Localized Model Order Reduction via Multiscale-Spectral Generalized Finite Elements

Robert Scheichl

Institute for Mathematics & Interdisciplinary Center for Scientific Computing

UNIVERSITÄT HEIDELBERG
ZUKUNFT
SEIT 1386

Collaborators: Chupeng Ma (Great Bay Univ.), Jean Bénézech (ONERA), **Christian Alber**, Peter Bastian, Lukas Holbach, Arne Strehlow (HD), **Markus Bachmayr, Huqing Yang** (RWTH Aachen), ...

"Reduced Order & Surrogate Modelling for Digital Twins", **IMSI**, Chicago

Motivation

Many scientific/engineering problems involve multiscale phenomena:

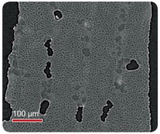
- Flow and transport in **porous media**
- Structural mechanics in **composites**
- **Geodynamics** (e.g. crystal-melt in earth mantel)
- **High-frequency** waves in heterogeneous media

Motivation

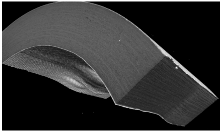
Many scientific/engineering problems involve multiscale phenomena:

- Flow and transport in **porous media**
- Structural mechanics in **composites**
- **Geodynamics** (e.g. crystal-melt in earth mantel)
- **High-frequency** waves in heterogeneous media

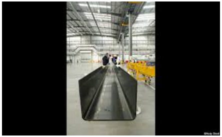
- Coefficients vary on many (unseparated) scales in a complicated way,



Ply with individual fibres



Coupon with defect



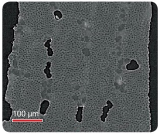
wing spar

Motivation

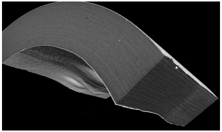
Many scientific/engineering problems involve multiscale phenomena:

- Flow and transport in **porous media**
- Structural mechanics in **composites**
- **Geodynamics** (e.g. crystal-melt in earth mantel)
- **High-frequency** waves in heterogeneous media

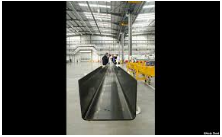
- Coefficients vary on many (unseparated) scales in a complicated way, often depending on **time**, the **solution** or a (high-dim.) **parameter** y



Ply with individual fibres



Coupon with defect



wing spar

Key Questions

- How to set up a digital twin for such multiscale problems?
- How to make it computational feasible?
- How to deal with the inherent high dimensionality?
- How to build reduced order models or surrogates **with theory**?

Challenges and Aims (e.g. for diffusion)

Challenges for classical FEs: e.g. for $a(u,v) = \int_\Omega A(y;x)\nabla u \cdot \nabla v \, \mathrm{d}x$

$$\|u - u_H\|_{H^1(\Omega)} \leq C_A \inf_{v \in V_H} \|u - v\|_{H^1(\Omega)} \qquad \text{(Cea's lemma)}.$$

Accuracy depends solely on the approximation properties of the FE space V_h.

Challenges and Aims (e.g. for diffusion)

Challenges for classical FEs: e.g. for $a(u,v) = \int_\Omega A(y;x)\nabla u \cdot \nabla v \, \mathrm{d}x$

$$\|u - u_H\|_{H^1(\Omega)} \le C_A \inf_{v \in V_H} \|u - v\|_{H^1(\Omega)} \qquad \text{(Cea's lemma)}.$$

Accuracy depends solely on the approximation properties of the FE space V_h.

But if coefficient $A(y;\cdot) \in L^\infty(\Omega)$ is strongly varying on fine scale, then for a "geometric" **coarse** space V_H the constant C_A may be **very bad**.

- $\boxed{\textbf{Aim 1}}$ Use fine mesh and solve efficiently, using a multilevel solver

  robust w.r.t. mesh size h + w.r.t. coefficient variation $A(y;x)$ **!**

Challenges and Aims (e.g. for diffusion)

Challenges for classical FEs: e.g. for $a(u,v) = \int_\Omega A(y;x)\nabla u \cdot \nabla v \, dx$

$$\|u - u_H\|_{H^1(\Omega)} \le C_A \inf_{v \in V_H} \|u - v\|_{H^1(\Omega)} \qquad \text{(Cea's lemma).}$$

Accuracy depends solely on the approximation properties of the FE space V_h.

But if coefficient $A(y;\cdot) \in L^\infty(\Omega)$ is strongly varying on fine scale, then for a "geometric" **coarse** space V_H the constant C_A may be **very bad**.

- $\boxed{\textbf{Aim 1}}$ Use fine mesh and solve efficiently, using a multilevel solver

  robust w.r.t. mesh size h + w.r.t. coefficient variation $A(y;x)$ **!**

- $\boxed{\textbf{Aim 2}}$ Simulate on coarse mesh where $A(y;x)$ is **not resolved !**

  need to find **bespoke** (adapted) A-dependent space $V_H(y)$

Key Question (for both): **Robust coarse space** $V_H(y)$ or $\overline{V}_H$

Multiscale methods

Multiscale methods are effective techniques to reduce computational cost by incorporating fine-scale information into coarse model/space.

Traditionally: Asymptotic homogenization (Lions, Oleinik, Allaire,...)

Various multiscale methods (focussing only on related ones):

1. Multiscale Finite Element Method (MsFEM) (Hou, Wu, Efendiev,...)
2. Generalized Multiscale FE Method (GMsFEM) (Hou, Efendiev,...)
3. Localized Orthogonal Decomposition (LOD) (Peterseim, Malqvist,...)
4. *Multiscale Spectral Generalized FE Method (MS-GFEM)*
   [**Babuska & Lipton**, *Multiscale Model Sim (SIAM)* **9**, 2011], . . .

Based on enriching FE space V_H with pre-computed multiscale basis fcts. containing fine-scale information

Multiscale methods $\longleftrightarrow$ localized model reduction

Multiscale methods are effective techniques to reduce computational cost by incorporating fine-scale information into coarse model/space.

Traditionally: Asymptotic homogenization (Lions, Oleinik, Allaire,...)

Various multiscale methods (focussing only on related ones):

1. Multiscale Finite Element Method (MsFEM) (Hou, Wu, Efendiev,...)
2. Generalized Multiscale FE Method (GMsFEM) (Hou, Efendiev,...)
3. Localized Orthogonal Decomposition (LOD) (Peterseim, Malqvist,...)
4. *Multiscale Spectral Generalized FE Method (MS-GFEM)*
   [**Babuska & Lipton**, *Multiscale Model Sim (SIAM)* **9**, 2011], . . .

Based on enriching FE space V_H with pre-computed multiscale basis fcts. containing fine-scale information $\longleftrightarrow$ $\boxed{\text{localized } \textbf{model order reduction}}$

Some context

- Setup is expensive! Without **reuse** (multiple times) **no clear gains** over efficient, scalable multilevel parallel solver!
- For **wave problems** efficient solvers (w. theory) still hot topic!!
- Can use **MS-GFEM** coarse space in **multilevel iterative meth.**
- Can use for **localized model reduction** [Smetana, Patera, 2016], ...
- Embed all in efficient offline/online, parallel, adaptive scheme.

Some context

- Setup is expensive! Without **reuse** (multiple times) **no clear gains** over efficient, scalable multilevel parallel solver!
- For **wave problems** efficient solvers (w. theory) still hot topic!!
- Can use **MS-GFEM** coarse space in **multilevel iterative meth.**
- Can use for **localized model reduction** [Smetana, Patera, 2016], ...
- Embed all in efficient offline/online, parallel, adaptive scheme.

THIS TALK: Localized model reduction via GFEM + **new analysis!**

Coefficient-robust theory for **multiscale approximation** (continuous & discrete), **wavenumber-explicit** theory for **Helmholtz** + extension to nonsym./indef. PDEs **AND** exponential convergence in space+parameter via **Grassmann interpolation**

Generalized Finite Element Methods (GFEM)

- Developed by Babuska & Melenk [Melenk, PhD Thesis, 1995], $\cdots$
- **Extension** of standard FEMs based on a domain decomposition technique and the partition of unity (POU) approach.
- **Generalised FE methods (GFEM)** consist of three steps:
  1. Construct local spaces with good approximation properties;
  2. Glue together local spaces by PoU to form global approximation;
  3. Solve the problem over the global approximation space.

Generalized Finite Element Methods (GFEM)

- Developed by Babuska & Melenk [Melenk, PhD Thesis, 1995], $\cdots$
- **Extension** of standard FEMs based on a domain decomposition technique and the partition of unity (POU) approach.
- **Generalised FE methods (GFEM)** consist of three steps:
  1. Construct local spaces with good approximation properties;
  2. Glue together local spaces by PoU to form global approximation;
  3. Solve the problem over the global approximation space.
- Global error fully determined by local approximation errors.
- Can design **special approximation spaces** tailored to problem:

  Multiscale-Spectral Generalized FE Method (**MS-GFEM**)

  [Babuska & Lipton, 2011], [Ma, **RS**, Dodwell, 2022], [Ma, 2025],...

Types of Problems – Abstract Formulation

Find $u^e \in \mathcal{H}_0(\Omega) \subset \mathcal{H}(\Omega) \subset \mathcal{L}(\Omega)$ Hilbert on $\Omega \subset \mathbb{R}^d$ bdd. Lipschitz s.t.

$$a(u^e, v) = F(v), \quad \forall v \in \mathcal{H}_0(\Omega).$$

with $a : \mathcal{H}(\Omega) \times \mathcal{H}(\Omega) \to \mathbb{C}$ sesquilinear.

