Cooperation, Competition, and Common Pool Resources in Mean Field Games and Extensions with Inverse Learning

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Tragedy of the Commons and Common Pool Resources

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Some examples of common pool resources: fish stocks, coffee, public parks, water...


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[1] Picture Credits: 1. https://blogs.ntu.edu.sg/hp3203-2017-29/tragedy-of-the-commons/ 2. https://online.hbs.edu/blog/post/tragedy-of-the-commons-impact-on-sustainability-issues

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Our motivation is to model possible altruism of the individuals in large populations.

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Mean Field Games vs Mean Field Control

Due to the exponentially increasing number of interactions among agents, it is challenging to find solutions* in the game problems.

*: Social Optimum (full cooperation) or Nash equilibrium (full competition)

Approach: using mean field approximations ⇒ Mean field game & Mean field control

Similarities:

→ Assume $N \to \infty$.

→ Agents are identical and infinitesimal.

→ Agents interact symmetrically.

→ There are mean field interactions.

→ **Idea:** Focus on representative agent and her interactions with the distribution.

Differences:

→ Mean field game: non-cooperative agents; Mean field control: cooperative agents

Mathematical Formulation Example of MFG and MFC

The representative agent's state process $\boldsymbol{X}^{\boldsymbol{\alpha}}$ has the following dynamics:

$$dX_t^{\boldsymbol{\alpha}} = b(t, X_t, \alpha_t, \mu_t)dt + \sigma dW_t, \quad t \in [0, T], \qquad X_0^{\boldsymbol{\alpha}} \sim \mu_0. \tag{1}$$

Mean Field Game:

For each fixed flow of probability measures $\boldsymbol{\mu} = (\mu_t)_{0 \leq t \leq T}$, the representative agent has the cost:

$$J(\boldsymbol{\alpha}; \boldsymbol{\mu}) = \mathbb{E}\Big[\int_0^T f(t, X_t^{\boldsymbol{\alpha}}, \alpha_t, \mu_t)dt + g(X_T^{\boldsymbol{\alpha}}, \mu_T)\Big], \tag{2}$$

Def: An MFG solution (i.e., MFG (Nash) equilibrium) is a pair $(\boldsymbol{\alpha}^{\mathrm{MFG}}, \boldsymbol{\mu}^{\mathrm{MFG}})$ s.t.:

- → **Best response:** $\boldsymbol{\alpha}^{\mathrm{MFG}}$ minimizes $J(\cdot; \boldsymbol{\mu}^{\mathrm{MFG}})$ defined in (2),
- → **Fixed point:** $\mu_t^{\mathrm{MFG}} = \mathcal{L}(X_t^{\mathrm{MFG}})$ for all $t \in [0, T]$, where $\boldsymbol{X}^{\mathrm{MFG}}$ is the solution to (1) controlled by $\boldsymbol{\alpha}^{\mathrm{MFG}}$.

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→ **Fixed point:** $\mu_t^{\mathrm{MFG}} = \mathcal{L}(X_t^{\mathrm{MFG}})$ for all $t \in [0, T]$, where $\boldsymbol{X}^{\mathrm{MFG}}$ is the solution to (1) controlled by $\boldsymbol{\alpha}^{\mathrm{MFG}}$.

Mean Field Control:

Def: An MFC solution (i.e., MFC optimum) is a minimizer of the following cost:

$$J^{\mathrm{MFC}}(\boldsymbol{\alpha}) = \mathbb{E}\Big[\int_0^T f\big(t, X_t^{\boldsymbol{\alpha}}, \alpha_t, \mathcal{L}(X_t^{\boldsymbol{\alpha}})\big)dt + g\big(X_T^{\boldsymbol{\alpha}}, \mathcal{L}(X_T^{\boldsymbol{\alpha}})\big)\Big] \tag{3}$$

where $\boldsymbol{X}^{\boldsymbol{\alpha}}$ is determined by (1) using the control $\boldsymbol{\alpha}$.

Define the following notation: $\mu_t^{\boldsymbol{\alpha}} := \mathcal{L}(X_t^{\boldsymbol{\alpha}})$

Mean Field Games vs Mean Field Control with Common Pool Resources

$\rightarrow$ In **MFG**: Agents are insignificant and they do not see their effect. They overuse the resources. $\Rightarrow$ Tragedy of the commons is **inevitable**.

$\rightarrow$ In **MFC**: Agents are fully altruistic or the decisions are prescribed by outside regulator. $\Rightarrow$ Not a good representation of human **nature**.

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Motivation: In real life, populations are a **mixture** of cooperative and non-cooperative individuals, or even further, the individuals could have cooperative and non-cooperative tendencies **within themselves**.

We want to model such situations.

Related Literature

Some related literature:

→ Angiuli, Detering, Fouque, Laurière, Lin (2022a, 2022b): Mean field control games

→ Barreiro-Gomez, Duncan, Tembine (2020): Co-opetitive LQ MFG

In today's talk, we will discuss topics mainly from:

→ *"Cooperation, Competition, and Common-Pool Resources in Mean Field Games"* by **G.D.** and Laurière (2025) (ArXiv: 2504.09043)

→ *"Inverse Learning of the Altruism and Cost Level in Mixed-Individual Mean Field Games"* by Cao, **G.D.**, Shi (2026)

Some other related work:

→ *"From Nash Equilibrium to Social Optimum and vice versa: a Mean Field Perspective"* by Carmona, **G.D.**, Delarue, Laurière. Applied Mathematics and Optimization (ArXiv: 2312.10526)

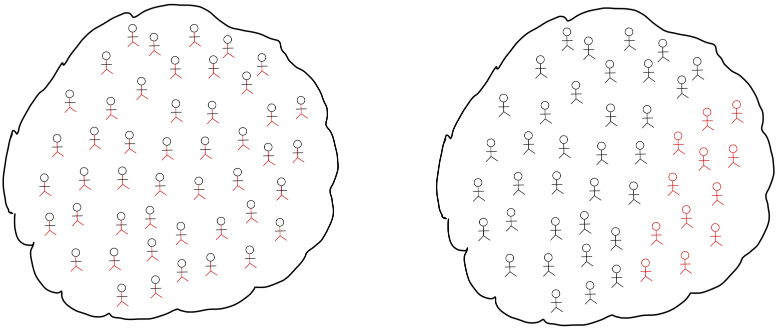
→ *"How can the tragedy of the commons be prevented?: Introducing Linear Quadratic Mixed Mean Field Games"* by **G.D.** and Laurière. In Proceedings of IEEE Conference in Control and Decision (CDC) 2024 (ArXiv: 2409.08235)

→ *"Finite-State Mixed-Individual Mean Field Games with an Application to Epidemic Modeling"* by Jia, **G.D.** (2026)

Two population setups will be considered:

→ Mixed Individual Mean Field Models

→ Mixed Population Mean Field Models



(a) Mixed Individuals

(b) Mixed Populations

Mixed Individual
Mean Field Models

MI: General Model Form

Idea: We have large number of **homogeneous** agents and they each see their *individual effect* as well as the p*opulation effect*.

