Learning dynamical systems from frequency-response data

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Reduced Order and Surrogate Modeling for Digital Twins

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Input Load Estimation for Deployable Space Structures

Need for CFRP Booms in Deployable Space Structures:

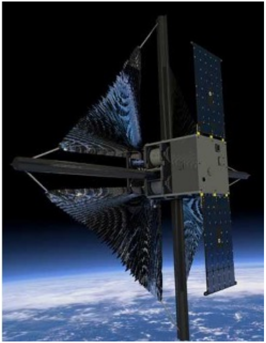
- CFRP Boom are viable alternative to massive truss structures for deploying optical and communication systems accurately and can assist in opening solar sails
- lightweight and cheaper than standard metallic booms, reduce the fuel mass requirements

Need for Modeling Dynamics of Deployable Booms:

- Passive deployments can be violent and unpredictable, can impact satellite dynamics
- Performance prediction and ensuring reliability in harsh environmental conditions
- Mitigating risk of damage to the satellite components

Difficulties in Modeling:

- Uncertainties in mass and stiffness
- Non-uniform cross section and shape with deployer boundary conditions

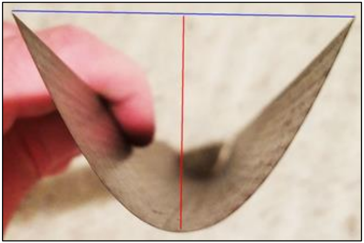


Advanced Composite Solarsail System (ACS)-3

- Risks of uncontrolled deployments:
  - Excessive vibration & resonance excitation
  - Coupling with spacecraft attitude control
  - Sensor/instrument misalignment
  - Fatigue and long-term reliability issues
  - Interference with other spacecraft elements
- Measuring deployment forces directly is often impractical due to challenges in placing a force sensor precisely at the point of force application.
- Advantages of prediction of these deployment forces:
  - Protect structure & avoid dynamic failure
  - Accurate controller design & safe deployment sequencing

Boom under study

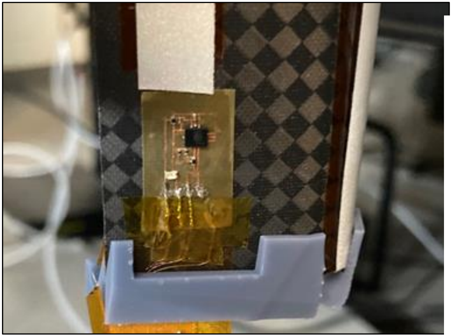
- Bi-stable CFRP boom with parabolic cross section
- Composite layup properties [45PWc/0C/-45PWc] as shown in table below.
- Dimensions:
    - Flattened Width and Coil Height: 70 mm
    - Bistable Coil Diameter: ~78.3 mm
    - Thickness: 0.17 mm
    - Parabola Tip Separation: 52.63 mm
    - Parabola Height: 31.56 mm
    - Length: 1219.2 mm (4 ft)


Boom geometry

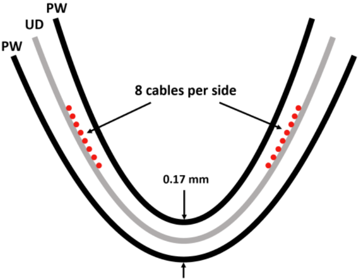
Ply Material	Fiber/Resin	E_1 [GPa]	E_2 [GPa]	ν_{12}	G_{12} [GPa]	Thickness t [μm]
Unidirectional Carbon Fiber	MR60H/PMT-F7	144.1	5.2	0.335	2.8	40.0
Plain Weave Carbon Fiber	M30S/PMT-F7	89.0	89.0	0.035	4.2	58.2

