Painting a picture of model polytopes

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Conclusion Slide

- Model polytopes carry information about statistical models.
- They exhibit rich and elegant combinatorics.
- Multiset permutations are cool.
- Some model polytopes relate closely to multiset permutations.

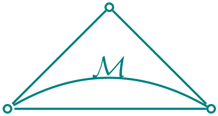
Probability Simplex

- $X_1, \ldots, X_n$ discrete random variables with outcomes $[m_1]_0, \ldots, [m_n]_0$ respectively, $\quad [m_j]_0 := \{0, \ldots, m_j\}$.
- $\mathcal{R} := [m_1]_0 \times \cdots \times [m_n]_0$ set of possible outcomes.

The joint distribution of $X_1, \ldots, X_n$ lies in the $(|\mathcal{R}| - 1)$-dimensional probability simplex

$$\Delta_{|\mathcal{R}|-1} = \{p \in \mathbb{R}^{|\mathcal{R}|} : p_i \geq 0, \text{ for all } i \in \mathcal{R} \text{ and } \sum_{i \in \mathcal{R}} p_i = 1\}.$$

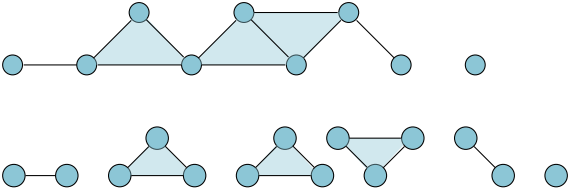
A (discrete) statistical model $\mathcal{M}$ is a subset of $\Delta_{|\mathcal{R}|-1}$.



Decomposable Simplicial Complexes

A simplicial complex is called decomposable if we can split it into two along a face such that each component is either decomposable or a simplex.

Example



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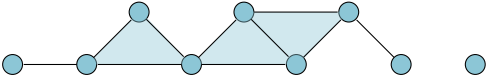
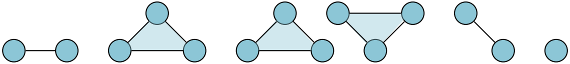
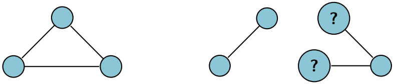




Non-example



Discrete Decomposable Models

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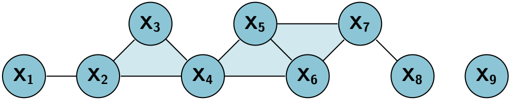
- Γ *decomposable simplicial complex* on $[n]$.

  F facet of Γ, $\mathcal{R}_F := \prod_{j \in F} [m_j]_0$ set of facet outcomes.



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The (discrete) decomposable model $\mathcal{M}_\Gamma$ associated with Γ is

$$\mathcal{M}_\Gamma = \{p \in \Delta^{\circ}_{|\mathcal{R}|-1} : p_i = \frac{1}{Z(y)} \prod_{F \in \text{facets}(\Gamma)} y^{(F)}_{i_F} \text{ for all } i \in \mathcal{R}\},$$

for $y^{(F)}_{i_F}$ positive parameters and $Z(y)$ normalizing constant.

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The (discrete) decomposable model $\mathcal{M}_\Gamma$ associated with Γ is

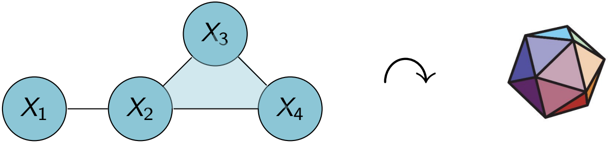
$$\mathcal{M}_\Gamma = \mathcal{V}(\ker(\phi_\Gamma)) \cap \Delta^\circ_{|\mathcal{R}|-1},$$

$$\phi_\Gamma : \mathbb{C}[z_i : i \in \mathcal{R}] \longrightarrow \mathbb{C}[y^F_{i_F} : i_F \in \mathcal{R}_F, F \in \text{facets}(\Gamma)];$$

$$\phi_\Gamma : z_i \longmapsto \prod_{F \in \text{facets}(\Gamma)} y^F_{i_F}.$$

From the model to the Polytope

- There is a (toric) variety $\mathcal{V}_{\mathcal{M}_\Gamma}$ and a lattice polytope $P_{\mathcal{M}_\Gamma}$ associated to a decomposable model $\mathcal{M}_\Gamma$. We can read off $P_{\mathcal{M}_\Gamma}$ directly from Γ and $m_1, \ldots, m_n$.