Types of Problems – Abstract Formulation

Find $u^e \in \mathcal{H}_0(\Omega) \subset \mathcal{H}(\Omega) \subset \mathcal{L}(\Omega)$ Hilbert on $\Omega \subset \mathbb{R}^d$ bdd. Lipschitz s.t.

$$a(u^e, v) = F(v), \quad \forall v \in \mathcal{H}_0(\Omega).$$

with $a : \mathcal{H}(\Omega) \times \mathcal{H}(\Omega) \to \mathbb{C}$ sesquilinear.

We assume

$$|a(u,v)| \leq C_a \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \qquad \forall u, v \in \mathcal{H}(\Omega) \qquad \text{(Boundedness)}$$

$$C_1 \|u\|_{\mathcal{H}}^2 - C_0 \|u\|_{\mathcal{L}}^2 \leq |a(u,u)| \qquad \forall u \in \mathcal{H}(\Omega) \qquad \text{(Gårding Ineq.)}$$

(If a coercive $\Rightarrow$ $C_0 = 0$. If a symmetric $\Rightarrow$ choose $\|u\|_{\mathcal{H}} = \sqrt{a(u,u)}$.)

In addition, we assume unique solvability and stability:

$$\|u^e\|_{\mathcal{H}} + \|w^e\|_{\mathcal{H}} \leq C_{\text{stab}} \|F\|_{\mathcal{L}} \qquad \forall F \in \mathcal{L}(\Omega)$$

where $w^e \in \mathcal{H}_0(\Omega)$ is the adjoint solution s.t. $a(v, w^e) = F(v), \quad \forall v \in \mathcal{H}_0(\Omega)$.

Types of Problems – Abstract Formulation

Find $u^e \in \mathcal{H}_0(\Omega) \subset \mathcal{H}(\Omega) \subset \mathcal{L}(\Omega)$ Hilbert on $\Omega \subset \mathbb{R}^d$ bdd. Lipschitz s.t.

$$a_y(u^e, v) = F(v), \quad \forall v \in \mathcal{H}_0(\Omega).$$

with $a_y : \mathcal{H}(\Omega) \times \mathcal{H}(\Omega) \to \mathbb{C}$ sesquilinear depending on parameter $y \in \mathbb{R}^s$.

We assume

$$|a(u,v)| \le C_a \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \qquad \forall u, v \in \mathcal{H}(\Omega) \qquad \text{(Boundedness)}$$

$$C_1 \|u\|_{\mathcal{H}}^2 - C_0 \|u\|_{\mathcal{L}}^2 \le |a(u,u)| \qquad \forall u \in \mathcal{H}(\Omega) \qquad \text{(Gårding Ineq.)}$$

(If a coercive $\Rightarrow$ $C_0 = 0$. If a symmetric $\Rightarrow$ choose $\|u\|_{\mathcal{H}} = \sqrt{a(u,u)}$.)

In addition, we assume unique solvability and stability:

$$\|u^e\|_{\mathcal{H}} + \|w^e\|_{\mathcal{H}} \le C_{\mathrm{stab}} \|F\|_{\mathcal{L}} \qquad \forall F \in \mathcal{L}(\Omega)$$

where $w^e \in \mathcal{H}_0(\Omega)$ is the adjoint solution s.t. $a(v, w^e) = F(v), \ \forall v \in \mathcal{H}_0(\Omega)$.

Examples

- (High-Peclet) Convection-Diffusion:

$$a_y(u,v) = \int_\Omega A(x,y) \nabla u \cdot \nabla v + v\boldsymbol{b}(x,y) \cdot \nabla u \, \mathrm{d}x$$

- (Non-)Linear Elasticity: $a_y(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega C(x,y)\, \epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{v}) \, \mathrm{d}x$

- (High-frequency) Helmholtz:

$$a_y(u,v) = \int_\Omega \left(A(x,y) \nabla u \cdot \nabla \overline{v} - k^2 V^2(x,y) u \overline{v} \right) \mathrm{d}x - \mathrm{i}k \int_{\partial\Omega} \beta(x,y) u \overline{v} \, d\boldsymbol{s}$$

- Heterogeneous Stokes:

$$a_y(u,v) = \int_\Omega \nu(x,y) \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \gamma \nu(x,y) \,\mathrm{div}\boldsymbol{u} \,\mathrm{div}\boldsymbol{v} \, \mathrm{d}x$$

Examples

- (High-Peclet) Convection-Diffusion:

$$a_y(u,v) = \int_\Omega A(x,y)\nabla u \cdot \nabla v + v\boldsymbol{b}(x,y) \cdot \nabla u \, \mathrm{d}x$$

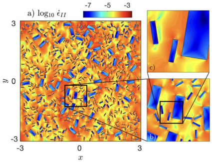
- (Non-)Linear Elasticity: $a_y(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega C(x,y)\,\epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{v}) \, \mathrm{d}x$

- (High-frequency) Helmholtz:

$$a_y(u,v) = \int_\Omega \left(A(x,y)\nabla u \cdot \nabla \overline{v} - k^2 V^2(x,y) u\overline{v}\right) \mathrm{d}x - \mathrm{i}k \int_{\partial\Omega} \beta(x,y) u\overline{v} \, d\boldsymbol{s}$$

- Heterogeneous Stokes:

$$a_y(u,v) = \int_\Omega \nu(x,y)\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \gamma\nu(x,y)\,\mathrm{div}\boldsymbol{u}\;\mathrm{div}\boldsymbol{v} \, \mathrm{d}x$$



Multiscale coefficients varying on many scales

Global Approximation Theorem (coercive case)

Theorem (Ma, **RS**, Dodwell, 2022)

For each $i = 1...N$ *let* $u_i^p \in \mathcal{H}(\omega_i^*)$ *be a particular function* (satisfying BCs & RHS on ω_i^*) *and let* $S_{n_i}(\omega_i^*) \subset \mathcal{H}(\omega_i^*)$ *be an* n_i*-dimensional subspace s.t.*

$$u^p := \sum_{i=1}^{N} \chi_i u_i^p \in \mathcal{H}_0(\Omega), \quad S_n(\Omega) := \left\{ \sum_{i=1}^{N} \chi_i \phi_i : \phi_i \in S_{n_i}(\omega_i^*) \right\} \subset \mathcal{H}_0(\Omega).$$

Assume that there exist $\phi_i \in S_{n_i}(\omega_i^*)$ *such that* (local approximation)

$$\|\chi_i(u^e - u_i^p - \phi_i)\|_{a,\omega_i^*} \le \varepsilon_i \, \|u^e\|_{a,\omega_i^*} \qquad (\dagger)$$

Global Approximation Theorem (coercive case)

Theorem (Ma, **RS**, Dodwell, 2022)

For each $i = 1 \ldots N$ *let* $u_i^p \in \mathcal{H}(\omega_i^*)$ *be a particular function* (satisfying BCs & RHS on ω_i^*) *and let* $S_{n_i}(\omega_i^*) \subset \mathcal{H}(\omega_i^*)$ *be an* n_i*-dimensional subspace s.t.*

$$u^p := \sum_{i=1}^{N} \chi_i u_i^p \in \mathcal{H}_0(\Omega), \quad S_n(\Omega) := \left\{ \sum_{i=1}^{N} \chi_i \phi_i : \phi_i \in S_{n_i}(\omega_i^*) \right\} \subset \mathcal{H}_0(\Omega).$$

Assume that there exist $\phi_i \in S_{n_i}(\omega_i^*)$ *such that* (local approximation)

$$\|\chi_i (u^e - u_i^p - \phi_i)\|_{a,\omega_i^*} \leq \varepsilon_i \, \|u^e\|_{a,\omega_i^*} \qquad (\dagger)$$

Then, for $u^F = u^p + \sum_{i=1}^{M} \chi_i \phi_i$

$$\|u^e - u^F\|_a \leq \kappa \max_{i=1}^{N} \varepsilon_i \, \|u^e\|_a .$$

Here we assume that each point $\boldsymbol{x} \in \Omega$ *belongs to at most* κ *subdomains* ω_i^*.

Global Approximation Theorem (coercive case)

Theorem (Ma, **RS**, Dodwell, 2022)

For each $i = 1 \ldots N$ *let* $u_i^p \in \mathcal{H}(\omega_i^*)$ *be a particular function* (satisfying BCs & RHS on ω_i^*) *and let* $S_{n_i}(\omega_i^*) \subset \mathcal{H}(\omega_i^*)$ *be an* n_i*-dimensional subspace s.t.*

$$u^p := \sum_{i=1}^{N} \chi_i u_i^p \in \mathcal{H}_0(\Omega), \quad S_n(\Omega) := \left\{ \sum_{i=1}^{N} \chi_i \phi_i : \phi_i \in S_{n_i}(\omega_i^*) \right\} \subset \mathcal{H}_0(\Omega).$$

Assume that there exist $\phi_i \in S_{n_i}(\omega_i^*)$ *such that* (local approximation)

$$\|\chi_i(u^e - u_i^p - \phi_i)\|_{a,\omega_i^*} \le \varepsilon_i \, \|u^e\|_{a,\omega_i^*} \qquad (\dagger)$$

Then, for $u^F = u^p + \sum_{i=1}^{M} \chi_i \phi_i$ *and the* Galerkin solution $u^G \in u^p + S_n(\Omega)$ *we have*

$$\|u^e - u^G\|_a \le \|u^e - u^F\|_a \le \kappa \max_{i=1}^{N} \varepsilon_i \, \|u^e\|_a .$$

Here we assume that each point $\boldsymbol{x} \in \Omega$ *belongs to at most* κ *subdomains* ω_i^*.