Mathematically: Add the distribution of agent's own state on the cost function and dynamics.

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Mathematically: Add the distribution of agent's own state on the cost function and dynamics.

The cost of the **representative agent** who chooses control $\boldsymbol{\alpha} = (\alpha_t)_{t\in[0,T]}$ given the population distribution $\boldsymbol{\mu} = (\mu_t)_{t\in[0,T]}$ is

$$J(\boldsymbol{\alpha};\boldsymbol{\mu}) := \mathbb{E}\left[\int_0^T f(t, X_t^{\boldsymbol{\alpha}}, \alpha_t, \mu_t, \mu_t^{\boldsymbol{\alpha}})dt + g(X_T^{\boldsymbol{\alpha}}, \mu_T, \mu_T^{\boldsymbol{\alpha}})\right],$$

where $\mu_t^{\boldsymbol{\alpha}} = \mathcal{L}(X_t^{\boldsymbol{\alpha}})$ (i.e., individual effect). The representative agent's state, $\boldsymbol{X}^{\boldsymbol{\alpha}}$, has dynamics:

$$dX_t^{\boldsymbol{\alpha}} = b(t, X_t^{\boldsymbol{\alpha}}, \alpha_t, \mu_t, \mu_t^{\boldsymbol{\alpha}})dt + \sigma dW_t,$$

with $\mu_0 = \mu_0^{\boldsymbol{\alpha}} = \mathcal{L}(X_0)$.

Def: We will call $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\mu}})$ a mixed individual mean field Nash equilibrium if

i. **Best response:** $\hat{\boldsymbol{\alpha}}$ is the **best response** of the representative agent given the mean field distribution $\hat{\boldsymbol{\mu}}$.
$\Rightarrow \hat{\boldsymbol{\alpha}} \in \arg\min_{\boldsymbol{\alpha} \in \mathbb{A}} J(\boldsymbol{\alpha}; \hat{\boldsymbol{\mu}})$,

ii. **Fixed point:** For all $t \in [0, T]$, we have $\hat{\mu}_t = \mu_t^{\hat{\boldsymbol{\alpha}}}$.

MI: Equilibrium Definition and Analysis (II)

Theorem:[2] Let $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\mu}})$ be a mixed individual mean field Nash equilibrium. Then, it is given by:

$$\hat{\alpha}_t = \arg\min_{\alpha} H^{\mathrm{MI}}(t, X_t, \alpha, \mu_t, \mu_t, Y_t), \qquad \hat{\mu}_t = \mu_t,$$

where $\mu_t = \mathcal{L}(X_t)$ and $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ solve the MKV FBSDE:

$$dX_t = b(t, X_t, \hat{\alpha}_t, \mu_t, \mu_t)dt + \sigma dW_t$$

$$dY_t = \Big(- \underbrace{\partial_x H^{\mathrm{MI}}(t, X_t, \hat{\alpha}_t, \mu_t, \mu_t, Y_t)}_{\text{Similar to MFG and MFC}} - \underbrace{\tilde{\mathbb{E}}\big[\partial_{\mu_2} H^{\mathrm{MI}}(t, \tilde{X}_t, \tilde{\hat{\alpha}}_t, \mu_t, \mu_t, \tilde{Y}_t)(X_t)\big]}_{\text{Similar to MFC}} \Big) dt + Z_t dW_t$$

$$X_0 \sim \mu_0, \qquad Y_T = \partial_x g(X_T, \mu_T, \mu_T) + \tilde{\mathbb{E}}[\partial_{\mu_2} g(\tilde{X}_T, \mu_T, \mu_T)(X_T)].$$

[2] Under suitable conditions, such as the ones given in Chapter 6.3 Carmona-Delarue Vol I.

[3] Under suitable conditions such as drift is linear in control, running cost is bounded, quadratic in control and lipshitz in state and pop. distribution and terminal cost is bounded.

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Theorem:[3] Mixed individual mean field Nash equilibrium is an ϵ-Nash equilibrium for the finite player mixed individual game.

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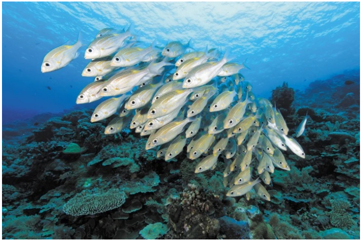
[3] Under suitable conditions such as drift is linear in control, running cost is bounded, quadratic in control and lipshitz in state and pop. distribution and terminal cost is bounded.

Modeling Common Pool Resources with Mixed Individuals

Common Pool Resource will be denoted by $\boldsymbol{K} = (K_t)_{t\in[0,T]}$ with dynamics:

$$dK_t = \Big(\underbrace{b_0^K(t, K_t)}_{\text{replenishment}} - \underbrace{b_1^K(t, K_t, \mu_t, \mu_t^{\alpha})}_{\text{decay}} \Big) dt.$$



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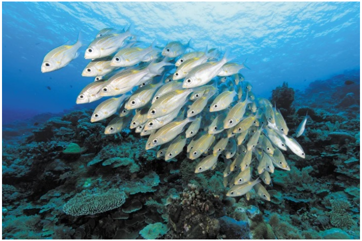

With the addition of common pool resources the model becomes:

$$J(\boldsymbol{\alpha}; \boldsymbol{\mu}) := \mathbb{E}\left[\int_0^T f(t, X_t^{\boldsymbol{\alpha}}, K_t, \alpha_t, \mu_t, \mu_t^{\boldsymbol{\alpha}}) dt + g(X_T^{\boldsymbol{\alpha}}, K_T, \mu_T, \mu_T^{\boldsymbol{\alpha}}) \right],$$

where $\mu_t^{\boldsymbol{\alpha}} = \mathcal{L}(X_t^{\boldsymbol{\alpha}})$ and with the dynamics:

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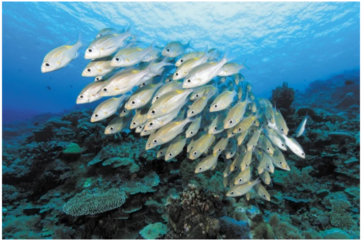

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Remark: This model can be written in the general form where the modified state $\check{X}_t = [X_t, K_t]^\top \Rightarrow$ Same technical analysis applies.