$P_{\mathcal{M}_\Gamma}$

- We call $P_{\mathcal{M}_\Gamma}$ the model polytope.

We record the support vectors of each $\phi_\Gamma(z_i)$ as the columns of a $(N_\Gamma \times |\mathcal{R}|)$ 0/1-matrix A_Γ, where $N_\Gamma = \sum_{F \in \text{facets}(\Gamma)} |\mathcal{R}_F|$. Then $P_{\mathcal{M}_\Gamma} = \text{conv}\big(\text{columns}(A_\Gamma)\big) \subset \mathbb{R}^{N_\Gamma}$.

Example

Γ

$[m_1]_0 = \{0, 1, 2\}, \ [m_2]_0 = [m_3]_0 = \{0, 1\}$

$A_\Gamma =$

	$\underline{00}\bar{1}$	001	010	011	100	101	110	111	200	201	210	211
00·	1	1	0	0	0	0	0	0	0	0	0	0
01·	0	0	1	1	0	0	0	0	0	0	0	0
10·	0	0	0	0	1	1	0	0	0	0	0	0
11·	0	0	0	0	0	0	1	1	0	0	0	0
20·	0	0	0	0	0	0	0	0	1	1	0	0
21·	0	0	0	0	0	0	0	0	0	0	1	1
··0	1	0	1	0	1	0	1	0	1	0	1	0
··1	0	1	0	1	0	1	0	1	0	1	0	1

$P_{\mathcal{M}_\Gamma} = \text{conv}\big(e_1 + e_7, e_1 + e_8, e_2 + e_7, \ldots, e_6 + e_8\big) \subset \mathbb{R}^8.$

Example 2: Independence Model

- X_1, X_2 binary variables ($m_1 = m_2 = 1$), Γ empty graph.
- $\mathcal{M}_\Gamma$ consists of positive joint distributions $p = (p_{00}, p_{01}, p_{10}, p_{11})$ for which X_1, X_2 are *independent*.

Γ

$$A_\Gamma = \begin{array}{c} \\ 0\cdot \\ 1\cdot \\ \cdot 0 \\ \cdot 1 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

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$$P_{\mathcal{M}_\Gamma} = \mathrm{conv}\big(\{1,0,1,0\}, \{1,0,0,1\}, \{0,1,1,0\}, \{0,1,0,1\}\big) \subset \mathbb{R}^4$$

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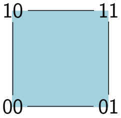
But $P_{\mathcal{M}_\Gamma}$ is 2-dimensional and isomorphic to the unit square.



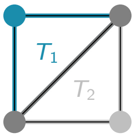
When investigating a polytope's combinatorics, there are several questions to be explored, such as

1) What are the facets of the polytope $P_{\mathcal{M}_\Gamma}$?

2) Does $P_{\mathcal{M}_\Gamma}$ admit a unimodular triangulation?
– What combinatorial information does this triangulation carry?

3) Enumerative combinatorics of $P_{\mathcal{M}_\Gamma}$.

4) Subpolytopes of interest.



What are the facets of the polytope $P_{\mathcal{M}_\Gamma}$?

Develin-Sullivant (2003): An H-representation of $P_{\mathcal{M}_\Gamma}$ is given by $y_{i_F}^F \geq 0$ for all $F \in \text{facets}(\Gamma)$ and $i_F \in \mathcal{R}_F$.

Hoşten-Sullivant (2002): $P_{\mathcal{M}_\Gamma}$ has a unimodular triangulation T.