Global Approximation Theorem (non-coercive case)

Theorem (Ma, FoCM, 2025)

For each $i = 1 \ldots N$ *let* $u_i^p \in \mathcal{H}(\omega_i^*)$ *be a particular function* (satisfying BCs & RHS on ω_i^*) *and let* $S_{n_i}(\omega_i^*) \subset \mathcal{H}(\omega_i^*)$ *be an* n_i*-dimensional subspace s.t.*

$$u^p := \sum_{i=1}^{N} \chi_i u_i^p \in \mathcal{H}_0(\Omega), \quad S_n(\Omega) := \left\{ \sum_{i=1}^{N} \chi_i \phi_i : \phi_i \in S_{n_i}(\omega_i^*) \right\} \subset \mathcal{H}_0(\Omega).$$

Assume that there exist $\phi_i \in S_{n_i}(\omega_i^*)$ *such that* (local approximation)

$$\|\chi_i (u^e - u_i^p - \phi_i)\|_{a^+,\omega_i^*} \leq \varepsilon_i \, \|u^e\|_{a^+,\omega_i^*} \qquad (\dagger)$$

Then, for $u^F = u^p + \sum_{i=1}^{M} \chi_i \phi_i$ *and the Galerkin solution* $u^G \in u^p + S_n(\Omega)$ *we have*

$$\|u^e - u^G\|_{a^+} \;\leq\; \kappa \frac{2C_a}{C_1} \max_{i=1}^{N} \varepsilon_i \, \|u^e\|_{a^+}.$$

with a suitable **inner product** $a^+(\cdot,\cdot)$ *(+ under some resolution condition).*

Optimal Local Approximation (only interior ω_i^*)

- Particular solution u_i^p with artificial BC $u_i^p|_{\partial\omega_i^*} = 0$ s.t. locally

$$a_{\omega_i^*}(u_i^p, v) = F_{\omega_i^*}(v) \quad v \in \mathcal{H}_0(\omega_i^*)\,.$$

- The **'remainder'** $u|_{\omega_i^*} - u_i^p$ is locally a-harmonic and lies in

$$\mathcal{H}_a(\omega_i^*) = \left\{ v \in \mathcal{H}(\omega_i^*) \,:\, a_{\omega_i^*}(v,\varphi) = 0 \;\; \forall \varphi \in \mathcal{H}_0(\omega_i^*) \right\}.$$

- Consider the **PoU operator** $P_i : \mathcal{H}_a(\omega_i^*) \to \mathcal{H}_0(\omega_i)$ s.t.

$$P_i v(\boldsymbol{x}) = \chi_i(\boldsymbol{x}) v(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \omega_i, \;\; \forall v \in \mathcal{H}_a(\omega_i^*)\,,$$

Optimal Local Approximation (only interior ω_i^*)

- Particular solution u_i^p with artificial BC $u_i^p|_{\partial\omega_i^*} = 0$ s.t. locally

$$a_{\omega_i^*}(u_i^p, v) = F_{\omega_i^*}(v) \quad v \in \mathcal{H}_0(\omega_i^*)\,.$$

- The **'remainder'** $u|_{\omega_i^*} - u_i^p$ is locally a-harmonic and lies in

$$\mathcal{H}_a(\omega_i^*) = \left\{ v \in \mathcal{H}(\omega_i^*) \,:\, a_{\omega_i^*}(v,\varphi) = 0 \;\; \forall \varphi \in \mathcal{H}_0(\omega_i^*) \right\}.$$

- Consider the **PoU operator** $P_i : \mathcal{H}_a(\omega_i^*) \to \mathcal{H}_0(\omega_i)$ s.t.

$$P_i v(\boldsymbol{x}) = \chi_i(\boldsymbol{x}) v(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \omega_i, \;\; \forall v \in \mathcal{H}_a(\omega_i^*)\,,$$

Crucial observation: P_i is **compact**. Optimal local approximation space $S_{n_i}(\omega_i^*)$ can be computed via SVD in the a^+-inner product.

Key Tools: Cacciopoli inequality and a weak approximation property

Local Particular Solution & Approximation Space

Theorem (Ma, **RS**, Dodwell, 2022; Ma, 2025)

Local particular solutions u_i^p *& local approximation spaces defined as*

$$S_n(\omega_i^*) := \mathrm{span}\{v_1, \cdots, v_n\},$$

where $v_k \in \mathcal{H}_a(\omega_i^*)$ *denotes* k*-th eigenfunction of the eigenproblem*

$$a^+_{\omega_i^*}(\chi_i v, \chi_i \varphi) = \lambda\, a^+_{\omega_i^*}(v, \varphi), \quad \forall \varphi \in \mathcal{H}_a(\omega_i^*).$$

Local Particular Solution & Approximation Space

Theorem (Ma, **RS**, Dodwell, 2022;   Ma, 2025)

Local particular solutions u_i^p *& local approximation spaces defined as*

$$S_n(\omega_i^*) := \mathrm{span}\{v_1, \cdots, v_n\},$$

where $v_k \in \mathcal{H}_a(\omega_i^*)$ *denotes* k*-th eigenfunction of the eigenproblem*

$$a^+_{\omega_i^*}(\chi_i v, \chi_i \varphi) = \lambda\, a^+_{\omega_i^*}(v, \varphi), \quad \forall \varphi \in \mathcal{H}_a(\omega_i^*).$$

similiar to **GenEO***-eigenproblem, but in* $\mathcal{H}_a(\omega_i^*)$ *and with* $\mathrm{supp}\chi_i \subsetneq \omega_i^*$

[Spillane, Dolean, Hauret, Nataf, Pechstein, **RS**, 2014]

Local Particular Solution & Approximation Space

Theorem (Ma, **RS**, Dodwell, 2022; Ma, 2025)

Local particular solutions u_i^p *& local approximation spaces defined as*

$$S_n(\omega_i^*) := \mathrm{span}\{v_1, \cdots, v_n\},$$

where $v_k \in \mathcal{H}_a(\omega_i^*)$ *denotes* k*-th eigenfunction of the eigenproblem*

$$a^+_{\omega_i^*}(\chi_i v, \chi_i \varphi) = \lambda\, a^+_{\omega_i^*}(v, \varphi), \quad \forall \varphi \in \mathcal{H}_a(\omega_i^*).$$

similiar to **GenEO***-eigenproblem, but in* $\mathcal{H}_a(\omega_i^*)$ *and with* $\mathrm{supp}\chi_i \subsetneq \omega_i^*$

[Spillane, Dolean, Hauret, Nataf, Pechstein, **RS**, 2014]

Then, there exist $\phi_i \in S_n(\omega_i^*)$ *such that*

$$\|\chi_i(u^e - u_i^p - \phi_i)\|_{a^+,\omega_i^*} \le \lambda_{n+1}^{1/2}\, \|u^e\|_{a^+,\omega_i^*} \quad \textit{(Kolmogorov n-width).}$$

If $\mathrm{supp}\chi_i \subsetneq \omega_i^*$ *then* $\lambda_{n+1}^{1/2} = \mathcal{O}(e^{-cn^{1/d}})$ *for all the examples above.*

Local Particular Solution & Approximation Space

Theorem (Ma, **RS**, Dodwell, 2022; Ma, 2025)

Local particular solutions u_i^p & *local approximation spaces defined as*

$$S_n(\omega_i^*) := \mathrm{span}\{v_1, \cdots, v_n\},$$

where $v_k \in \mathcal{H}_a(\omega_i^*)$ *denotes* k*-th eigenfunction of the eigenproblem*

$$a^+_{\omega_i^*}(\chi_i v, \chi_i \varphi) = \lambda \, a^+_{\omega_i^*}(v, \varphi), \quad \forall \varphi \in \mathcal{H}_a(\omega_i^*).$$

similiar to **GenEO***-eigenproblem, but in* $\mathcal{H}_a(\omega_i^*)$ *and with* $\mathrm{supp}\chi_i \subsetneq \omega_i^*$

[Spillane, Dolean, Hauret, Nataf, Pechstein, **RS**, 2014]

Then, there exist $\phi_i \in S_n(\omega_i^*)$ *such that* $\boxed{\textbf{continuous \& discrete}}$

$$\|\chi_i(u^e - u_i^p - \phi_i)\|_{a^+, \omega_i^*} \le \lambda_{n+1}^{1/2} \, \|u^e\|_{a^+, \omega_i^*} \quad \textit{(Kolmogorov n-width).}$$

If $\mathrm{supp}\chi_i \subsetneq \omega_i^*$ *then* $\lambda_{n+1}^{1/2} = \mathcal{O}(e^{-cn^{1/d}})$ *for all the examples above.*

How to solve eigenproblems in discrete *a*-harmonic space?

Main computational cost in earlier works: Building (discrete) $\mathcal{H}_a^h(\omega_i^*)$

- [Babuska, Lipton '11] *a*-harmonic extension of all boundary hat functions

How to solve eigenproblems in discrete *a*-harmonic space?

Main computational cost in earlier works: Building (discrete) $\mathcal{H}_a^h(\omega_i^*)$

- [Babuska, Lipton '11] *a*-harmonic extension of all boundary hat functions
- [Calo et al '16; Chen et al '20] random sampling techniques
- [Babuska, Lipton et al '20] subset of boundary functions with wider support
- [Ma, **RS**, Dodwell '22] Steklov (DtN) eigenproblem to approximate $\mathcal{H}_a^h(\omega_i^*)$.