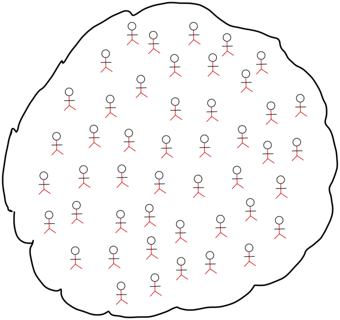
MI Special Case: λ-interpolated MFGs

Everyone will have the same and constant level, $(1-\lambda) \in [0,1]$, of altruism.

→ When $\lambda = 0 \Rightarrow$ fully cooperative.

→ When $\lambda = 1 \Rightarrow$ fully non-cooperative (i.e., competitive).

Then, the cost functions and drifts will depend on: $\lambda\mu_t + (1-\lambda)\mu_t^{\boldsymbol{\alpha}}$



Mixed Population

Mean Field Models

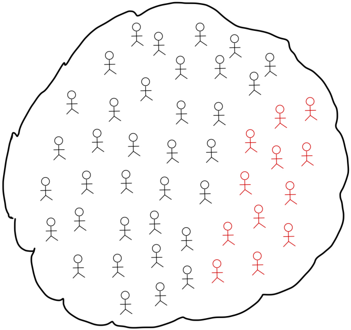
MP: General Model Form (I)

Idea: Now we have a large population of fully non-cooperative and cooperative agents in the population mixed together.

Mathematically: Focus on two representative agents: Non-cooperative & Cooperative

→ For the non-cooperative: MFG model given the distribution of cooperative agents.

→ For the cooperative: MFC model given the distribution of the non-cooperative agents.



MP: General Model Form (II)

The cost of the **representative non-cooperative agents**:

$$J^{\mathrm{NC}}(\boldsymbol{\alpha}^{\mathrm{NC}};\boldsymbol{\mu}^{\mathrm{NC}},\boldsymbol{\mu}^{\mathrm{C}}) := \mathbb{E}\left[\int_0^T f^{\mathrm{NC}}(t, X_t^{\boldsymbol{\alpha}^{\mathrm{NC}}}, \alpha_t^{\mathrm{NC}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}})dt + g^{\mathrm{NC}}(X_T^{\boldsymbol{\alpha}^{\mathrm{NC}}}, \mu_T^{\mathrm{NC}}, \mu_T^{\mathrm{C}})\right].$$

The state of the representative non-cooperative agent, $\boldsymbol{X}^{\boldsymbol{\alpha}^{\mathrm{NC}}}$, has dynamics:

$$dX_t^{\boldsymbol{\alpha}^{\mathrm{NC}}} = b^{\mathrm{NC}}(t, X_t^{\boldsymbol{\alpha}^{\mathrm{NC}}}, \alpha_t^{\mathrm{NC}}, \underbrace{\mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}}_{\text{exogenous}})dt + \sigma^{\mathrm{NC}} dW_t^{\mathrm{NC}}.$$

The cost of the **representative cooperative agents**:

$$J^{\mathrm{C}}(\boldsymbol{\alpha}^{\mathrm{C}};\boldsymbol{\mu}^{\mathrm{NC}}) := \mathbb{E}\left[\int_0^T f^{\mathrm{C}}(t, X_t^{\boldsymbol{\alpha}^{\mathrm{C}}}, \alpha_t^{\mathrm{C}}, \mu_t^{\mathrm{NC}}, \mu_t^{\boldsymbol{\alpha}^{\mathrm{C}}})dt + g^{\mathrm{C}}(X_T^{\boldsymbol{\alpha}^{\mathrm{NC}}}, \mu_T^{\mathrm{NC}}, \mu_T^{\boldsymbol{\alpha}^{\mathrm{C}}})\right],$$

The state of the representative cooperative agent, $\boldsymbol{X}^{\boldsymbol{\alpha}^{\mathrm{C}}}$, has dynamics:

$$dX_t^{\boldsymbol{\alpha}^{\mathrm{C}}} = b^{\mathrm{C}}(t, X_t^{\boldsymbol{\alpha}^{\mathrm{C}}}, \alpha_t^{\mathrm{C}}, \underbrace{\mu_t^{\mathrm{NC}}}_{\text{exogenous}}, \mu_t^{\boldsymbol{\alpha}^{\mathrm{C}}})dt + \sigma^{\mathrm{C}} dW_t^{\mathrm{C}}.$$

where $\mu_t^{\boldsymbol{\alpha}^{\mathrm{C}}} = \mathcal{L}(X_t^{\boldsymbol{\alpha}^{\mathrm{C}}})$,

MP: Equilibrium Definition and Analysis (I)

Def: We will call $(\hat{\boldsymbol{\alpha}}^{\mathrm{NC}}, \hat{\boldsymbol{\mu}}^{\mathrm{NC}}, \hat{\boldsymbol{\alpha}}^{\mathrm{C}})$ a mixed population mean field equilibrium if

i. **Non-cooperative best response:** $\hat{\boldsymbol{\alpha}}^{\mathrm{NC}}$ is the **best response** of the representative **non-cooperative** agent given the mean field distributions of non-cooperative & cooperative agents: $\hat{\boldsymbol{\mu}}^{\mathrm{NC}}$, $\boldsymbol{\mu}^{\hat{\boldsymbol{\alpha}}^{\mathrm{C}}}$
$\Rightarrow \hat{\boldsymbol{\alpha}}^{\mathrm{NC}} \in \arg\min_{\boldsymbol{\alpha}^{\mathrm{NC}} \in \mathbb{A}} J^{\mathrm{NC}}(\boldsymbol{\alpha}^{\mathrm{NC}}; \hat{\boldsymbol{\mu}}^{\mathrm{NC}}, \boldsymbol{\mu}^{\hat{\boldsymbol{\alpha}}^{\mathrm{C}}})$,

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ii. **Non-cooperative fixed point:** For all $t \in [0, T]$, we have $\hat{\mu}_t^{\mathrm{NC}} = \mu_t^{\hat{\boldsymbol{\alpha}}^{\mathrm{NC}}}$,

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ii. **Non-cooperative fixed point:** For all $t \in [0, T]$, we have $\hat{\mu}_t^{\mathrm{NC}} = \mu_t^{\hat{\boldsymbol{\alpha}}^{\mathrm{NC}}}$,

iii. **Cooperative optimality:** $\hat{\boldsymbol{\alpha}}^{\mathrm{C}}$ is the **optimal control** of the representative **cooperative** agent given the mean field distribution of non-cooperative agents: $\hat{\boldsymbol{\mu}}^{\mathrm{NC}}$
$\Rightarrow \hat{\boldsymbol{\alpha}}^{\mathrm{C}} \in \arg\min_{\boldsymbol{\alpha}^{\mathrm{C}} \in \mathbb{A}} J^{\mathrm{C}}(\boldsymbol{\alpha}^{\mathrm{C}}; \hat{\boldsymbol{\mu}}^{\mathrm{NC}})$.