Boom layup properties

Boom under study

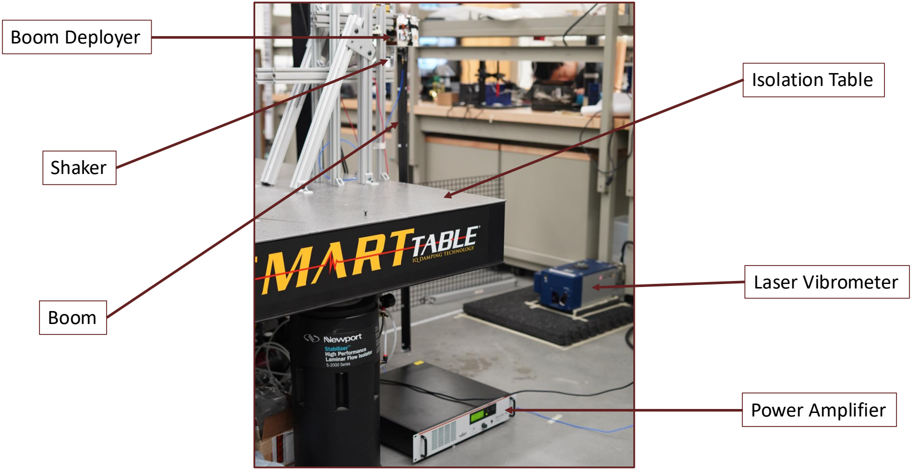


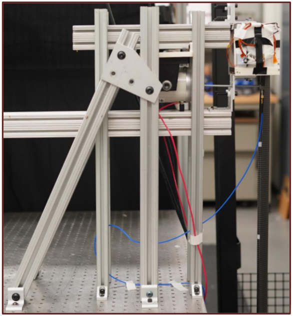
Boom tip IMU circuit



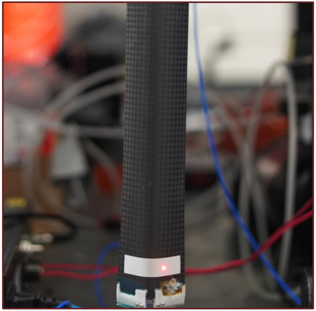
Cross-section

Experimental Setup

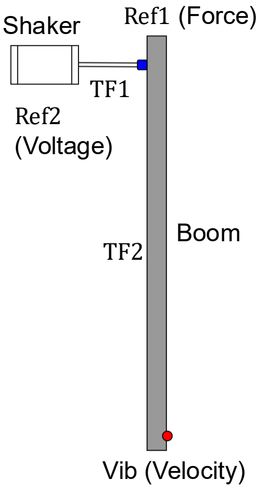




Side view showing shaker



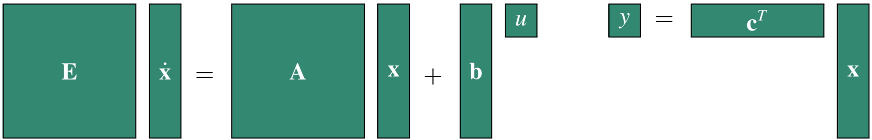
Boom Tip Velocity Measurement



Linear dynamical systems with inputs and outputs

$$\mathcal{S}: \qquad u(t) \longrightarrow \boxed{\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t) \\ y(t) &= \mathbf{c}^T\mathbf{x}(t) \end{aligned}} \longrightarrow y(t)$$

1. $\mathbf{x}(t) \in \mathbb{R}^n$: Internal degrees of freedom (position and velocity)
2. $u(t) \in \mathbb{R}$: input (force due to the end of deployment shock)
3. $y(t) \in \mathbb{R}$: output (velocity at the tip (near IMU))
4. $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n \times n}, \quad \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$



Data-driven modeling for dynamical systems

- In many instances, we only have

$$u(t) \longrightarrow \boxed{\text{Black-box dynamical system}} \longrightarrow y(t)$$

- Dynamics are not available; only access to input/output $(u(t)/y(t))$ data.

Construct
$$\boxed{\begin{aligned} \mathbf{E}_r \dot{\mathbf{x}}_r(t) &= \mathbf{A}_r \mathbf{x}_r(t) + \mathbf{b}_r u(t) \\ y_r(t) &= \mathbf{c}_r^T \mathbf{x}_r(t) \end{aligned}}$$
directly from input/output data

- Learn the reduced order operators $(\mathbf{E}_r, \mathbf{A}_r, \mathbf{b}_r, \mathbf{c}_r)$ (without access to the full-order operators $(\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c})$) such that $y_r(t) \approx y(t)$ for a wide range of input functions

- Data in this talk: *Frequency-domain samples*

Outline of the talk

- Data-driven rational least-square approximants
- Application to boom deployment:

  With Deven Mhadgut and Austin Phoenix from VT
- Learning parametric dynamical systems from data:

  With Andrea Carracedo Rodriguez and Linus Balicki from VT

Rational approximants for learning dynamical systems

$$\boxed{\begin{array}{c}\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) = \mathbf{c}^T\mathbf{x}(t)\end{array}} \Longrightarrow y(t) = (\mathcal{S}u)(t) = \int_0^t h(t-\tau)u(\tau)d\tau$$

- $h(t) = \mathbf{c}^T e^{\mathbf{A}t}\mathbf{b}$ when $\mathbf{E} = \mathbf{I}$.
- Transform the problem into the frequency domain via Laplace transform:

$$H(s) = \int_0^\infty h(t)e^{-st}dt = \mathbf{c}^T(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}$$

- $H(s)$ is called the transfer function. In the frequency domain, we have

$$\hat{y}(\omega) = H(\imath\omega)\hat{u}(\omega)$$

- $H(s)$ is a degree-n rational function:

$$H(s) = \mathbf{c}^T(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{b} = \frac{p_0 + p_1 s + p_2 s^2 + \cdots + p_{n-1}s^{n-1}}{1 + q_1 s + +q_2 s^2 + \cdots + q_n s^n}$$

Enforce the same structure in the learned model

$$\boxed{\begin{aligned} \mathbf{E}_r \dot{\mathbf{x}}_r(t) &= \mathbf{A}_r \mathbf{x}_r(t) + \mathbf{b}_r u(t) \\ y_r(t) &= \mathbf{c}_r^T \mathbf{x}_r(t) \end{aligned}}$$

- Enforce the learned-transfer function to be a degree-r rational function:

$$H_r(s) = \mathbf{c}_r{}^T (s\mathbf{E}_r - \mathbf{A}_r)^{-1} \mathbf{b}_r = \frac{\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \cdots + \alpha_{r-1} s^{r-1}}{1 + \beta_1 s + +\beta_2 s^2 + \cdots + +\beta_r s^r}$$

Original model: $\hat{y}(\omega) = H(\imath\omega)\hat{u}(\omega)$ Learned model: $\hat{y}_r(\omega) = H_r(\imath\omega)\hat{u}(\omega)$