How to solve eigenproblems in discrete *a*-harmonic space?

Main computational cost in earlier works: Building (discrete) $\mathcal{H}_a^h(\omega_i^*)$

- [Babuska, Lipton '11] *a*-harmonic extension of all boundary hat functions
- [Calo et al '16; Chen et al '20] random sampling techniques
- [Babuska, Lipton et al '20] subset of boundary functions with wider support
- [Ma, **RS**, Dodwell '22] Steklov (DtN) eigenproblem to approximate $\mathcal{H}_a^h(\omega_i^*)$.

Idea [Ma, **RS**, 2022]: Eigensolver based on mixed formulation

Find $\lambda \in \mathbb{R}$, $v \in V_h(\omega_i^*)$ and $\overbrace{p \in V_{h,0}(\omega_i^*)}^{\text{Lagrange multiplier}}$ such that

$$
\begin{aligned}
a^+_{\omega_i^*}(v,\varphi) + a_{\omega_i^*}(\varphi,p) &= \lambda^{-1}\, a^+_{\omega_i^*}(\chi_i v, \chi_i \varphi) && \forall \varphi \in V_h(\omega_i^*),\\
a_{\omega_i^*}(v,\xi) &= 0 && \forall \xi \in V_{h,0}(\omega_i^*).
\end{aligned}
$$

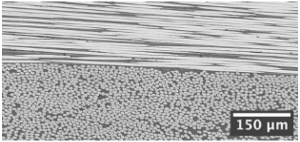
Note. The *a*-harmonic constraint is incorporated into the eigenproblem.

Example 1: (Non-)linear elasticity

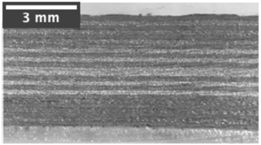
Carbon fibre aerospace composites (C heterog./anisotropic) [Benezech et al, 2024]

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega C(\mathbf{x})\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Gamma (\sigma\cdot\mathbf{n})\cdot\mathbf{v}\; d\mathbf{x} \quad \forall \mathbf{v}\in V$$

Carbon fibres



inner ply scale (microscopic)

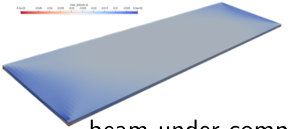


inner structure (mesoscopic)

Example 1: (Non-)linear elasticity

Carbon fibre aerospace composites (C heterog./anisotropic) [Benezech et al, 2024]

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega C(\mathbf{x})\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Gamma (\sigma\cdot\mathbf{n})\cdot\mathbf{v}\;dx \quad \forall\mathbf{v}\in V$$



beam under compression

(laminate: 3 layers + thin interlayers)

Example 1: (Non-)linear elasticity

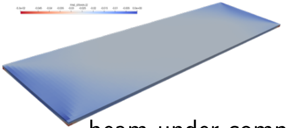
Carbon fibre aerospace composites (C heterog./anisotropic) [Benezech et al, 2024]

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega C(\mathbf{x})\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Gamma (\sigma\cdot\mathbf{n})\cdot\mathbf{v}\;d\mathbf{x} \quad \forall \mathbf{v}\in V$$

Theory [Ma, 2025]:

$$\frac{\|u-u^G\|_a}{\|u\|_a} \le \kappa \max_{j=1,\cdots,M} \left(\lambda_j^{m_j+1}\right)^{-1/2}$$

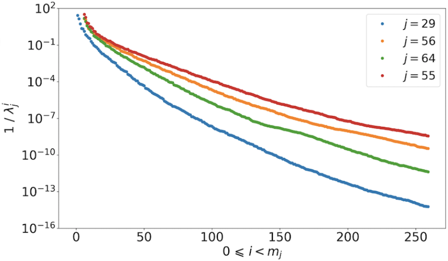
(can be extended to nonlinear case)



beam under compression

(laminate: 3 layers + thin interlayers)

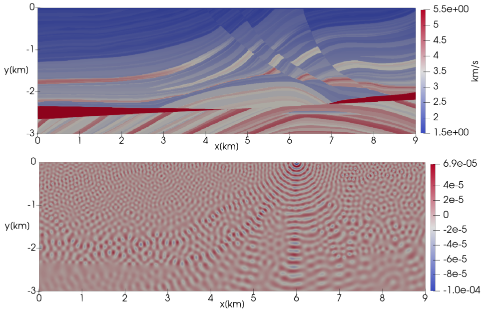
(a)

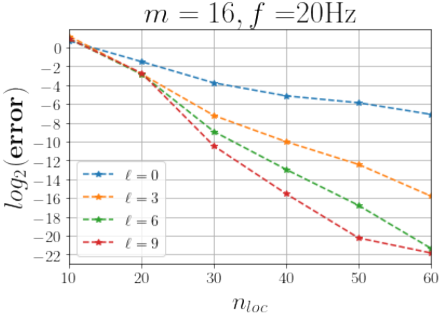


(b)

Example 2: High-frequency heterogeneous Helmholtz

Marmousi benchmark model with $A \equiv I$ and V^{-1} below [Ma, Alber, **RS**, 2023]





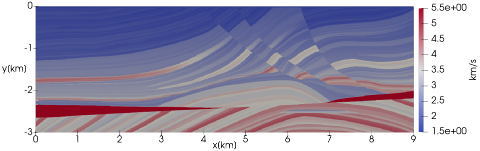
23000 basis functions $\longrightarrow$ **error** $< 10^{-3}$ $(m = 16,\ \ell = 9,\ n_{\mathrm{loc}} = 30)$

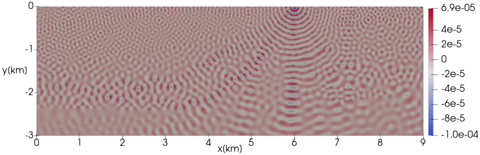
Fully coefficient robust! No pollution effect!

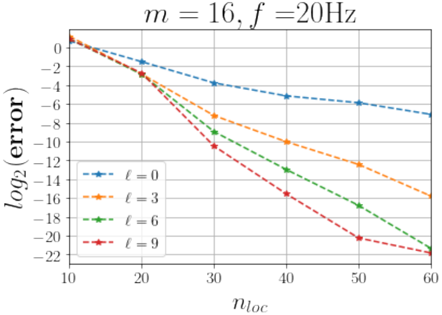
(**quasi-optimal** convergence for $H = \mathcal{O}(1/k)$)

Example 2: High-frequency heterogeneous Helmholtz

Marmousi benchmark model with $A \equiv I$ and V^{-1} below [Ma, Alber, **RS**, 2023]







23000 basis functions $\longrightarrow$ **error** $< 10^{-3}$ $(m = 16,\ \ell = 9,\ n_{\mathrm{loc}} = 30)$

Fully coefficient robust! No pollution effect!

(**quasi-optimal** convergence for $H = \mathcal{O}(1/k)$)

Crucial: Cacciopoli-ineq., generalized 'Helmholtz'-harmonic eigenproblem

Corollary: Two-Level Hybrid Restricted Additive Schwarz

Aim 1 Robust and scalable preconditioner for (discretised) $A\boldsymbol{u} = \boldsymbol{b}$

$$P \;:=\; R_0^T A_0^{-1} R_0 \left(I - \sum\nolimits_i R_i^T X_i A_i^{-1} R_i \right) \;+\; \sum\nolimits_i R_i^T X_i A_i^{-1} R_i$$

Theorem (Strehlow, Ma, **RS**, 2024; Ma, Alber, **RS**, Zhang, 2024)

Let $\Lambda := \kappa \frac{2C_a}{C_1} \max_i \lambda_{i,n+1}^{1/2} < 1$. *The sequence* $\{\mathbf{u}^j\}_{j\geq 1}$ *of approximations generated by preconditioned GMRES (in the b-inner product) satisfies*

$$\|PA(\mathbf{u}-\mathbf{u}^j)\|_b \leq \Lambda^j \left(\frac{1+\Lambda}{1-\Lambda}\right) C \|PA(\mathbf{u}-\mathbf{u}^0)\|_b, \quad \forall j \geq 1$$

where $C = 1$ *for* $b = a^+$ *and* $C = h^{-1} a_{\max}/a_{\min}$ *for Euclidean inner product.*

Corollary: Two-Level Hybrid Restricted Additive Schwarz

Aim 1 Robust and scalable preconditioner for (discretised) $A\boldsymbol{u} = \boldsymbol{b}$

$$P \ := \ R_0^T A_0^{-1} R_0 \left(I - \sum\nolimits_i R_i^T X_i A_i^{-1} R_i \right) \ + \ \sum\nolimits_i R_i^T X_i A_i^{-1} R_i$$

Theorem (Strehlow, Ma, **RS**, 2024; Ma, Alber, **RS**, Zhang, 2024)

Let $\Lambda := \kappa \frac{2C_a}{C_1} \max_i \lambda_{i,n+1}^{1/2} < 1$. The sequence $\{\mathbf{u}^j\}_{j \geq 1}$ of approximations generated by preconditioned GMRES (in the b-inner product) satisfies

$$\|PA(\mathbf{u} - \mathbf{u}^j)\|_b \leq \Lambda^j \left(\frac{1+\Lambda}{1-\Lambda} \right) C \|PA(\mathbf{u} - \mathbf{u}^0)\|_b, \quad \forall j \geq 1$$

where $C = 1$ for $b = a^+$ and $C = h^{-1} a_{\max}/a_{\min}$ for Euclidean inner product.