MP: Equilibrium Definition and Analysis (II)

Theorem:[4] Let $(\hat{\boldsymbol{\alpha}}^{\mathrm{NC}}, \hat{\boldsymbol{\mu}}^{\mathrm{NC}}, \hat{\boldsymbol{\alpha}}^{\mathrm{C}})$ be a mixed population MF equilibrium. Then:

$$\hat{\alpha}_t^{\mathrm{NC}} = \arg\min_{\alpha} H^{\mathrm{MP,NC}}(t, X_t^{\mathrm{NC}}, \alpha, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}, Y_t^{\mathrm{NC}})$$

$$\hat{\alpha}_t^{\mathrm{C}} = \arg\min_{\alpha} H^{\mathrm{MP,C}}(t, X_t^{\mathrm{C}}, \alpha, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}, Y_t^{\mathrm{C}}), \qquad \hat{\mu}_t^{\mathrm{NC}} = \mu_t^{\mathrm{NC}},$$

where $\mu_t^{\mathrm{NC}} = \mathcal{L}(X_t^{\mathrm{NC}})$, $\mu_t^{\mathrm{C}} = \mathcal{L}(X_t^{\mathrm{C}})$, and $(\boldsymbol{X}^{\mathrm{NC}}, \boldsymbol{Y}^{\mathrm{NC}}, \boldsymbol{Z}^{\mathrm{NC}}, \boldsymbol{X}^{\mathrm{C}}, \boldsymbol{Y}^{\mathrm{C}}, \boldsymbol{Z}^{\mathrm{C}})$ solve

$$\begin{aligned}
dX_t^{\mathrm{NC}} &= b^{\mathrm{NC}}(t, X_t^{\mathrm{NC}}, \hat{\alpha}_t^{\mathrm{NC}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}})dt + \sigma^{\mathrm{NC}} dW_t^{\mathrm{NC}} \\
dY_t^{\mathrm{NC}} &= -\partial_x H^{\mathrm{MP,NC}}(t, X_t^{\mathrm{NC}}, \hat{\alpha}_t^{\mathrm{NC}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}, Y_t^{\mathrm{NC}})dt + Z_t^{\mathrm{NC}} dW_t^{\mathrm{NC}} \\
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&\qquad \boxed{-\tilde{\mathbb{E}}[\partial_{\mu^{\mathrm{C}}} H^{\mathrm{MP,C}}(t, \tilde{X}_t^{\mathrm{C}}, \tilde{\hat{\alpha}}_t^{\mathrm{C}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}, \tilde{Y}_t^{\mathrm{C}})(X_t^{\mathrm{C}})]\Big) dt} + Z_t^{\mathrm{C}} dW_t^{\mathrm{C}} \\
X_0^{\mathrm{NC}} &\sim \mu_0^{\mathrm{NC}}, \quad Y_T^{\mathrm{NC}} = \partial_x g(X_T^{\mathrm{NC}}, \mu_T^{\mathrm{NC}}, \mu_T^{\mathrm{C}}), \\
X_0^{\mathrm{C}} &\sim \mu_0^{\mathrm{C}}, \quad Y_T^{\mathrm{C}} = \partial_x g(X_T^{\mathrm{C}}, \mu_T^{\mathrm{NC}}, \mu_T^{\mathrm{C}}) \boxed{+\tilde{\mathbb{E}}[\partial_{\mu^{\mathrm{C}}} g(\tilde{X}_T^{\mathrm{C}}, \mu_T^{\mathrm{NC}}, \mu_T^{\mathrm{C}})(X_T^{\mathrm{C}})]}.
\end{aligned}$$

[4] Under suitable conditions, such as the ones given in Chapters 3.3 & 6.3 Carmona-Delarue Vol I.

MP: Equilibrium Definition and Analysis (II)

Theorem:[4] Let $(\hat{\boldsymbol{\alpha}}^{\mathrm{NC}}, \hat{\boldsymbol{\mu}}^{\mathrm{NC}}, \hat{\boldsymbol{\alpha}}^{\mathrm{C}})$ be a mixed population MF equilibrium. Then:

$$\hat{\alpha}_t^{\mathrm{NC}} = \arg\min_{\alpha} H^{\mathrm{MP,NC}}(t, X_t^{\mathrm{NC}}, \alpha, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}, Y_t^{\mathrm{NC}})$$

$$\hat{\alpha}_t^{\mathrm{C}} = \arg\min_{\alpha} H^{\mathrm{MP,C}}(t, X_t^{\mathrm{C}}, \alpha, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}, Y_t^{\mathrm{C}}), \qquad \hat{\mu}_t^{\mathrm{NC}} = \mu_t^{\mathrm{NC}},$$

where $\mu_t^{\mathrm{NC}} = \mathcal{L}(X_t^{\mathrm{NC}})$, $\mu_t^{\mathrm{C}} = \mathcal{L}(X_t^{\mathrm{C}})$, and $(\boldsymbol{X}^{\mathrm{NC}}, \boldsymbol{Y}^{\mathrm{NC}}, \boldsymbol{Z}^{\mathrm{NC}}, \boldsymbol{X}^{\mathrm{C}}, \boldsymbol{Y}^{\mathrm{C}}, \boldsymbol{Z}^{\mathrm{C}})$ solve

$$\begin{aligned}
dX_t^{\mathrm{NC}} &= b^{\mathrm{NC}}(t, X_t^{\mathrm{NC}}, \hat{\alpha}_t^{\mathrm{NC}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}})dt + \sigma^{\mathrm{NC}} dW_t^{\mathrm{NC}} \\
dY_t^{\mathrm{NC}} &= -\partial_x H^{\mathrm{MP,NC}}(t, X_t^{\mathrm{NC}}, \hat{\alpha}_t^{\mathrm{NC}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}, Y_t^{\mathrm{NC}})dt + Z_t^{\mathrm{NC}} dW_t^{\mathrm{NC}} \\
dX_t^{\mathrm{C}} &= b^{\mathrm{C}}(t, X_t^{\mathrm{C}}, \hat{\alpha}_t^{\mathrm{C}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}})dt + \sigma^{\mathrm{C}} dW_t^{\mathrm{C}} \\
dY_t^{\mathrm{C}} &= \Big(-\partial_x H^{\mathrm{MP,C}}(t, X_t^{\mathrm{C}}, \hat{\alpha}_t^{\mathrm{C}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}, Y_t^{\mathrm{C}}) \\
&\qquad \boxed{-\tilde{\mathbb{E}}[\partial_{\mu^{\mathrm{C}}} H^{\mathrm{MP,C}}(t, \tilde{X}_t^{\mathrm{C}}, \tilde{\hat{\alpha}}_t^{\mathrm{C}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}}, \tilde{Y}_t^{\mathrm{C}})(X_t^{\mathrm{C}})]\Big) dt} + Z_t^{\mathrm{C}} dW_t^{\mathrm{C}} \\
X_0^{\mathrm{NC}} &\sim \mu_0^{\mathrm{NC}}, \quad Y_T^{\mathrm{NC}} = \partial_x g(X_T^{\mathrm{NC}}, \mu_T^{\mathrm{NC}}, \mu_T^{\mathrm{C}}), \\
X_0^{\mathrm{C}} &\sim \mu_0^{\mathrm{C}}, \quad Y_T^{\mathrm{C}} = \partial_x g(X_T^{\mathrm{C}}, \mu_T^{\mathrm{NC}}, \mu_T^{\mathrm{C}}) \boxed{+\tilde{\mathbb{E}}[\partial_{\mu^{\mathrm{C}}} g(\tilde{X}_T^{\mathrm{C}}, \mu_T^{\mathrm{NC}}, \mu_T^{\mathrm{C}})(X_T^{\mathrm{C}})]}.
\end{aligned}$$