- The error is determined by the mismatch between $H(s)$ and $H_r(s)$:

$$\|y - y_r\| \leq \|H - H_r\| \, \|u\|$$

- Constructing $H_r(s)$ becomes a rational approximation problem

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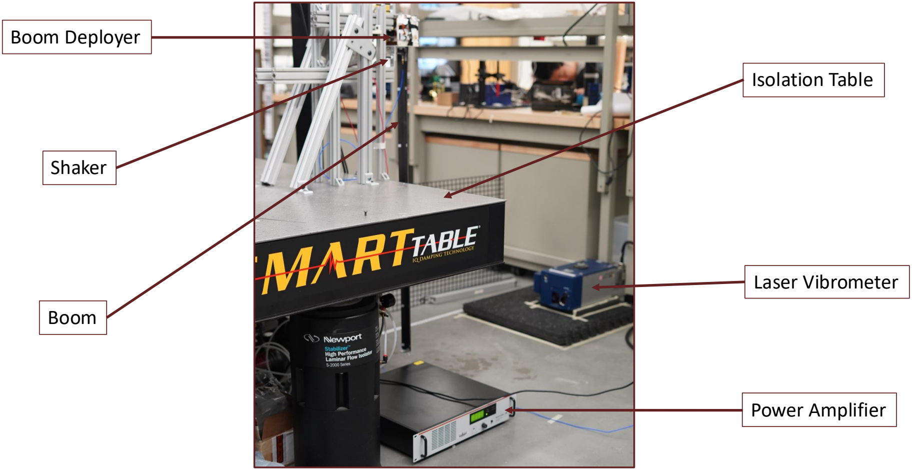
Goal: Given the samples $\{H(s_1), H(s_2), \ldots, H(s_N)\}$, construct/learn $H_r(s)$

Recall: 3D Laser Vibrometer Measurements



Rational least-squares fitting for learning dynamics

- Experiments measure the transfer function at the IMU tip location (excitation from the input force)

$$H(s) \text{ has a single input and single output}$$

- Input: Force. Output: Velocity response at a single location
- Measurements are in the $[0, 100]$ Hz range.
- Overall $N = 12800$ samples

$$H(s_k) \in \mathbb{C}, \quad s_k = \imath\omega_k, \quad \text{for } \; k = 1, 2, \ldots, N$$

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Given $\{H(s_1), H(s_2), \ldots, H(s_N)\}$, construct $H_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$ such that the least-squares error is minimized:

$$\sum_{i=1}^{N} |H_r(s_i) - H(s_i)|^2 \longrightarrow \min.$$

Choosing the form of the rational approximant

$$\boxed{\begin{aligned} \mathbf{E}_r\,\dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\,\mathbf{x}_r(t) + \mathbf{b}_r\,u(t) \\ y_r(t) &= \mathbf{c}_r^T\,\mathbf{x}_r(t) \end{aligned}}$$

- We can learn $H_r(s)$ in many different forms: So far we have seen

$$H_r(s) = \underbrace{\mathbf{c}_r{}^T}_{1\times r}(s\,\underbrace{\mathbf{E}_r}_{r\times r} - \underbrace{\mathbf{A}_r}_{r\times r})^{-1}\,\underbrace{\mathbf{b}_r}_{r\times 1} = \frac{\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \cdots + \alpha_{r-1}s^{r-1}}{1 + \beta_1 s + +\beta_2 s^2 + \cdots + +\beta_r s^r}$$

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- We can also use the *pole-residue* form:

$$H_r(s) = \sum_{i=1}^{r}\frac{\psi_i}{s-\lambda_i}$$

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- Solve the nonlinear rational least-squares problem using the chosen form:

$$\sum_{i=1}^{N} |H_r(s_i) - H(s_i)|^2 \longrightarrow \min.$$

Barycentric form of the rational approximant

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$$H_r(s) = \frac{s+1}{s+2} = \frac{-4\,\dfrac{s-4}{\ell(s)} + 5\,\dfrac{s-3}{\ell(s)}}{-5\,\dfrac{s-4}{\ell(s)} + 6\,\dfrac{s-3}{\ell(s)}} = \frac{\dfrac{-4}{s-3} + \dfrac{5}{s-4}}{\dfrac{-5}{s-3} + \dfrac{6}{s-4}} = \frac{\displaystyle\sum_{j=1}^{2} \frac{\phi_j}{s-\sigma_j}}{\displaystyle\sum_{j=1}^{2} \frac{\varphi_j}{s-\sigma_j}}$$

- In the general case, define

$$\ell_j(s) = \prod_{\substack{k=1 \\ k\neq j}}^{r+1} (s-\sigma_k) \quad \text{and} \quad \ell(s) = \prod_{k=1}^{r+1} (s-\sigma_k)$$

- Then, write $H_r(s)$ in the barycentric form:

$$H_r(s) = \frac{p_r(s)}{q_r(s)} = \frac{\sum_{j=1}^{r+1} \phi_j \ell_j(s)}{\sum_{j=1}^{r+1} \varphi_j \ell_j(s)} = \frac{\sum_{j=1}^{r+1} \phi_j \frac{\ell_j(s)}{\ell(s)}}{\sum_{j=1}^{r+1} \varphi_j \frac{\ell_j(s)}{\ell(s)}} = \frac{\sum_{j=1}^{r+1} \frac{\phi_j}{s-\sigma_j}}{\sum_{j=1}^{r+1} \frac{\varphi_j}{s-\sigma_j}}$$

- Given the data $H(s_i)$, for $i = 1, \ldots, N$, find $\quad H_r(s) = \dfrac{n(s)}{d(s)} = \dfrac{\displaystyle\sum_{j=1}^{r} \frac{\phi_j}{s - \sigma_j}}{\displaystyle\sum_{j=1}^{r} \frac{\varphi_j}{s - \sigma_j} + 1}$

  with $r < N$ such that the least-squares error is minimized:

$$\sum_{i=1}^{N} |H_r(s_i) - H(s_i)|^2 = \sum_{i=1}^{N} \left| \frac{n(s_i) - d(s_i)H(s_i)}{d(s_i)} \right|^2 \longrightarrow \min .$$

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- Solve a sequence of linear least-squares problems ([Sanathanan/Koerner,63]:)

$$H_r^{(k)} = \frac{n^{(k)}(s)}{d^{(k)}(s)}, \qquad \sum_{i=1}^{N} \left| \frac{n^{(k+1)}(s_i) - d^{(k+1)}(s_i)H(s_i)}{d^{(k)}(s_i)} \right|^2 \longrightarrow \min.$$