Recall. $\lambda_{i,n+1}^{1/2} = \mathcal{O}(e^{-cn^{1/d}}) \implies$ **Tuneable convergence rate!**

see also [Chartier, Falgout et al, *Spectral AMGe*, SISC, 2003]

Example 3: 3D convection-diffusion (high Peclet number)

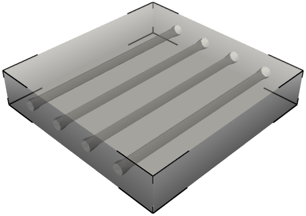
Iterative MS-GFEM (parallel RAS preconditioner) [Bastian, Holbach, **RS**, 2025]

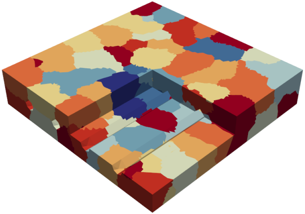
$$a(u,v) = \int_\Omega A\nabla u \cdot \nabla v + v\boldsymbol{b}\cdot\nabla u \,\mathrm{d}x \qquad \text{with} \quad \mathrm{Pe} = \frac{h\|\boldsymbol{b}\|_\infty}{a_{\min}} \gg 1$$

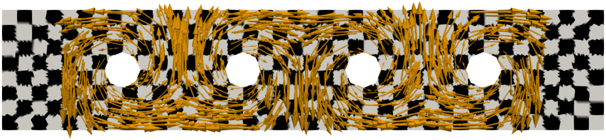
Example 3: 3D convection-diffusion (high Peclet number)

Iterative MS-GFEM (parallel RAS preconditioner) [Bastian, Holbach, **RS**, 2025]

$$a(u,v) = \int_\Omega A\nabla u \cdot \nabla v + v\boldsymbol{b}\cdot\nabla u \,\mathrm{d}x \qquad \text{with} \quad \mathrm{Pe} = \frac{h\|\boldsymbol{b}\|_\infty}{a_{\min}} \gg 1$$







- 64 subdomains, 2 016 256 dofs
- $u = 0$ on $\partial\Omega$
- $80 \times 40 \times 8$ checkerboard diffusion ($a_{\max} = 1$)
- rotating velocity field around holes

Example 3: 3D convection-diffusion (high Peclet number)

Iterative MS-GFEM (parallel RAS preconditioner) [Bastian, Holbach, **RS**, 2025]

	GAMG		BoomerAMG		PoU (cst+lin)		MS-GFEM		
$a_{\min}$	#IT	t_{total}	#IT	t_{total}	#IT	t_{total}	#IT	t_{setup}	t_{total}
10^{-3}	2 061	52.7	165	5.7	179	70.9	13	78.1	81.6
10^{-4}	-	-	-	-	365	134.3	13	77.1	80.7
10^{-5}	-	-	-	-	-	-	12	77.4	80.4
10^{-6}	-	-	-	-	-	-	13	77.6	81.5

- Preconditioned GMRES, MPI-parallel on dual AMD474 EPYC 7713
- 2 layers overlap + 1 oversampling (3 overlap for PoU coarse space)

Example 3: 3D convection-diffusion (high Peclet number)

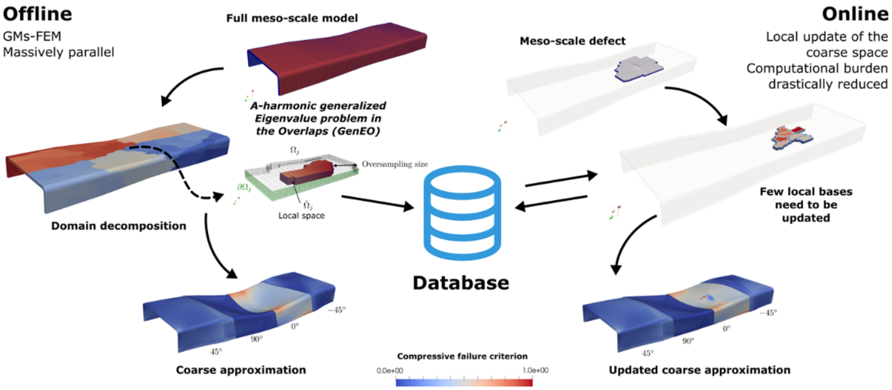
Iterative MS-GFEM (parallel RAS preconditioner) [Bastian, Holbach, **RS**, 2025]

	GAMG		BoomerAMG		PoU (cst+lin)		MS-GFEM		
a_{min}	#IT	t_{total}	#IT	t_{total}	#IT	t_{total}	#IT	t_{setup}	t_{total}
10^{-3}	2 061	52.7	165	5.7	179	70.9	13	78.1	81.6
10^{-4}	-	-	-	-	365	134.3	13	77.1	80.7
10^{-5}	-	-	-	-	-	-	12	77.4	80.4
10^{-6}	-	-	-	-	-	-	13	77.6	81.5

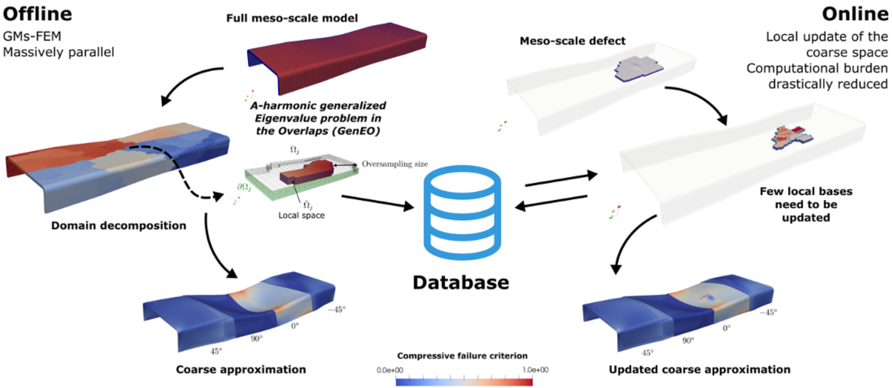
- Preconditioned GMRES, MPI-parallel on dual AMD474 EPYC 7713
- 2 layers overlap + 1 oversampling (3 overlap for PoU coarse space)
- Setup cost can be significantly reduced by formulating *eigenproblems on rings* instead of full subdomains $\rightsquigarrow$ speed-up by factor 10!

[Alber, Bastian, Hauck, **RS**, 2025]

Offline–Online Method for UQ



Offline–Online Method for UQ



Can use now to study **localised wrinkling defects**

Studying Impact of Localised Wrinkling Defects

Strain field variations (pristine vs. defective)

On **1 CPU core**: 2-3min to recompute local bases & 16s for coarse solve

Parametric Setting – Localized MOR

BUT in a **digital twin** (or in many other target applications):

- Do not want to compute $V_H(y)$ again for **each new** value of y (solving an eigenproblem in each ω_i^* is **very costly**).
- Instead would like the method to **"generalise"** to new parameters y with **theoretical guarantees** (ideally retaining exponential convergence)
- **Utopian?** **Maybe not!**

Parametric Setting – Localized MOR

BUT in a **digital twin** (or in many other target applications):

- Do not want to compute $V_H(y)$ again for **each new** value of y (solving an eigenproblem in each ω_i^* is **very costly**).
- Instead would like the method to **“generalise”** to new parameters y with **theoretical guarantees** (ideally retaining exponential convergence)
- **Utopian?** **Maybe not!**

Localized Model Order Reduction

e.g., survey paper [Buhr, Iapichino, Ohlberger, Rave, Schindler, Smetana, 2021]
[Maday, Ronquist, Hetmaniuk, Lehoucq, Efendiev, Galvis, Hou, Patera, Rozza,. . .]

Parametric Setting – Localized MOR

BUT in a **digital twin** (or in many other target applications):

- Do not want to compute $V_H(y)$ again for **each new** value of y (solving an eigenproblem in each ω_i^* is **very costly**).
- Instead would like the method to **"generalise"** to new parameters y with **theoretical guarantees** (ideally retaining exponential convergence)
- **Utopian?** **Maybe not!**

Localized Model Order Reduction (with optimal local spaces)

e.g., survey paper [Buhr, Iapichino, Ohlberger, Rave, Schindler, Smetana, 2021]
[Maday, Ronquist, Hetmaniuk, Lehoucq, Efendiev, Galvis, Hou, Patera, Rozza,. . .]
[Patera, Smetana, 2016], [Buhr,Smetana, 2019]

Parametric Setting – Localized MOR

BUT in a **digital twin** (or in many other target applications):

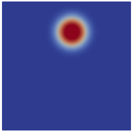
- Do not want to compute $V_H(y)$ again for **each new** value of y (solving an eigenproblem in each ω_i^* is **very costly**).
- Instead would like the method to **"generalise"** to new parameters y with **theoretical guarantees** (ideally retaining exponential convergence)
- **Utopian? Maybe not!**

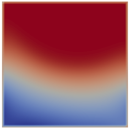
Localized Model Order Reduction (with optimal local spaces)

e.g., survey paper [Buhr, Iapichino, Ohlberger, Rave, Schindler, Smetana, 2021] [Maday, Ronquist, Hetmaniuk, Lehoucq, Efendiev, Galvis, Hou, Patera, Rozza,. . .] [Patera, Smetana, 2016], [Buhr,Smetana, 2019]

+ Significantly reduce effective dimension + cost (locally active params.)
− One "average" space $\overline{V}_H$ (size?) + **no full** theoretical guarantees!

