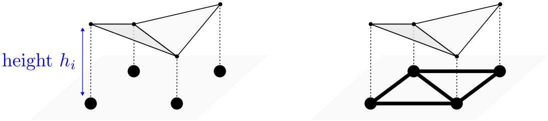
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$





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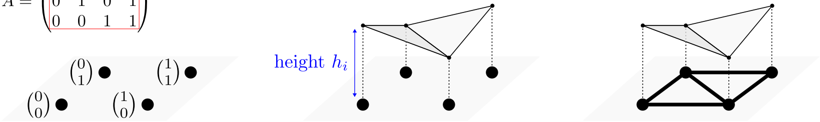

Definition (Regular subdivision)

The **regular subdivision** $\Delta(A, h)$ of the point configuration A induced by height function $h \in \mathbb{R}^n$ is the collection of cells $\sigma \subseteq [n]$ for which $\text{conv}(\binom{a_i}{h_i} \mid i \in \sigma)$ is a lower face of $\text{conv}(\binom{a_i}{h_i} \mid i \in [n])$.

- **Three definitions** (2)

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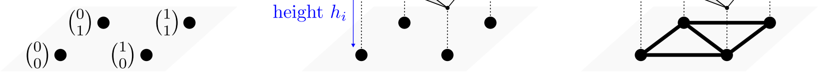

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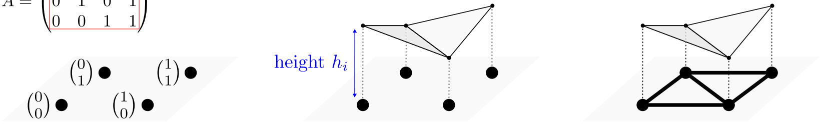

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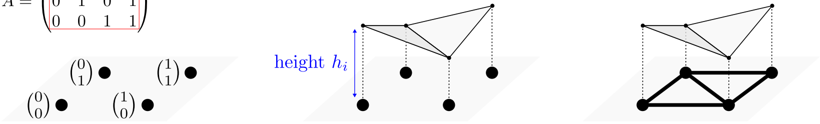

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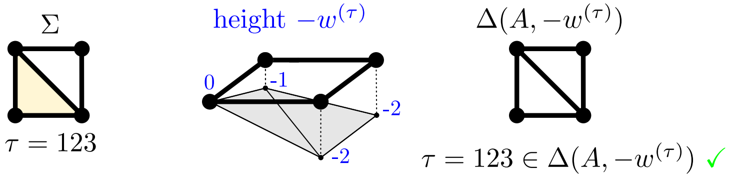

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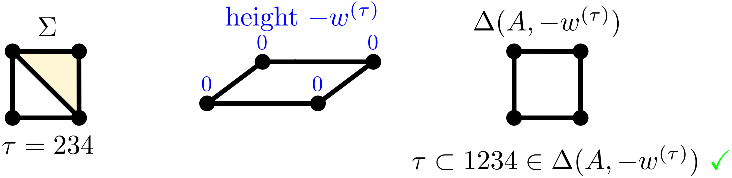

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- **Main result**

Theorem (Boniface-Devriendt-H)

If Σ and w are compatible, then the tropical critical points for the data pair (A, w) are given by the vectors

$$\widehat{q}(\tau) := A^T (A_\tau^T)^{-1} w_\tau^{(\tau)} \quad \textit{with mult. } \mathrm{vol}(\tau),$$

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$\widehat{q}($  $) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = (0, 2, 1, 3)$ with mult. 1

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Sanity check:

1. $\sum_{\tau\in\Sigma} \mathrm{mult}(\widehat{q}(\tau)) = \mathrm{deg}(X_A)$, by Kushnirenko's theorem
2. $\widehat{q}(\tau) \in \mathrm{row}(A)$, by definition

Running example:

▻ Two independent binary random variables

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad w = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$$

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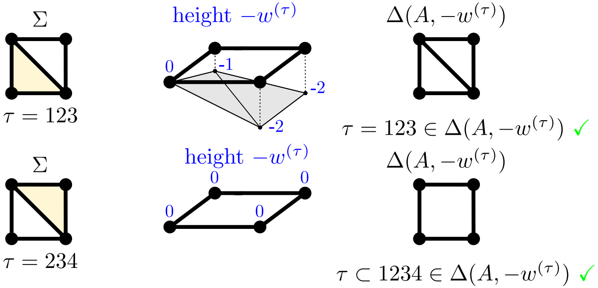

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$\rightsquigarrow$ any Σ that refines $\Delta(A, \chi(\text{supp}(w)))$

- **Sketch of the proof:**

Step 1: Main and most difficult technical result:

For every $\tau \in M(A)$, tropical affine space $L_{A,w}$ contains the cone

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Step 3: Rest follows by bookkeeping with Σ

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