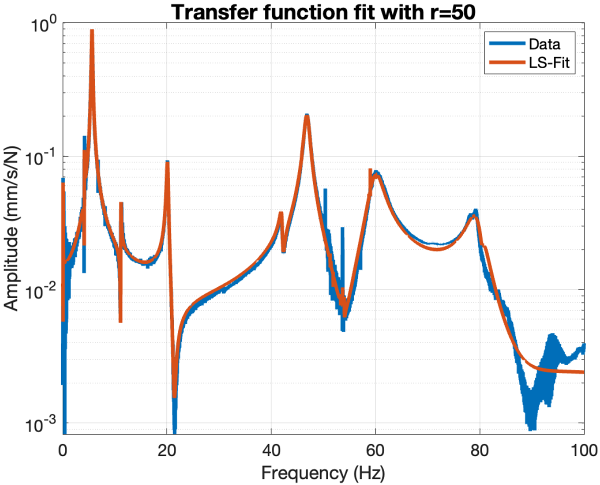
- At the k^{th} step: Solve

$$\|\Delta^{(k)}(\mathcal{A}x^{(k+1)} - h)\|_2 \to \min$$

- [Sanathanan/Koerner, 63], [Gustavsen/Semlyen, 99], [Drmač/G./Beattie, 15]

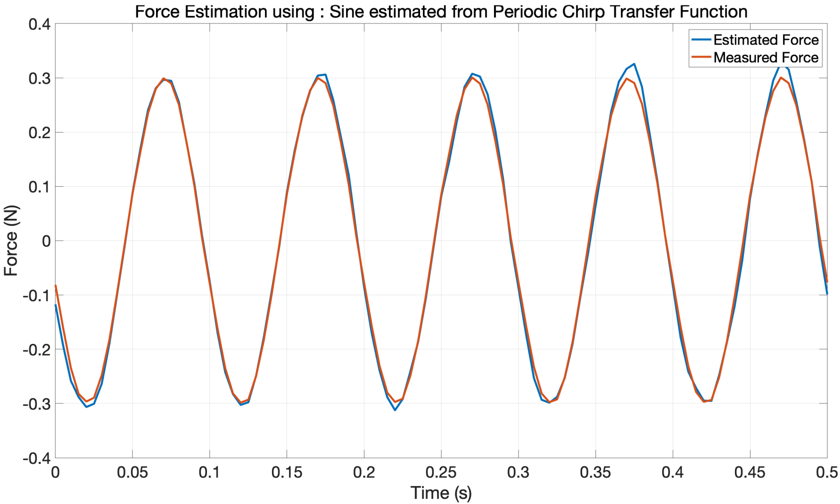
Back to Boom Deployment: Matching the frequency response

- Data: $H(s_k) \in \mathbb{C}$ $s_k = \imath\omega_k$, for $k = 1, 2, \ldots, 12800$
- We fit an order $r = 50$ rational function $H_r(s)$ to the data



Force Estimation

- “Invert" the observed velocity output $y(t)$ using the learn model to estimate the experimental (measured) forcing.



Interpolation via Barycentric form

- For the dynamical system $H(s)$, as before, assume access to its samples

$$h_i = H(s_i), \qquad s_i \in \mathbb{C}, \qquad \text{for} \ \ i = 1, \ldots, N.$$

- Build a rational function $H_r(s)$ that interpolates the data

$$H_r(s_i) = H(s_i) = h_i \ \ \text{for} \ \ \ i = 1, 2, \ldots, N.$$

- We will again use the barycentric form of a rational approximant:

$$H_r(s) = \frac{\displaystyle\sum_{j=1}^{k} \frac{\phi_j}{s - \sigma_j}}{\displaystyle\sum_{j=1}^{k} \frac{\varphi_j}{s - \sigma_j}} = \frac{n_{k-1}(s)}{d_{k-1}(s)} \qquad (\Longrightarrow H_r(s) = \mathbf{c}_r^\top (s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r, \ \ r = k - 1)$$

where $\sigma_j \in \mathbb{C}$ are the support (interpolation) points, a subset of the sampling points $\{s_1, \ldots, s_N\}$, and $\phi_j, \varphi_j \in \mathbb{C}$ to be determined.

Barycentric rational interpolation via Loewner matrices

$$H_r(s) = \sum_{j=1}^{k} \frac{\phi_j}{s-\sigma_j} \Bigg/ \sum_{j=1}^{k} \frac{\varphi_j}{s-\sigma_j}$$

- Construct $H_r(s)$ such that

$$H_r(s_i) = H(s_i) = h_i \quad \text{for} \quad i = 1, 2, \ldots, N.$$

- Partition the sampling points and the corresponding function values:

$$\begin{aligned} \text{sampling points:} \quad & \{s_1, \ldots, s_N\} = \{\sigma_1, \ldots, \sigma_k\} \cup \{\hat{\sigma}_1, \ldots, \hat{\sigma}_{N-k}\}, \\ \text{sampled values:} \quad & \{h_1, \ldots, h_N\} = \{g_1, \ldots, g_k\} \cup \{\hat{g}_1, \ldots, \hat{g}_{N-k}\}. \end{aligned}$$