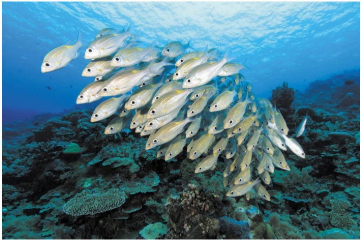
Theorem: Mixed population MF equilibrium is an (ϵ_1, ϵ_2) equilibrium/optimum for the finite player mixed population problem.

[4] Under suitable conditions, such as the ones given in Chapters 3.3 & 6.3 Carmona-Delarue Vol I.

Modeling Common Pool Resources with Mixed Populations

Similarly to MI, **Common Pool Resource** will be denoted by $\boldsymbol{K} = (K_t)_{t\in[0,T]}$.

Different than MI, non-cooperative and cooperative agents will **perceive** different dynamics for the common pool resource:



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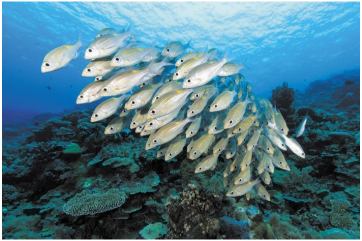

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Non-cooperative agents:

$$dK_t^{\mathrm{NC}} = \Big(\underbrace{b_0^K(t, K_t^{\mathrm{NC}})}_{\text{replenishment}} - \underbrace{b_1^K(t, K_t^{\mathrm{NC}}, \mu_t^{\mathrm{NC}}, \mu_t^{\mathrm{C}})}_{\text{decay}}\Big)dt.$$

Cooperative agents:

$$dK_t^{\mathrm{C}} = \Big(\underbrace{b_0^K(t, K_t^{\mathrm{C}})}_{\text{replenishment}} - \underbrace{b_1^K(t, K_t^{\mathrm{C}}, \mu_t^{\mathrm{NC}}, \mu_t^{\boldsymbol{\alpha}^{\mathrm{C}}})}_{\text{decay}}\Big)dt.$$

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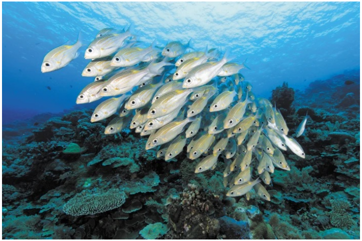

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However, **at the equilibrium:** $\mu_t^{\boldsymbol{\alpha}^{\mathrm{C}}} = \mu_t^{\mathrm{C}}$

$\rightarrow$ **Therefore:** we will have only one flow ($\boldsymbol{K}$) for the common pool resource.

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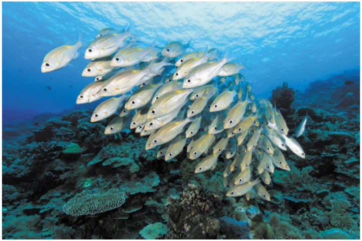

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However, **at the equilibrium:** $\mu_t^{\boldsymbol{\alpha}^{\mathrm{C}}} = \mu_t^{\mathrm{C}}$

$\to$ **Therefore:** we will have only one flow $(\boldsymbol{K})$ for the common pool resource.

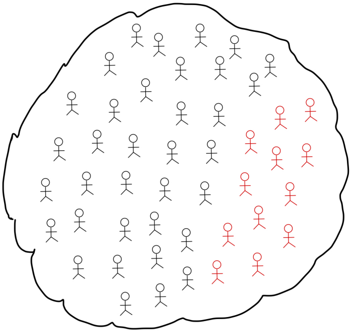
Remark: This model can be written in the general form where the modified states $\check{X}_t^{\mathrm{NC}} = [X_t^{\mathrm{NC}}, K_t^{\mathrm{NC}}]^\top$ and $\check{X}_t^{\mathrm{C}} = [X_t^{\mathrm{C}}, K_t^{\mathrm{C}}]^\top \Rightarrow$ Same technical analysis applies.

MP Special Case: p-Mixed MFGs

The population consists of constant p proportion of non-cooperative agents and $(1-p)$ proportion of cooperative agents.

Then, the cost functions and drifts will depend on

- $\rightarrow$ $p\mu_t^{\mathrm{NC}} + (1-p)\mu_t^{\mathrm{C}}$ for non-cooperative agents,
- $\rightarrow$ $p\mu_t^{\mathrm{NC}} + (1-p)\mu_t^{\boldsymbol{\alpha}^{\mathrm{C}}}$ for cooperative agents



Motivating Example with Common Pool Resources (CPRs):

Modeling Fishers and Fish Stock

Mixed Individual Fisher and Fish Stock Model[5]

We model a large population of fishers with an altruism level of $(1 - \lambda) \in [0, 1]$.

- $\rightarrow$ **State:** Fishing effort X_t, **Control:** Change in fishing effort α_t
- $\rightarrow$ **Common pool resource:** Fish stock level K_t
- $\rightarrow$ **Management Concern Level (TOTC):** Following WWF, around 25-35% of Spawning Stock Biomass ($\mathcal{K}$) creates management concerns. We set it 30%.

[5]The model is motivated by the Ph.D. thesis "Integrating Ecology and Economics in the Mathematical Modelling of Marine Ecosystems" R. McPike. (2021, University of Strathclyde, Glasgow).