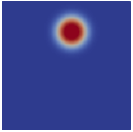
Example Diffusion $a_y(u, v) = \int_\Omega A(y; x)\nabla u \cdot \nabla v \, dx$

$A(y; x)|_{\omega_i^*} =$ 

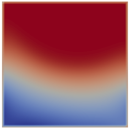
$S_n(\omega_i^*) = \mathrm{span}\Big\{ \mathbf{1}_{\omega_i^*},$  $,$  $, \cdots \Big\}$

$\max_x A = 10^4$, $\min_x A = 1$

Example Diffusion $a_y(u,v) = \int_\Omega A(y;x)\nabla u \cdot \nabla v \, dx$

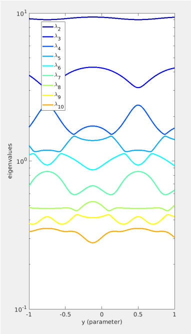
$A(y;x)|_{\omega_i^*} =$ 

$\max_x A = 10^4, \; \min_x A = 1$

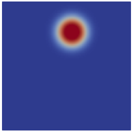
$S_n(\omega_i^*) = \mathrm{span}\Big\{ \mathbf{1}_{\omega_i^*},$  $,$  $, \cdots \Big\}$

Now **vary parameter** y = location of Gaussian bump:

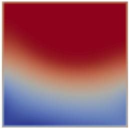
Eigenvalues



Example Diffusion $a_y(u,v) = \int_\Omega A(y;x)\nabla u \cdot \nabla v \, dx$

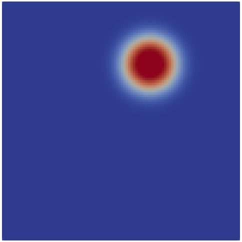
$A(y;x)|_{\omega_i^*} =$ 

$\max_x A = 10^4, \; \min_x A = 1$

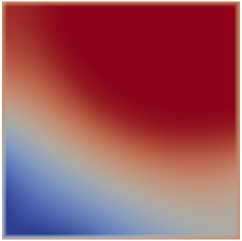
$S_n(\omega_i^*) = \text{span}\Big\{ \mathbf{1}_{\omega_i^*},$  $,$  $, \cdots \Big\}$

Now **vary parameter** y = location of Gaussian bump:

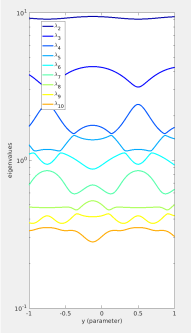
Coefficient



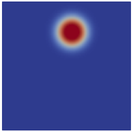
Dominant eigenfunction



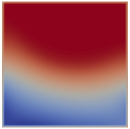
Eigenvalues



Example Diffusion $a_y(u,v) = \int_\Omega A(y;x)\nabla u \cdot \nabla v \, dx$

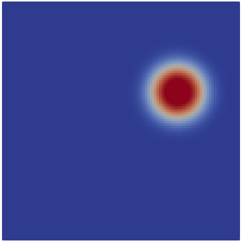
$A(y;x)|_{\omega_i^*} =$ 

$\max_x A = 10^4, \ \min_x A = 1$

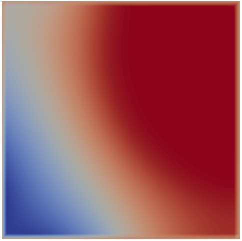
$S_n(\omega_i^*) = \mathrm{span}\Big\{ \mathbf{1}_{\omega_i^*},$  $,$  $, \cdots \Big\}$

Now **vary parameter** y = location of Gaussian bump:

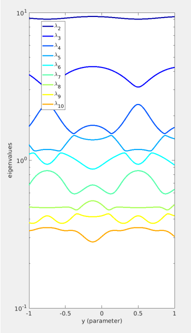
Coefficient



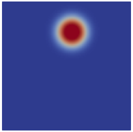
Dominant eigenfunction



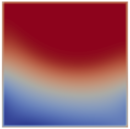
Eigenvalues



Example Diffusion $a_y(u, v) = \int_\Omega A(y; x)\nabla u \cdot \nabla v \, dx$

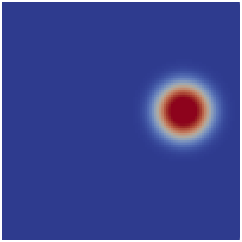
$A(y; x)|_{\omega_i^*} =$ 

$\max_x A = 10^4, \; \min_x A = 1$

$S_n(\omega_i^*) = \text{span}\Big\{ \mathbf{1}_{\omega_i^*},$  ,  $, \cdots \Big\}$

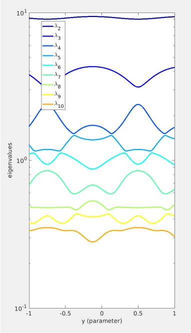
Now **vary parameter** y = location of Gaussian bump:

Coefficient



Dominant eigenfunction

Eigenvalues

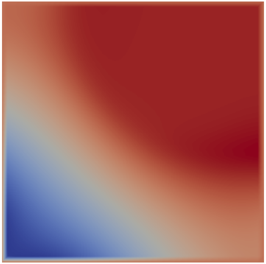


Comparison of Projection Errors (diffusion example)

How does $S_2(\omega_i^*)$ vary with y? Can we approximate it from snapshots?

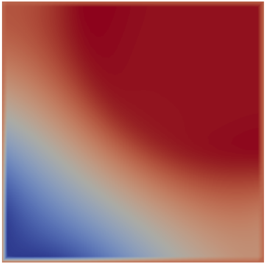
2 snapshots / interpolation points

PCA (dim=3)



$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.908$

Grassmannian interpol. (dim=2)



$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.951$

$\Pi_{\widetilde{S}}$ = projection to approximate space $\widetilde{S}$, v_2 = exact 2nd eigenfunction

Comparison of Projection Errors (diffusion example)

How does $S_2(\omega_i^*)$ vary with y? Can we approximate it from snapshots?

4 snapshots / interpolation points

PCA (dim=5)

$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.304$

Grassmannian interpol. (dim=2)

$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.498$

$\Pi_{\widetilde{S}}$ = projection to approximate space $\widetilde{S}$, v_2 = exact 2nd eigenfunction

Comparison of Projection Errors (diffusion example)

How does $S_2(\omega_i^*)$ vary with y? Can we approximate it from snapshots?

6 snapshots / interpolation points

PCA (dim=7)

$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.077$

Grassmannian interpol. (dim=2)

$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.134$

$\Pi_{\widetilde{S}}$ = projection to approximate space $\widetilde{S}$, v_2 = exact 2nd eigenfunction

Comparison of Projection Errors (diffusion example)

How does $S_2(\omega_i^*)$ vary with y? Can we approximate it from snapshots?

8 snapshots / interpolation points

PCA (dim=9)

$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.010$

Grassmannian interpol. (dim=2)

$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.017$

$\Pi_{\widetilde{S}}$ = projection to approximate space $\widetilde{S}$, v_2 = exact 2nd eigenfunction

Comparison of Projection Errors (diffusion example)

How does $S_2(\omega_i^*)$ vary with y? Can we approximate it from snapshots?

10 snapshots / interpolation points

PCA (dim=11)	Grassmannian interpol. (dim=2)
	
$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.007$	$\|(I - \Pi_{\widetilde{S}})v_2\|_a / \|v_2\|_a = 0.011$

Exploiting holomorphic dependence on y !

Grassmannian Interpolation of ROMs

[Amsallem, Farhat, AIAA Journal, 2008]

Parametric family M of all n-dimensional local approximation spaces is the so-called Grassmannian manifold.

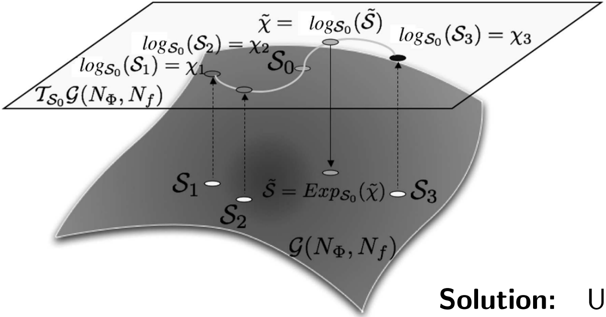
Main problem: M is not a linear vector space. How to interpolate on M?

Grassmannian Interpolation of ROMs

[Amsallem, Farhat, AIAA Journal, 2008]

Parametric family M of all n-dimensional local approximation spaces is the so-called Grassmannian manifold.

Main problem: M is not a linear vector space. How to interpolate on M?



Solution: Use the tangent space to M at a basepoint S_0 to interpolate (use $\log_{S_0}$ / $\exp_{S_0}$ to move between them)

Rigorous Error Analysis: A Roadmap (coercive case)
[Alber, Bachmayr, **RS**, Yang, 2025+]

Assume gap in the spectrum $\lambda_{n+1}/\lambda_n \leq \sigma < 1$ uniform in y.

1. In that case, extension to higher dimensions of perturbation theory in [Kato, 2013] gives holomorphic dependence of $S_n(\omega_i^*)$ on y.
2. Polynomial interpolation of holomorphic functions (in tangent space = vector space) converges exponentially in 1D [Trefethen, 2019]

Rigorous Error Analysis: A Roadmap (coercive case)

[Alber, Bachmayr, **RS**, Yang, 2025+]

Assume gap in the spectrum $\lambda_{n+1}/\lambda_n \leq \sigma < 1$ uniform in y.

1. In that case, extension to higher dimensions of perturbation theory in [Kato, 2013] gives holomorphic dependence of $S_n(\omega_i^*)$ on y.
2. Polynomial interpolation of holomorphic functions (in tangent space = vector space) converges exponentially in 1D [Trefethen, 2019]
3. and nearly exponentially/algebraically in higher dimensions using anisotropic sparse grids [Nobile, Tempone, Webster, 2008].