- To enforce interpolation at $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$:

$$\phi_j = \varphi_j h_j, \quad \varphi_j \neq 0 \quad \Longrightarrow \quad H_r(s) = \sum_{j=1}^{k} \frac{\varphi_j h_j}{s-s_j} \Bigg/ \sum_{j=1}^{k} \frac{\varphi_j}{s-s_j}$$

$$\begin{aligned}\{s_1,\dots,s_N\} &= \{\sigma_1,\dots,\sigma_k\}\cup\{\hat\sigma_1,\dots,\hat\sigma_{N-k}\},\\ \{h_1,\dots,h_N\} &= \{g_1,\dots,g_k\}\cup\{\hat g_1,\dots,\hat g_{N-k}\}.\end{aligned}$$

$$H_r(s) = \sum_{j=1}^{k}\frac{\varphi_j h_j}{s-s_j}\Bigg/\sum_{j=1}^{k}\frac{\varphi_j}{s-s_j}$$

- To enforce interpolation at $\hat\sigma_i$, for $i = 1,2,\dots,N-k$:

$$H(\hat\sigma_i) = H_r(\hat\sigma_i) \implies \sum_{j=1}^{k}\frac{\hat g_i - g_j}{\hat\sigma_i - \sigma_k}\varphi_j = 0 \implies \text{Solve}\quad \mathbb{L}\,\mathbf{a} = \mathbf{0}$$

$$\text{where}\quad \mathbb{L} = \underbrace{\begin{bmatrix} \frac{\hat g_1 - g_1}{\hat\sigma_1-\sigma_1} & \cdots & \frac{\hat g_1 - g_k}{\hat\sigma_1-\sigma_k}\\ \vdots & \ddots & \vdots \\ \frac{\hat g_{N-k} - g_1}{\hat\sigma_{N-k}-\sigma_1} & \cdots & \frac{\hat g_{N-k} - g_k}{\hat\sigma_{N-k}-\sigma_k}\end{bmatrix}}_{(N-k)\times k\ \text{Loewner matrix}} \quad\text{and}\quad \mathbf{a} = \begin{bmatrix}\varphi_1\\ \vdots\\ \varphi_k\end{bmatrix}$$

- Under some mild assumptions, a unique rational interpolant of minimal degree is obtained ([Antoulas/Anderson, 86])

- Optimal selection of interpolation points: [G./Antoulas/Beattie, 08]

A hybrid approach: Interpolate and LS

- partition data:

$$\text{(sampling points)} \quad \{s_1,\ldots,s_N\} = \{\sigma_1,\ldots,\sigma_k\} \;\cup\; \{\hat{\sigma}_1,\ldots,\hat{\sigma}_{N-k}\}$$
$$\text{(sampled values)} \quad \{h_1,\ldots,h_N\} = \{g_1,\ldots,g_k\} \;\cup\; \{\hat{g}_1,\ldots,\hat{g}_{N-k}\}$$

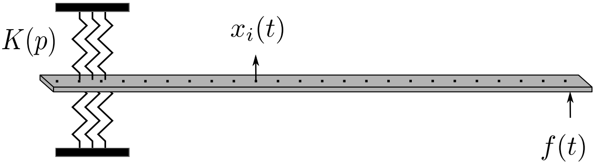
- Combine interpolation and LS:

  Adaptive Anderson-Antoulas (AAA) [Nakatsukasa/Sete/Trefethen, 2018]

$$H_r(s) = \sum_{j=1}^{k} \frac{\varphi_j g_i}{s-\sigma_i} \Bigg/ \sum_{j=1}^{k} \frac{\varphi_j}{s-\sigma_i}$$

- Interpolation on a subset of data
- φ_j: via LS fit on the rest of data: Solve $\min \|\mathbb{L}\,\mathbf{a}\|_2,\ \mathbf{a} \neq 0.$
- Greedy selection for the next interpolation point: $\sigma_{k+1} = \arg\max_{s_i} |h_i - H_r(s_i)|$

Parametric Dynamical Systems



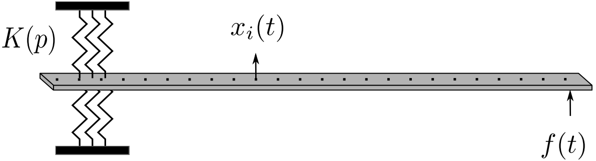
- A parametric beam model with equations of the form

$$\mathbf{M}\ddot{\mathbf{x}}(t;p) + \mathbf{D}\dot{\mathbf{x}}(t;p) + \mathbf{K}(p)\mathbf{x}(t;p) = \mathbf{b}\,u(t), \quad y(t) = \mathbf{c}^T\mathbf{x}(t;p)$$

- Laplace transform (frequency domain) $\rightsquigarrow$ transfer function:

$$Y(s;p) = \underbrace{\mathbf{c}^\top (s^2\mathbf{M} + s\mathbf{D} + \mathbf{K}(p))^{-1}\mathbf{b}}_{H(s;p)}\, U(s)$$

Parametric Dynamical Systems



- A parametric beam model with equations of the form

$$\mathbf{M}\ddot{\mathbf{x}}(t;p) + \mathbf{D}\dot{\mathbf{x}}(t;p) + \mathbf{K}(p)\mathbf{x}(t;p) = \mathbf{b}\,u(t), \quad y(t) = \mathbf{c}^T\mathbf{x}(t;p)$$

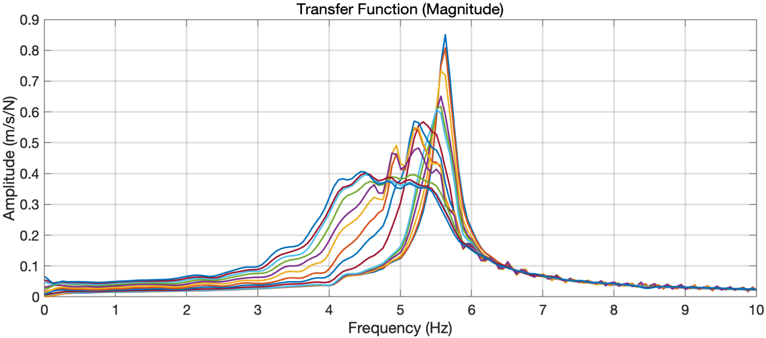
- Laplace transform (frequency domain) $\rightsquigarrow$ transfer function:

$$Y(s;p) = \underbrace{\mathbf{c}^\top\left(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K}(p)\right)^{-1}\mathbf{b}}_{H(s;p)}\,U(s)$$

The new goal:

$$\{H(s_1,p_1), H(s_1,p_2), \ldots, H(s_N,p_M)\} \quad \Longrightarrow \quad H_r(s,p) \approx H(s,p)$$

- In the boom deployment, vary the load level
- Fifteen load-levels (voltages) are tested



- Transfer function varies with the “load level".