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The state and the common pool resource dynamics are given as follows:

$$dX_t^{\boldsymbol{\alpha}} = \alpha_t dt + \sigma dW_t, \qquad X_0^{\boldsymbol{\alpha}} \sim \mu_0$$

$$dK_t = \Big[\underbrace{rK_t\big(1-\frac{K_t}{\mathcal{K}}\big)}_{\text{Logistic growth}} - \underbrace{qK_t\big(\lambda\bar{X}_t + (1-\lambda)\bar{X}_t^{\boldsymbol{\alpha}}\big)}_{\text{Decay due to fishing}}\Big]dt, \qquad K_0 = \rho\mathcal{K}.$$

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The representative fisher has the following cost:

$$\mathbb{E}\Big[\int_0^T \Big(\underbrace{-P\big(qK_t(\lambda\bar{X}_t + (1-\lambda)\bar{X}_t^{\boldsymbol{\alpha}})\big)K_tX_t^{\boldsymbol{\alpha}}}_{\text{Fishing revenue}} + \underbrace{c_1X_t^{\boldsymbol{\alpha}} + \frac{c_2}{2}(X_t^{\boldsymbol{\alpha}})^2}_{\text{High effort cost}} + \underbrace{\frac{c_3}{2}\alpha_t^2}_{\text{Ramping up/down cost}}\Big)dt\Big],$$

Price function is taken as a linear inverse supply function, i.e., $P(z) = p_0 - p_1 z$.

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Theorem: Mixed individual mean field equilibrium control is given by

$$\alpha_t^* = -Y_t^2/c_3$$

where $(\boldsymbol{K}, \boldsymbol{X}, \boldsymbol{Y}^1, \boldsymbol{Y}^2, \boldsymbol{Z}^1, \boldsymbol{Z}^2)$ solve the following FBSDE:

$$\begin{aligned}
dK_t &= \left[rK_t\left(1 - K_t/\mathcal{K}\right) - qK_t\bar{X}_t\right]dt \\
dX_t &= -\frac{Y_t^2}{c_3}dt + \sigma dW_t \\
dY_t^1 &= -\left[\left(r - \frac{2r}{\mathcal{K}}K_t - q\bar{X}_t\right)Y_t^1 + 2p_1qK_t\bar{X}_tX_t - p_0X_t\right]dt + Z_t^1 dW_t \\
dY_t^2 &= -\left[p_1q(2-\lambda)K_t^2\bar{X}_t + c_1 + c_2X_t - q(1-\lambda)K_t\bar{Y}_t^1 - p_0K_t\right]dt + Z_t^2 dW_t \\
K_0 &= \rho\mathcal{K}, \quad X_0 \sim \mu_0, \quad Y_T^1 = Y_T^2 = 0
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K_0 &= \rho\mathcal{K}, \quad X_0 \sim \mu_0, \quad Y_T^1 = Y_T^2 = 0
\end{aligned}$$

Theorem: Under small time condition, there exists a unique mixed individual mean field equilibrium $(\hat{\alpha}, \boldsymbol{K}, \bar{\boldsymbol{X}})$.

MI Numerical Results

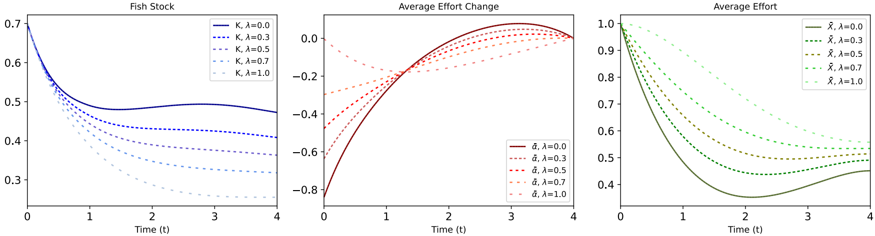

Figure 6: Left: Fish Stock over time: K_t; **Middle:** Average mixed individual mf equilibrium change in fishing effort over time $\bar{\alpha}_t^*$; **Right:** Average mixed individual mf equilibrium fishing effort over time $\bar{X}_t$;

MI Numerical Results

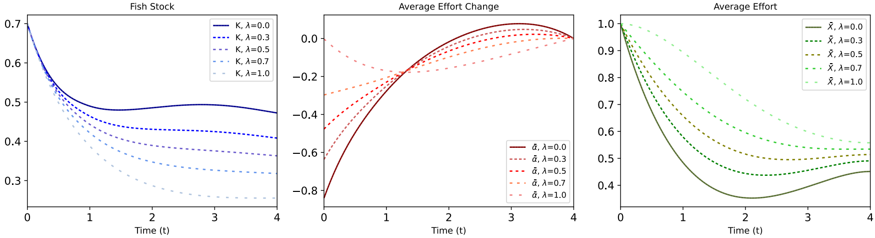

Figure 6: Left: Fish Stock over time: K_t; **Middle:** Average mixed individual mf equilibrium change in fishing effort over time $\bar{\alpha}_t^*$; **Right:** Average mixed individual mf equilibrium fishing effort over time $\bar{X}_t$;

As altruism increases (i.e., λ decreases), fishing effort is lower, fish stock levels are higher.

Mixed Population Fisher and Fish Stock Model (I)

We model a large population that is a mixture of p proportion of non-cooperative fishers and $(1-p)$ proportion of cooperative fishers.

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The **state** and the **common pool resource** dynamics are given as:

→ Representative non-cooperative agent:

$$dX_t^{\mathrm{NC}} = \alpha_t^{\mathrm{NC}} dt + \sigma dW_t^{\mathrm{NC}}, \qquad X_0^{\mathrm{NC}} \sim \mu_0^{\mathrm{NC}}$$

$$dK_t^{\mathrm{NC}} = rK_t^{\mathrm{NC}}\Big(1 - \frac{K_t^{\mathrm{NC}}}{\mathcal{K}}\Big) - qK_t^{\mathrm{NC}}\big(p\bar{X}_t^{\mathrm{NC}} + (1-p)\bar{X}_t^{\mathrm{C}}\big), \qquad K_0^{\mathrm{NC}} = \rho\mathcal{K}.$$

→ Representative cooperative agent:

$$dX_t^{\mathrm{C}} = \alpha_t^{\mathrm{C}} dt + \sigma dW_t^{\mathrm{C}}, \qquad X_0^{\mathrm{C}} \sim \mu_0^{\mathrm{C}}$$

$$dK_t^{\mathrm{C}} = rK_t^{\mathrm{C}}\Big(1 - \frac{K_t^{\mathrm{C}}}{\mathcal{K}}\Big) - qK_t^{\mathrm{C}}\big(p\bar{X}_t^{\mathrm{NC}} + (1-p)\bar{X}_t^{\boldsymbol{\alpha}^{\mathrm{C}}}\big), \qquad K_0^{\mathrm{C}} = \rho\mathcal{K}$$

Note: At the equilibrium: $K_t^{\mathrm{NC}} = K_t^{\mathrm{C}} = K_t$.