- Computed the number of facets in T.
- $\dim(P_{\mathcal{M}_\Gamma}) = \sum_{f \in \text{faces}(\Gamma) \setminus \{\emptyset\}} \prod_{f = \{j_1, \dots, j_t\}} m_{j_k}$.

D-Solus (2022): Described the facets of $P_{\mathcal{M}_\Gamma}$, using the above.

Special case. Let Γ be a disjoint union of simplices $F_1, \ldots, F_k$. A full-dimensional representation of $P_{\mathcal{M}_\Gamma}$ is given by

$$\sum_{i_{F_j} \in \mathcal{R}_{F_j} \setminus \{i_{\mathbb{0}}\}} y_{i_{F_j}}^{F_j} \leq 1, \qquad \forall\, j \in [k],$$

$$y_{i_{F_j}}^{F_j} \geq 0, \qquad \forall\, i_{F_j} \neq i_{\mathbb{0}},\ \forall\, j \in [k].$$

- $P_{\mathcal{M}_\Gamma}$ is a *chain polytope* (almost an *order polytope*)!

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$$y_{01}^{\textcircled{1}-\textcircled{2}} + y_{11}^{\textcircled{1}-\textcircled{2}} + y_{21}^{\textcircled{1}-\textcircled{2}} + y_{10}^{\textcircled{1}-\textcircled{2}} + y_{20}^{\textcircled{1}-\textcircled{2}} \leq 1, \qquad y_1^{\textcircled{3}} \leq 1,$$

$$y_{01}^{\textcircled{1}-\textcircled{2}}, y_{11}^{\textcircled{1}-\textcircled{2}}, y_{21}^{\textcircled{1}-\textcircled{2}}, y_{10}^{\textcircled{1}-\textcircled{2}}, y_{20}^{\textcircled{1}-\textcircled{2}}, y_1^{\textcircled{3}} \geq 0.$$

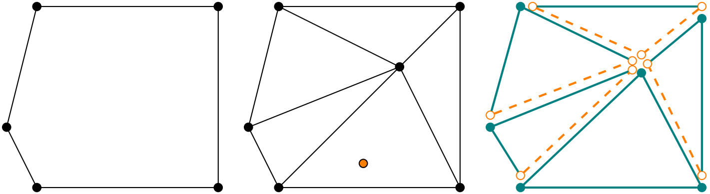
Definition

We define the h^*-polynomial of $P_{\mathcal{M}_\Gamma}$ as

$$h^*(P_{\mathcal{M}_\Gamma}; t) = h_0^* + h_1^* t + \cdots + h_d^* t^d,$$

where h_i^* is the number of facets *missing-i-facets* in a *half-open* unimodular triangulation T of $P_{\mathcal{M}_\Gamma}$, $i \in [d]_0$.

Note: $h^*(P_{\mathcal{M}_\Gamma}; t)$ is $f_T(P_{\mathcal{M}_\Gamma}; t)$ after a change of basis, enumerating faces of different dimensions in T.



$$h_0^* + h_1^* t + h_2^* t = 1 + 3t + t^2$$

Question: What can we say about $h^*(P_{\mathcal{M}_\Gamma}; t)$?

applications:

- ◇ lower bound on the weak maximum likelihood threshold of the model (Johnson-Sullivant, 2023);
- ◇ time-complexity bound for variable elimination (D-Solus, 2022).

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Palindromic $h^*(P_{\mathcal{M}_\Gamma}; t)$

If $h^*(P_{\mathcal{M}_\Gamma}; t)$ is palindromic, then $P_{\mathcal{M}_\Gamma}$ is called Gorenstein.

e.g.: If $h^*(P_{\mathcal{M}_\Gamma}; t) = 1 + 3t + t^2$ then $P_{\mathcal{M}_\Gamma}$ is Gorenstein.

Theorem [Johnson-Sullivant, 2023; D-Solus]:
$P_{\mathcal{M}_\Gamma}$ is Gorenstein if and only if $|\mathcal{R}_F|$ is fixed for every facet $F \in \text{facets}(\Gamma)$.