Barycentric forms for parametric dynamical systems and pAAA

- Recall the nonparametric case: $H_r(s) = \dfrac{\displaystyle\sum_{j=1}^{r} \frac{\phi_j}{s - \sigma_j}}{\displaystyle\sum_{j=1}^{r} \frac{\varphi_j}{s - \sigma_j}}$

- Find a bivariate rational approximation to $H(s,p)$: $H_r(s,p) = \dfrac{n(s,p)}{d(s,p)}$

$$\boxed{H_r(s,p) = \sum_{i=1}^{k} \sum_{j=1}^{q} \frac{\phi_{ij}}{(s - \sigma_i)(p - \pi_j)} \Bigg/ \sum_{i=1}^{k} \sum_{j=1}^{q} \frac{\varphi_{ij}}{(s - \sigma_i)(p - \pi_j)}}$$

- **pAAA Algorithm** ([Carracedo Rodriguez/Balicki/G., 23]): Extended the AAA algorithm of [Nakatsukasa/Sete/Treftehnen, 18] to parametric systems.

Parametric AAA (pAAA) [Carracedo Rodriguez/Balicki/G., 23]

$$\boxed{H_r(s,p) = \sum_{i=1}^{k}\sum_{j=1}^{q} \frac{\phi_{ij}}{(s-\sigma_i)(p-\pi_j)} \Bigg/ \sum_{i=1}^{k}\sum_{j=1}^{q} \frac{\varphi_{ij}}{(s-\sigma_i)(p-\pi_j)}}$$

- pAAA is an iterative algorithm and partitions the data at iteration k:

$$[s_1,\ldots,s_N] = [\sigma_1,\ldots,\sigma_k] \cup [\hat{\sigma}_1,\ldots,\hat{\sigma}_{N-k}] \qquad \text{freq. samples}$$

$$[p_1,\ldots,p_M] = [\pi_1,\ldots,\pi_q] \cup [\hat{\pi}_1,\ldots,\hat{\pi}_{M-q}] \qquad \text{par. samples}$$

$$[h_{ij}] = \left[\begin{array}{c|c} [H(\sigma_i,\pi_j)] & [H(\sigma_i,\hat{\pi}_j)] \\ \hline [H(\hat{\sigma}_i,\pi_j)] & [H(\hat{\sigma}_i,\hat{\pi}_j)] \end{array}\right] = \left[\begin{array}{c|c} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \hline \mathbf{H}_{21} & \mathbf{H}_{22} \end{array}\right] \qquad \text{values}$$

- Select greedily some of the data pairs (σ_i, p_j) for interpolation. At iteration k:

$$(\sigma_k,\pi_k) = \underset{(s_i,p_j)}{\operatorname{argmax}} \left| h_{ij} - H_r(s_i,p_j) \right| \quad \Longrightarrow \quad \phi_{ij} = \varphi_{ij} h_{ij}, \quad h_{ij} = H(\sigma_i,\pi_j)$$

- Minimize a linearized LS error on the rest of the data

pAAA

$$H_r(s,p) = \sum_{i=1}^{k}\sum_{j=1}^{q} \frac{\varphi_{ij}\, h_{ij}}{(s-\sigma_i)(p-\pi_j)} \Bigg/ \sum_{i=1}^{k}\sum_{j=1}^{q} \frac{\varphi_{ij}}{(s-\sigma_i)(p-\pi_j)}$$

Partition data at iteration k:

$$[s_1,\ldots,s_N] = [\sigma_1,\ldots,\sigma_k] \cup [\hat{\sigma}_1,\ldots,\hat{\sigma}_{N-k}] \quad \text{freq. samples}$$

$$[p_1,\ldots,p_M] = [\pi_1,\ldots,\pi_q] \cup [\hat{\pi}_1,\ldots,\hat{\pi}_{M-q}] \quad \text{par. samples}$$

$$[h_{ij}] = \left[\begin{array}{c|c} [H(\sigma_i,\pi_j)] & [H(\sigma_i,\hat{\pi}_j)] \\ \hline [H(\hat{\sigma}_i,\pi_j)] & [H(\hat{\sigma}_i,\hat{\pi}_j)] \end{array}\right] = \left[\begin{array}{c|c} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \hline \mathbf{H}_{21} & \mathbf{H}_{22} \end{array}\right] \quad \text{values}$$

Choose φ_{ij} to minimize the linearized LS error in the remainin data $\left[\begin{array}{c|c} & \mathbf{H}_{12} \\ \hline \mathbf{H}_{21} & \mathbf{H}_{22} \end{array}\right]$

Leads to a linear LS problem in every step. Algorithm stops after a tolerance value is reached. The complexity (order) and interpolation points are automatically chosen.