Mixed Population Fisher and Fish Stock Model (II)

The **cost functions** are as follows:

→ Representative non-cooperative agent:

$$\mathbb{E}\Big[\int_0^T \Big(\underbrace{-P\big(qK_t^{\mathrm{NC}}(p\bar{X}_t^{\mathrm{NC}} + (1-p)\bar{X}_t^{\mathrm{C}})\big)K_t^{\mathrm{NC}}X_t^{\boldsymbol{\alpha}^{\mathrm{NC}}}}_{\text{Fishing revenue}}$$

$$\underbrace{+c_1^{\mathrm{NC}}X_t^{\boldsymbol{\alpha}^{\mathrm{NC}}} + \frac{c_2^{\mathrm{NC}}}{2}(X_t^{\boldsymbol{\alpha}^{\mathrm{NC}}})^2 + \frac{c_3^{\mathrm{NC}}}{2}(\alpha_t^{\mathrm{NC}})^2}_{\text{Fishing costs}}\Big)dt\Big].$$

→ Representative cooperative agent:

$$\mathbb{E}\Big[\int_0^T \Big(-P\big(qK_t^{\mathrm{C}}(p\bar{X}_t^{\mathrm{NC}} + (1-p)\bar{X}_t^{\boldsymbol{\alpha}^{\mathrm{C}}})\big)K_t^{\mathrm{C}}X_t^{\boldsymbol{\alpha}^{\mathrm{C}}}$$

$$+ c_1^{\mathrm{C}}X_t^{\boldsymbol{\alpha}^{\mathrm{C}}} + \frac{c_2^{\mathrm{C}}}{2}(X_t^{\boldsymbol{\alpha}^{\mathrm{C}}})^2 + \frac{c_3^{\mathrm{C}}}{2}(\alpha_t^{\mathrm{C}})^2\Big)dt\Big].$$

MP Theoretical Analysis

Theorem: Mixed population mean field equilibrium controls are given by

$$\alpha_t^{\mathrm{NC},*} = -Y_t^{2,\mathrm{NC}}/c_3^{\mathrm{NC}} \qquad \alpha_t^{\mathrm{C},*} = -Y_t^{2,\mathrm{C}}/c_3^{\mathrm{C}}$$

where $(\boldsymbol{K}, \boldsymbol{X}^{\mathrm{NC}}, \boldsymbol{X}^{\mathrm{C}}, \boldsymbol{Y}^{1,\mathrm{NC}}, \boldsymbol{Y}^{1,\mathrm{C}}, \boldsymbol{Y}^{2,\mathrm{NC}}, \boldsymbol{Y}^{2,\mathrm{C}}, \boldsymbol{Z}^{1,\mathrm{NC}}, \boldsymbol{Z}^{1,\mathrm{C}}, \boldsymbol{Z}^{2,\mathrm{NC}}, \boldsymbol{Z}^{2,\mathrm{C}})$ solve:

$$
\begin{aligned}
dK_t &= \left[rK_t\left(1 - K_t/\mathcal{K}\right) - qK_t(p\bar{X}_t^{\mathrm{NC}} + (1-p)\bar{X}_t^{\mathrm{C}})\right]dt \quad \rightarrow \textbf{Common for both}\\
dX_t^{\mathrm{NC}} &= -Y_t^{2,\mathrm{NC}}/c_3^{\mathrm{NC}}dt + \sigma^{\mathrm{NC}}dW_t^{\mathrm{NC}}\\
dX_t^{\mathrm{C}} &= -Y_t^{2,\mathrm{C}}/c_3^{\mathrm{C}}dt + \sigma^{\mathrm{C}}dW_t^{\mathrm{C}}\\
dY_t^{1,\mathrm{NC}} &= -\Big[\left(r - \tfrac{2r}{\mathcal{K}}K_t - q(p\bar{X}_t^{\mathrm{NC}} + (1-p)\bar{X}_t^{\mathrm{C}})\right)Y_t^{1,\mathrm{NC}}\\
&\qquad + 2p_1qK_t(p\bar{X}_t^{\mathrm{NC}} + (1-p)\bar{X}_t^{\mathrm{C}})X_t^{\mathrm{NC}} - p_0X_t^{\mathrm{NC}}\Big]dt + Z_t^{1,\mathrm{NC}}dW_t^{\mathrm{NC}}\\
dY_t^{2,\mathrm{NC}} &= -\Big[-p_0K_t + p_1qK_t^2(p\bar{X}_t^{\mathrm{NC}} + (1-p)\bar{X}_t^{\mathrm{C}}) + c_1^{\mathrm{NC}} + c_2^{\mathrm{NC}}X_t^{\mathrm{NC}}\Big]dt + Z_t^{2,\mathrm{NC}}dW_t^{\mathrm{NC}}\\
dY_t^{1,\mathrm{C}} &= -\Big[\left(r - \tfrac{2r}{\mathcal{K}}K_t - q(p\bar{X}_t^{\mathrm{NC}} + (1-p)\bar{X}_t^{\mathrm{C}})\right)Y_t^{1,\mathrm{C}}\\
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&\qquad - qK_t(1-p)\bar{Y}_t^{1,\mathrm{C}} + p_1qK_t^2(1-p)\bar{X}_t^{\mathrm{C}}\Big]dt + Z_t^{2,\mathrm{C}}dW_t^{\mathrm{C}}\\
K_0 &= \rho\mathcal{K}, \quad X_0^{\mathrm{NC}} \sim \mu_0^{\mathrm{NC}}, X_0^{\mathrm{C}} \sim \mu_0^{\mathrm{C}}, \quad Y_T^{1,\mathrm{NC}} = Y_T^{2,\mathrm{NC}} = Y_T^{1,\mathrm{C}} = Y_T^{2,\mathrm{C}} = 0
\end{aligned}
$$

MP Numerical Results

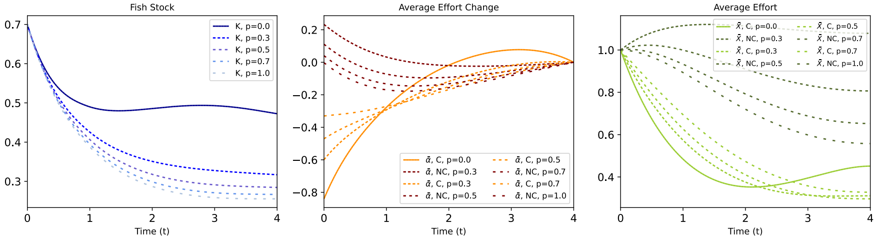


Figure 7: Left: Fish Stock over time: K_t; **Middle:** Average mixed population mf equilibrium change in fishing effort over time $\bar{\alpha}_t^*$; **Right:** Average mixed population mf equilibrium fishing effort over time $\bar{X}_t$;