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Γ

- $[m_1]_0 = \{0, 1, 2\}$, $[m_2]_0 = [m_3]_0 = \{0, 1\}$; $\mathcal{R}_{\textcircled{1}-\textcircled{2}} = 6$, $\mathcal{R}_{\textcircled{3}} = 2$ ✗
- $[m_1]_0 = \{0, 1, 2\}$, $[m_2]_0 = \{0, 1\}$, $[m_3]_0 = \{0, \ldots, 5\}$:

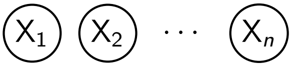
$\text{R}_{\textcircled{1}-\textcircled{2}} = 6$, $\mathcal{R}_{\textcircled{3}} = 6$. ✓

Computing $h^*(P_{\mathcal{M}_\Gamma}; t)$ for disjoint unions of simplices

Poset of chains

Independence model

Γ



Outcomes:

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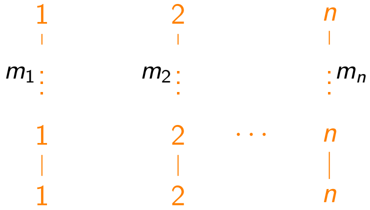

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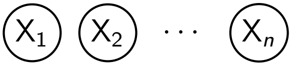

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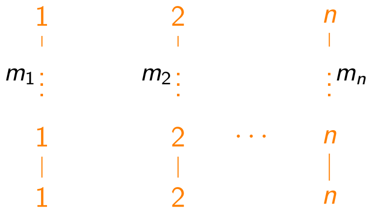

$$h^*(P_{\mathcal{M}_\Gamma}; t) = \sum_{\pi \in \mathcal{L}(\Pi)} t^{\mathrm{des}(\pi)} = \sum_{\pi \in S_M} t^{\mathrm{des}(\pi)}, \quad \text{where}$$

$\mathcal{L}(\Pi) = \{\text{linear extensions of } \Pi\}$

$S_M = \{\text{permutations of multiset } M = \{1, 1, \ldots, 2, 2, \ldots, n, n\}\}$

Multiset permutations - descent polynomial

- $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ multiset. For example, $M = \{1, 1, 1, 2, 2, 3, 3\}$ for $m_1 = 3,\ m_2 = m_3 = 2$.

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- S_M permutations of M. For example, $M = \{1,1,2\}$ gives $S_M = \{112, 121, 211\}$.

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- $\pi \in S_M$ permutation of M. The statistic $\mathrm{des}(\pi)$ counts descents in π, i.e., i's with $i + 1 < i$. For example, for $2\underline{3}111\underline{3}2$, $\mathrm{des}(\pi) = 2$.

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- descent polynomial of S_M: $\sum_{\pi \in S_M} t^{\mathrm{des}(\pi)}$. For example, for $M = \{112, 121, 211\}$, $\sum_{\pi \in S_M} t^{\mathrm{des}(\pi)} = 1 + 2t$.

Γ

- When $m_1 = m_2 = \cdots = m_n = 1$ (binary variables),

$$h^*(P_{\mathcal{M}_\Gamma}; t) = h^*(\square_n; t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)}.$$

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- For general $m_1, m_2, \ldots, m_n$, $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$,

$$h^*(P_{\mathcal{M}_\Gamma}; t) = h^*(\Delta_{m_1} \times \cdots \times \Delta_{m_n}; t) = \sum_{\pi \in S_M} t^{\mathrm{des}(\pi)}.$$

Bijection between simplices in the triangulation T of $P_{\mathcal{M}_\Gamma}$ and permutations in S_M (missing facets correspond to descents).

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[D-Han-Solus, 2024^+] For $r\Delta_m$ the r-th dilate of m-simplex,

$$h^*(r_1\Delta_{m_1} \times \cdots \times r_n\Delta_{m_n}; t) = \sum_{\pi \in S_{M^{\mathbf{r}}}} t^{\mathrm{des}(\pi)}.$$

- ◇ Characterized when the above polynomial is **palindromic**.
- ◇ Showed that it satisfies several **distributional properties** if $r_j > m_j$.

- Multiset permutations enumerate facets in the model polytope triangulation for the independence model.