Algorithm (pAAA)

Given $\{s_i\}$, $\{p_j\}$, *and* $\{h_{ij}\} = \{H(s_i, p_j)\}$
Initialize: $k = 0$ *and* $q = 0$
For 1st iteration, define: $H_r(s,p) = average(h_{ij})$ *and set error* $\leftarrow \frac{\|[h_{ij}]-[H_r(s_i,p_j)]\|_\infty}{\|[h_{ij}]\|_\infty}$
while *error* $>$ *desired tolerance* ***do***
  Select $(s_{\hat{\imath}}, p_{\hat{\jmath}})$ *by greedy search*
  Update the data partitioning
  if $s_{\hat{\imath}}$ *was not selected at a previous iteration* ***then***
    $k \leftarrow k + 1$
    $\sigma_k \leftarrow s_{\hat{\imath}}$
  end if
  if $p_{\hat{\jmath}}$ *was not selected at a previous iteration* ***then***
    $q \leftarrow q + 1$
    $\pi_q \leftarrow p_{\hat{\jmath}}$
  end if
  Build the Loewner matrix $\mathbb{L}_2$ *and solve* $\min \|\mathbb{L}_2\mathbf{a}\|_2$ *s.t.* $\|\mathbf{a}\|_2 = 1$
  Update the rational approximation $H_r(s,p)$
  error $\leftarrow \frac{\|[h_{ij}]-[H_r(s_i,p_j)]\|_\infty}{\|[h_{ij}]\|_\infty}$
end while
return $H_r(s,p)$

- Multi-parameter extension is (theoretically) straightforward

$$H_r(s,p,z) = \sum_{i=1}^{k}\sum_{j=1}^{q}\sum_{\ell=1}^{o} \frac{\phi_{ij\ell}}{(s-\sigma_i)(p-\pi_j)(z-\zeta_\ell)} \Bigg/ \sum_{i=1}^{k}\sum_{j=1}^{q}\sum_{\ell=1}^{o} \frac{\varphi_{ij\ell}}{(s-\sigma_i)(p-\pi_j)(z-\zeta_\ell)}$$

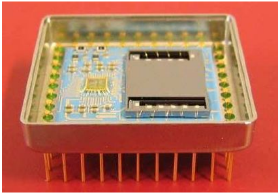
- Multi-input/multi-output problems are handled via tangential sampling:

$$H(s_i,p_j) \in \mathbb{C}^{n_i \times n_o} \quad \Rightarrow \quad \mathbf{u}^T H(s_i,p_j)\mathbf{v} \in \mathbb{C}$$

  - Inspired by [Elsworth/Güttel,19] for the non-parametric AAA.
  - This leads to a type of tangential interpolation and weighted LS solution ([Carracedo Rodriguez/Balicki/G.,23])

Parameterized Gyroscope Model

- The butterfly gyroscope is a vibrating micro-mechanical structure used for inertia-based navigation.
- This is the parameterized Modified Gyroscope model from MORwiki (`http://modelreduction.org/`)



$$\mathbf{M}(p)\ddot{\mathbf{q}}(t) + \mathbf{D}(p,z)\dot{\mathbf{q}}(t) + \mathbf{K}(p)\mathbf{q}(t) = \mathbf{b}\,u(t), \quad y(t) = \mathbf{c}^\top \mathbf{q}(t)$$

- p: structural parameter, z : the rotation velocity
- Transfer function:

$$H(s,p,z) = \mathbf{c}^\top (s^2\mathbf{M}(p) + s\,\mathbf{D}(p,z) + \mathbf{K}(p))^{-1}\mathbf{b}$$

- $\mathbf{M}(p) = \mathbf{M}_1 + p\mathbf{M}_2, \quad \mathbf{D}(p,z) = z(\mathbf{G}_1 + p\mathbf{G}_2), \quad \mathbf{K}(p) = \mathbf{K}_1 + \frac{1}{p}\mathbf{K}_2 + p\mathbf{K}_3$
- 100 samples of s_i in $[0.025, 0, 25]2\pi\imath$, 10 samples of p_j in $[1,2]$, and 10 samples of z_ℓ in $[10^{-7}, 10^{-5}]$

- pAAA results in $(k, q, o) = (52, 7, 9)$ (orders in s, p, and z)

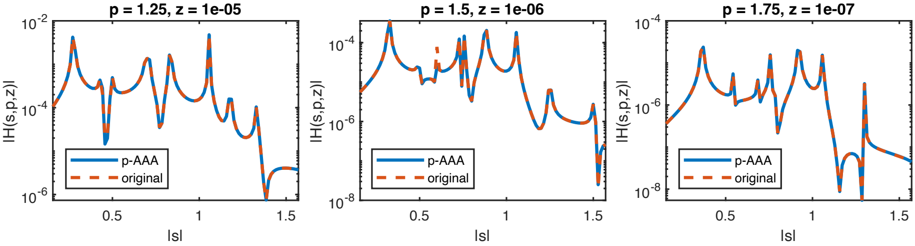
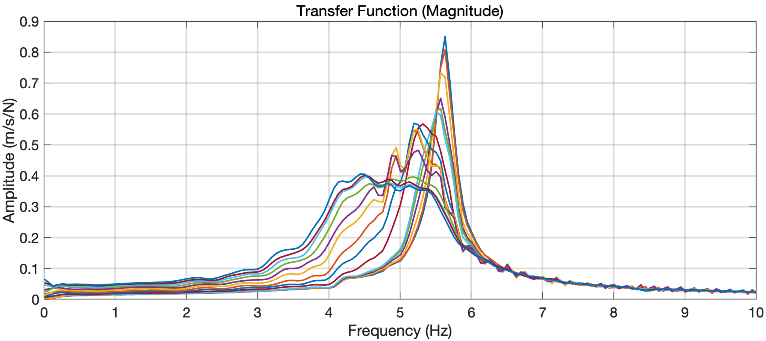


Figure: p-AAA approximation of gyroscope model for various parameter combinations.