Rigorous Error Analysis: A Roadmap (coercive case)

[Alber, Bachmayr, **RS**, Yang, 2025+]

Assume gap in the spectrum $\lambda_{n+1}/\lambda_n \leq \sigma < 1$ uniform in y.

1. In that case, extension to higher dimensions of perturbation theory in [Kato, 2013] gives holomorphic dependence of $S_n(\omega_i^*)$ on y.
2. Polynomial interpolation of holomorphic functions (in tangent space = vector space) converges exponentially in 1D [Trefethen, 2019]
3. and nearly exponentially/algebraically in higher dimensions using anisotropic sparse grids [Nobile, Tempone, Webster, 2008].
4. Grassmannnian has nonnegative sectional curvature and thus the interpolation error after retraction back to the manifold M (via $\exp_{S_0}$) is less than or equal the interpolation error on the tangent space [Jacobsson, Vandebril, van der Veken, Vannieuwenhoven, 2025].

Rigorous Error Analysis: A Roadmap (coercive case)

[Alber, Bachmayr, **RS**, Yang, 2025+]

Assume gap in the spectrum $\lambda_{n+1}/\lambda_n \leq \sigma < 1$ uniform in y.

1. In that case, extension to higher dimensions of perturbation theory in [Kato, 2013] gives holomorphic dependence of $S_n(\omega_i^*)$ on y.
2. Polynomial interpolation of holomorphic functions (in tangent space = vector space) converges exponentially in 1D [Trefethen, 2019]
3. and nearly exponentially/algebraically in higher dimensions using anisotropic sparse grids [Nobile, Tempone, Webster, 2008].
4. Grassmannnian has nonnegative sectional curvature and thus the interpolation error after retraction back to the manifold M (via $\exp_{S_0}$) is less than or equal the interpolation error on the tangent space [Jacobsson, Vandebril, van der Veken, Vannieuwenhoven, 2025].
5. Additional interpolation error can be estimated using cosine of angle between subspaces [Lipton, Sinz, Stübner, 2022].

Local approximation error

Theorem (Alber, Bachmayr, **RS**, Yang, 2025+) [Informal]

Assume that $\lambda_{n+1}/\lambda_n \geq \sigma > 1$ uniformly in y and let

$$\cos^2(\beta^a(y)) := \inf_{v \in S_n(y)} \frac{a_{\omega_i^*}(v, \Pi^a_{\widetilde{S}(y)} v)}{a_{\omega_i^*}(v, v)},$$

such that $\beta^a(y)$ is the angle (w.r.t. $a_{\omega_i^*}$) between the interpolated and the local approximation subspace. Then

$$\sup_{u \in H_{A(y)}(\omega_i^*)} \inf_{v \in \widetilde{S}(y)} \frac{\|u - v\|_{a,\omega_i^*}}{\|u\|_{a,\omega_i^*}} \leq \left(1 + \tfrac{1}{\cos \beta^a(y)}\right) \lambda_{n+1}^{1/2} \lesssim \left(1 + e^{-bN^\alpha}\right) e^{-cn^{1/d}}$$

where N is the number of interpolation points.

Local approximation error

Theorem (Alber, Bachmayr, **RS**, Yang, 2025+) [Informal]

Assume that $\lambda_{n+1}/\lambda_n \geq \sigma > 1$ uniformly in y and let

$$\cos^2(\beta^a(y)) := \inf_{v \in S_n(y)} \frac{a_{\omega_i^*}(v, \Pi^a_{\widetilde{S}(y)} v)}{a_{\omega_i^*}(v, v)},$$

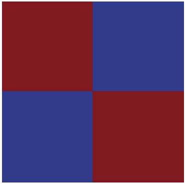
such that $\beta^a(y)$ is the angle (w.r.t. $a_{\omega_i^*}$) between the interpolated and the local approximation subspace. Then

$$\sup_{u \in H_{A(y)}(\omega_i^*)} \inf_{v \in \widetilde{S}(y)} \frac{\|u - v\|_{a,\omega_i^*}}{\|u\|_{a,\omega_i^*}} \leq \left(1 + \frac{1}{\cos \beta^a(y)}\right) \lambda_{n+1}^{1/2} \lesssim \left(1 + e^{-bN^\alpha}\right) e^{-cn^{1/d}}$$

where N is the number of interpolation points.

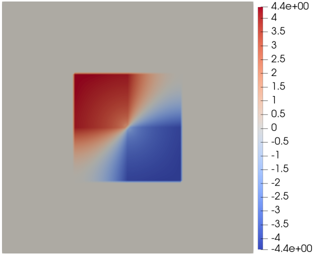
Note that for the theory we had to change the MS-GFEM eigenproblem slightly and use $P_i : H^{1/2}(\partial\omega_i^*) \to H_0^1(\omega_i)$ defined by $u \mapsto \chi_i E_a u$, where E_a is the a-harmonic extension from $H^{1/2}(\partial\omega_i^*)$ to $H_a(\omega_i^*)$.

Numerical Experiment (diffusion example with cross point)

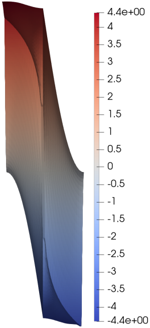


with $\quad A(y;x)|_{\omega_i^*} = \begin{cases} 1 - 0.99y, & \text{in red squares,} \\ 1 + 0.99y, & \text{in blue squares.} \end{cases}$

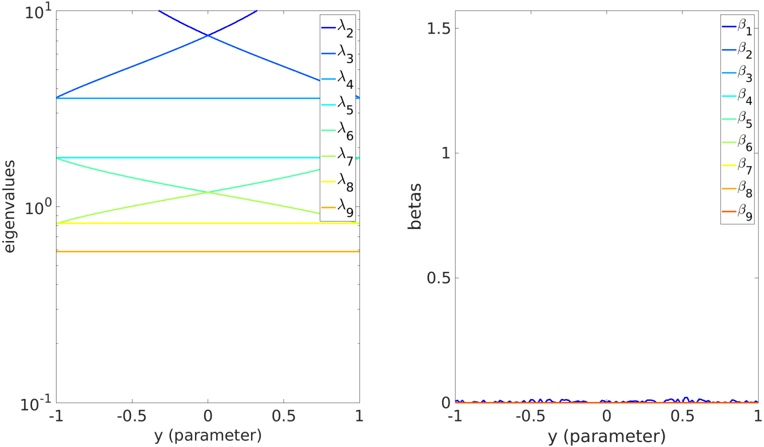
$v_2(-1) \;=$



$=$



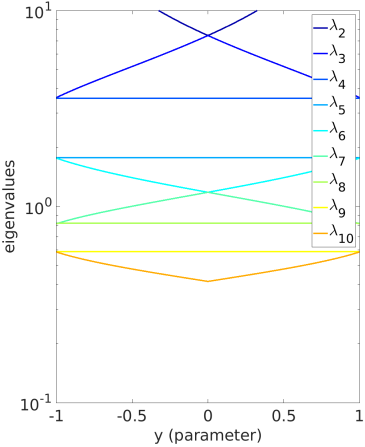
Numerical Experiment (diffusion example with cross point)

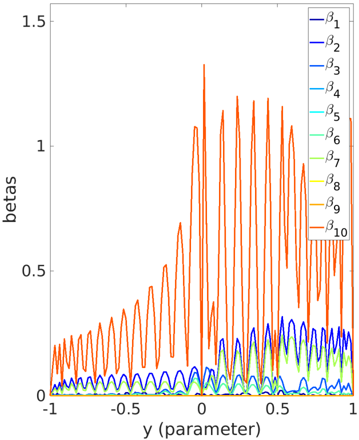


Angles for $S_9(\omega_i^*)$ (uniform gap between λ_9 and λ_{10})

(base point $S_n(\omega_i^*)$ at $y = -1$)

Numerical Experiment (diffusion example with cross point)

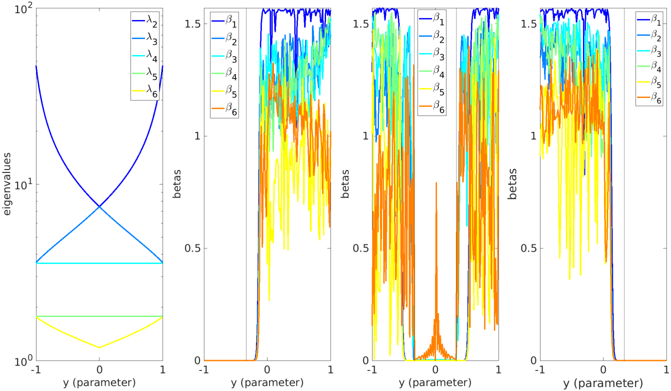




Angles for $S_{10}(\omega_i^*)$ (**no** uniform gap between λ_{10} and λ_{11})

(base point $S_n(\omega_i^*)$ at $y = -1$)

To deal with **eigenvalue crossings** use multiple base points :