- Is this a coincidence?

Split pair permutations

A split-pair permutation of the multiset $M = \{1, 1, 2, 2, ..., n, n\}$ is a permutation π of M such that, for each $i < n$, exactly one copy of $i + 1$ appears between the two copies of i in π (alternatively, they avoid $abba, aabb$ for $b = \pm 1$.)
E.g., 12134234.

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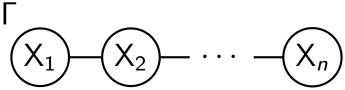
Question. (Graham-Zhang)

> Construct a bijection between split-pair permutations and the facets in triangulation T of $P_{\mathcal{M}_\Gamma}$ for the *binary Markov chain*.



$m_1 = \cdots = m_n = 1$ (binary)

Binary Markov chain

Conjecture [D-Solus, 2025^{+}]: For the binary Markov chain model,

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where SP_n denotes split-pair permutations of $\{1, 1, \ldots, n, n\}$.

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Recall. For general $m_1, \ldots, m_n$, $M = \{1^{m_1}, \ldots, n^{m_n}\}$,

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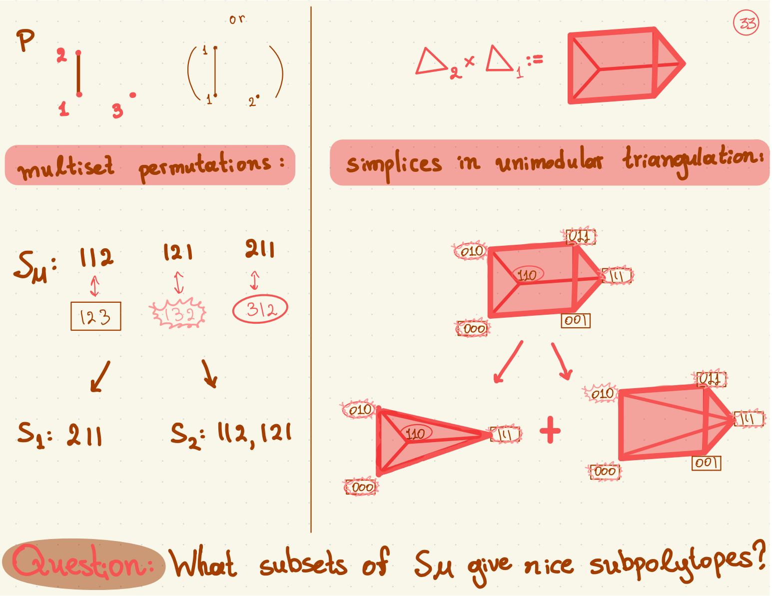
- Bijection between facets in the triangulation T of $P_{\mathcal{M}_\Gamma}$ and permutations in SP_n (missing facets correspond to descents)?
- How Markov chain polytope fits inside the independence model's.

Applications and hopes

Nice subpolytopes



Fix $M = \{1, 1, \ldots, n, n\}$.

$A \subseteq S_M$, where A has **palindromic descent polynomial**:

Carlitz-Hoggatt (1978): all of S_M

Elizalde (2024): canon permutations (motivated from Stirling permutations - see Julia's talk)

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Theorem

$P_{\mathcal{M}_\Gamma}$ is Gorenstein if and only if $|\mathcal{R}_F|$ is fixed for all facets F.

Corollary: $P_{\mathcal{M}_\Gamma}$ is Gorenstein for $\mathcal{M}_\Gamma$ binary Markov chain, i.e., $h^*(P_{\mathcal{M}_\Gamma}; t)$ is palindromic.

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Question. (Graham-Zhang, 2008)

"split-pair permutations" for $M = \{1, 1, 1, 2, 2, 2, \dots, n, n, n\}$?
(motivated from robotic scheduling)

hope: connection to $P_{\mathcal{M}_\Gamma}$ for suitable $\mathcal{M}_\Gamma$.

Conclusion Slide

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- They exhibit rich and elegant combinatorics.
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Thank you!

Some references

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