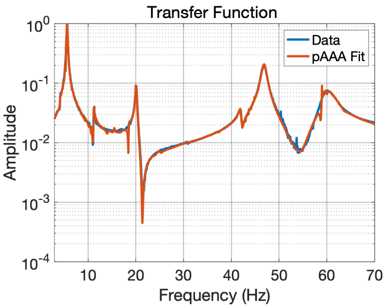
Back to the boom deployment

- We have parametrically varying dynamics based on the load level

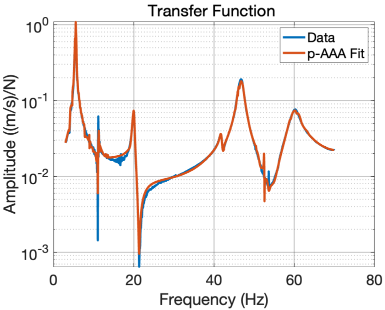


- Our parameter in this case is the “load level".
- We construct a data-driven parametric rational function $H_r(s,p)$ to fit the measurements using p-AAA.

- We have $H_r(s_i, p_j)$ samples for $i = 1, \ldots, 1600$ and $j = 1, \ldots, 15$.
- p-AAA results in $H_r(s, p)$ with order $k = 193$ in s and order $q = 13$ in p.



Load Case 1

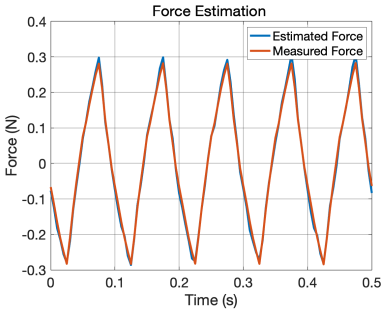


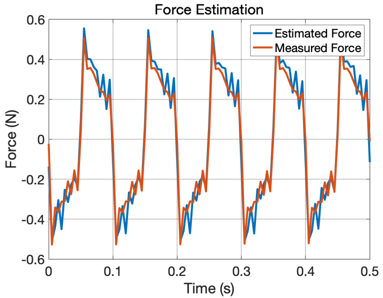
Load Case 6

- Neither load case was in the training data

Parametric Force Estimation

- “Invert" the observed velocity output $y(t)$ using the learn parametric model to estimate the experimental (measured) forcing.

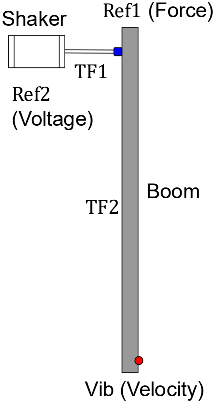


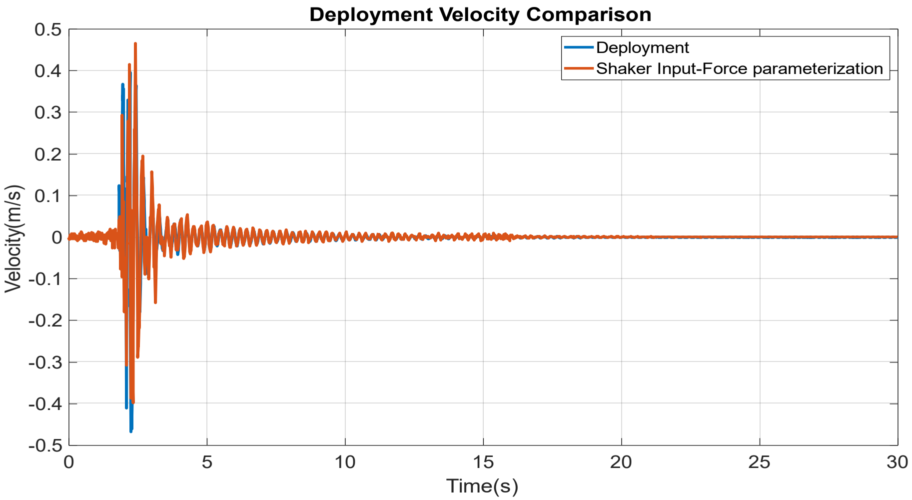


- Training was done using a chirp signal

Estimating the Deployment Force

- So far, we have been collecting data by exciting the system ourselves (using the shaker). The major goal is to estimate the forcing from deployment.
- Remove the shaker. Roll up the boom and release the spool.
- Measure the deployment velocity at the tip and calculate Ref1 (Force) based on the parametric TF2 transfer function model
- Calculate Ref2 Voltage from calculated Ref1 Force using another parametric model for TF1
- Reattach the shaker and apply this estimated Ref2 Voltage. Measure the velocity at the tip of the deployed boom
- Compare this velocity with the velocity from released spool.





Estimated forcing from the data-driven rational pAAA approximation leads to an output correctly predicting the observed velocity

Conclusions and outlook: Data-driven rational approximants

- Rational least-squares fit from data
- Application to boom deployment:

  "Input Load Estimation for Bistable Spacecraft Booms using System Identification", in preparation. Earlier version to appear in 2026 AIAA SciTech
- Extension to parametric dynamical systems (pAAA):
  - Original pAAA with grid data: Carrecado Rodriguez, G., and Balicki: SIAM SISC https://doi.org/10.1137/20M1322698.)
  - pAAA with low-rank barycentric form for multiple-parameters: Balicki and G.: ArXiV preprint arXiv:2502.03204.
  - pAAA for scarred data sets: Balicki and G.: ArXiV preprint arXiv:2510.22861.
- p-AAA Repository: `https://github.com/lbalicki/parametric-AAA`
- Currently under investigation:
  - Extending barycentric forms to structured and nonlinear dynamics