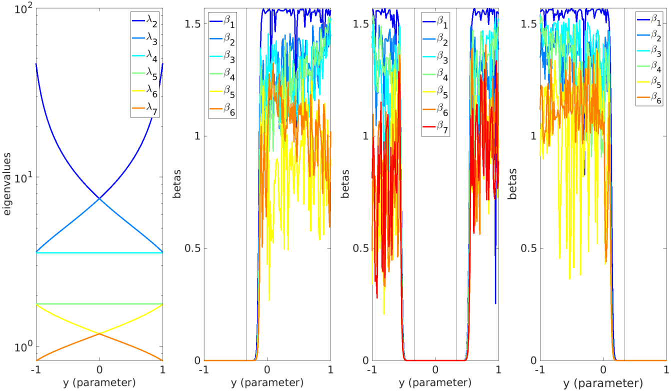


Angles for $S_6(\omega_i^*)$ (**<u>no</u>** uniform eigenvalue gap in middle)

(base points at $y = -1, 0, 1$)

Numerical Experiment (diffusion example with cross point)

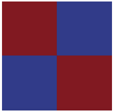
To deal with **eigenvalue crossings** use multiple base points + variable n:



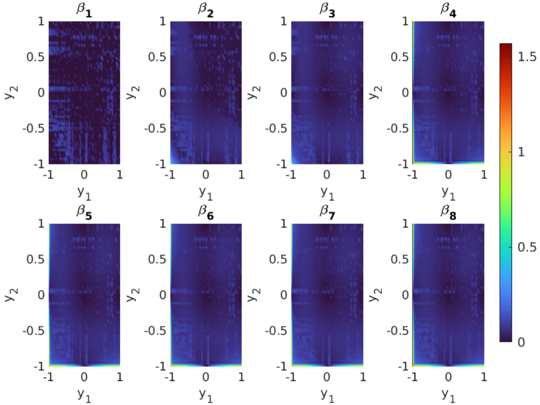
Angles for $S_6(\omega_i^*), S_7(\omega_i^*), S_6(\omega_i^*)$ (uniform eigenvalue gaps)

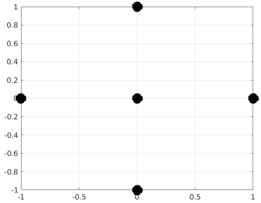
(base points at $y = -1, 0, 1$)

Numerical Experiment (two parameter cross point)



with $$A(y;x)|_{\omega_i^*} = \begin{cases} 1, & \text{in red squares}, \\ 1 + 0.99 y_1, & \text{in top right}, \\ 1 + 0.99 y_2, & \text{in bottom left}. \end{cases}$$

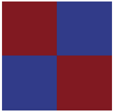




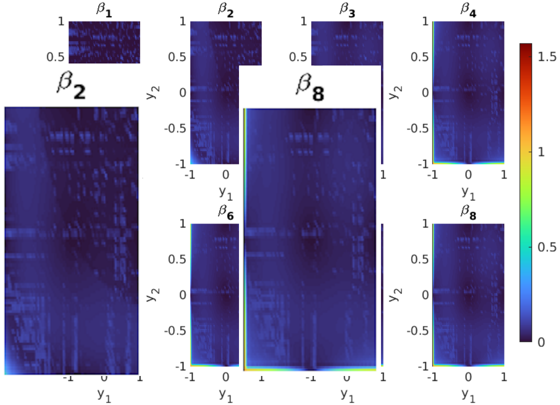
Clenshaw-Curtis sparse grid

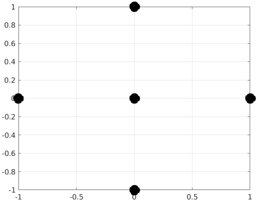
$S_8(\omega_i^*)$

Numerical Experiment (two parameter cross point)



with $$A(y;x)|_{\omega_i^*} = \begin{cases} 1, & \text{in red squares}, \\ 1 + 0.99 y_1, & \text{in top right}, \\ 1 + 0.99 y_2, & \text{in bottom left}. \end{cases}$$

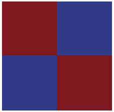




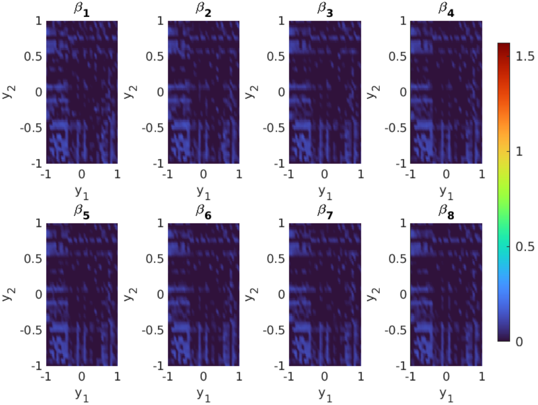
Clenshaw-Curtis sparse grid

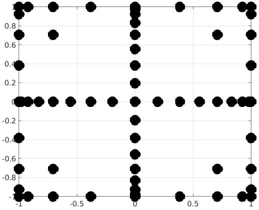
$S_8(\omega_i^*)$

Numerical Experiment (two parameter cross point)



with $A(y;x)|_{\omega_i^*} = \begin{cases} 1, & \text{in red squares}, \\ 1 + 0.99 y_1, & \text{in top right}, \\ 1 + 0.99 y_2, & \text{in bottom left}. \end{cases}$

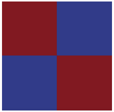




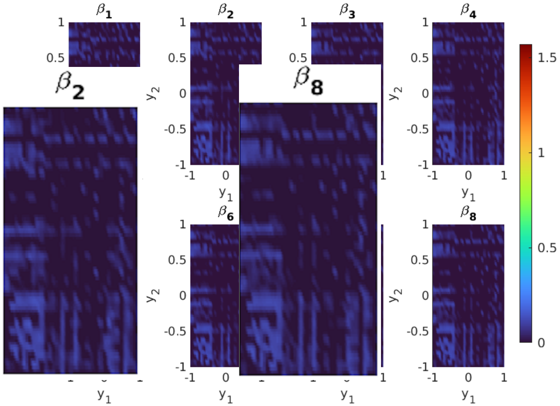
Clenshaw-Curtis sparse grid

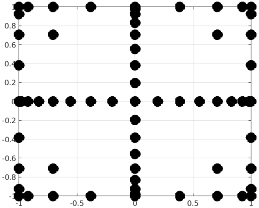
$S_8(\omega_i^*)$

Numerical Experiment (two parameter cross point)



with $A(y;x)|_{\omega_i^*} = \begin{cases} 1, & \text{in red squares}, \\ 1+0.99y_1, & \text{in top right}, \\ 1+0.99y_2, & \text{in bottom left}. \end{cases}$

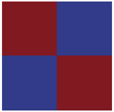




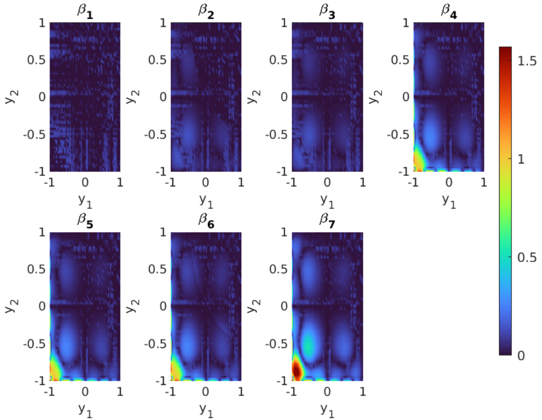
Clenshaw-Curtis sparse grid

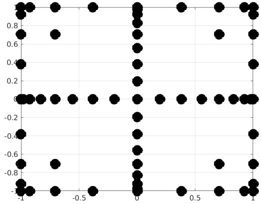
$S_8(\omega_i^*)$

Numerical Experiment (two parameter cross point)



with $\quad A(y;x)|_{\omega_i^*} = \begin{cases} 1, & \text{in red squares}, \\ 1 + 0.99 y_1, & \text{in top right}, \\ 1 + 0.99 y_2, & \text{in bottom left}. \end{cases}$





Clenshaw-Curtis
sparse grid

$S_7(\omega_i^*)$

Conclusions and Further Comments

- **Robust multiscale coarse space** via generalised FEM and **optimal** spectral local approximation spaces (w/o a priori regularity assumptions)
- **Crucial** to use **oversampling** and a-**harmonicity** (weakly enforced)
- Rigorous analysis for many generic **heterogeneous** PDEs, incl.
  - (non-)linear elasticity in composites [Benezech et al, 2024 & 2025+]
  - (high-Peclet) convection-diffusion [Bastian, Holbach, RS, 2025+]
  - (high-frequency) heterog. Helmholtz [Ma, Alber, RS, 2023 & 2024]
  - heterogeneous Stokes [Alber, Ma, RS, 2025+]
- General **abstract framework** in [Ma, FoCM, 2025]

Conclusions and Further Comments

- **Robust multiscale coarse space** via generalised FEM and **optimal** spectral local approximation spaces (w/o a priori regularity assumptions)
- **Crucial** to use **oversampling** and *a*-**harmonicity** (weakly enforced)
- Rigorous analysis for many generic **heterogeneous** PDEs, incl.
  - (non-)linear elasticity in composites [Benezech et al, 2024 & 2025+]
  - (high-Peclet) convection-diffusion [Bastian, Holbach, RS, 2025+]
  - (high-frequency) heterog. Helmholtz [Ma, Alber, RS, 2023 & 2024]
  - heterogeneous Stokes [Alber, Ma, RS, 2025+]
- General **abstract framework** in [Ma, FoCM, 2025]

Current and Future Work

- Nonlinear problems [Benzech et al, 2025+]
- Efficient Offline/Online implementation (e.g. for UQ) and adaptivity
- Complete theory for Grassmann interpolation [Alber et al, 2025+]
- Operator learning-based surrogates (again enforcing geometry)