MP Numerical Results

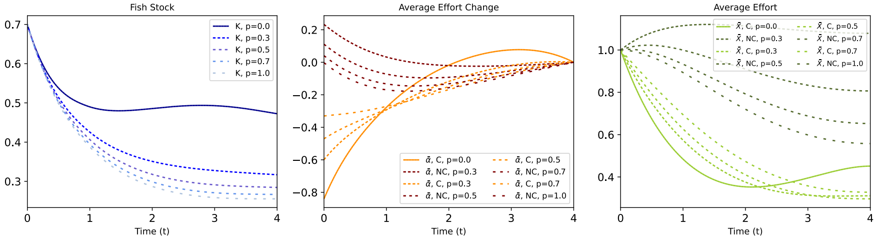
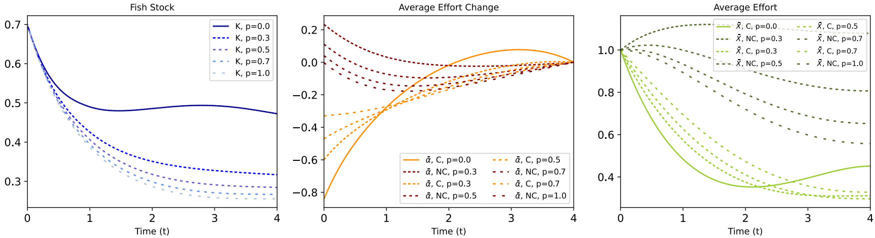


Figure 7: Left: Fish Stock over time: K_t; **Middle:** Average mixed population mf equilibrium change in fishing effort over time $\bar{\alpha}^*_t$; **Right:** Average mixed population mf equilibrium fishing effort over time $\bar{X}_t$;

i) Cooperative agents fish less to protect the fish stock levels. Non-cooperative agents enjoy *free-riding*.

MP Numerical Results



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i) Cooperative agents fish less to protect the fish stock levels. Non-cooperative agents enjoy *free-riding*. ii) As proportion of cooperative agents increases (i.e., p decreases), fish stock levels are higher.

Comparison of MI and MP Results

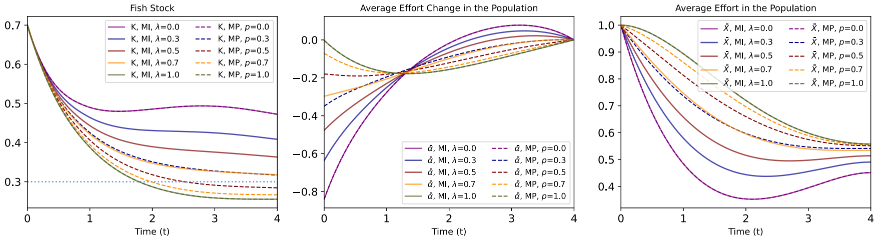


Figure 8: Left: Fish Stock over time: K_t; **Middle:** Average mixed individual and population mf equilibrium change in fishing effort over time $\bar{\alpha}_t^*$; **Right:** Average mixed individual and population mf equilibrium fishing effort over time $\bar{X}_t$;

Comparison of MI and MP Results

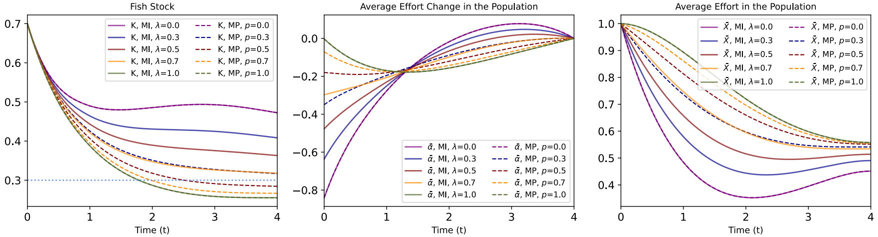


Figure 8: Left: Fish Stock over time: K_t; **Middle:** Average mixed individual and population mf equilibrium change in fishing effort over time $\bar{\alpha}_t^*$; **Right:** Average mixed individual and population mf equilibrium fishing effort over time $\bar{X}_t$;

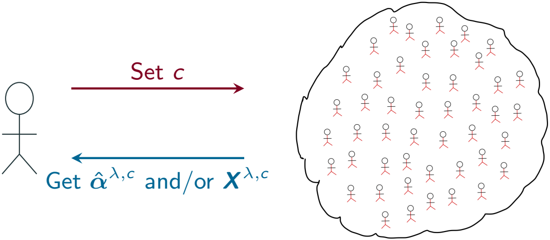
At the same level of λ and p, fish stock level is lower and fishing efforts are higher in the mixed population case. $\Rightarrow$ It is better to have everyone with some altruistic tendencies instead of having a mixture of fully cooperative and fully non-cooperative agents.

How can we learn
altruism levels from data?

Learning Altruism Levels from Data (I)

Motivation: Incentive design builds on full model information. However, altruism levels may not be externally observed.

Idea: Observe the state or control flows of agents, then inverse learn λ.[6]



In Mixed Individual MFG model, we saw that the model is characterized with a forward-backward differential equation system.

$\Rightarrow$ This system is parameterized with λ and characterizes $\hat{\alpha}$ and the corresponding $\boldsymbol{X}$.

[6] *"Inverse Learning of the Altruism and Cost Level in Mixed-Individual Mean Field Games"* by Cao, **G.D.**, Shi (2026)

Learning Altruism Levels from Data (II)

We start with the following model form:

$$J(\alpha;\boldsymbol{\mu}) = \frac{1}{2}\mathbb{E}\left[\int_0^T c_\alpha \alpha_t^2 + c_x\left(X_t^\alpha\right)^2 + c_\mu f(\mu_t^\alpha, \mu_t; \lambda)\, dt\right],$$

where the controlled process follows

$$dX_t^\alpha = (b_\alpha \alpha_t + b_x X_t)\, dt + \sigma dW_t, \qquad X_0 \sim \mu_0.$$

Our aim is to learn λ (and c_α) from observed data.

Theorem: MI-MFNE control profile $\hat{\boldsymbol{\alpha}}$ is given as:

$$\hat{\alpha}_t = -\frac{b_\alpha \partial_x u(t,x)}{c_\alpha}$$

where $(u,m) = (u_t, m_t)_{t\in[0,T]}$ solves the following forward-backward partial differential equation (FBPDE) system:

$$-\partial_t u_t = \tfrac{1}{2}\sigma^2 \partial_{xx}^2 u(t,x) - \tfrac{b_\alpha^2}{2c_\alpha}\left(\partial_x u(t,x)\right)^2 + b_x x \partial_x u(t,x) + \tfrac{c_x}{2}x^2 + \tfrac{c_\mu}{2}\bar{x}_t^2 + c_\mu(1-\lambda)\bar{x}_t x,$$

$$\partial_t m_t = \tfrac{\sigma^2}{2}\partial_{xx}^2 m(t,x) - \partial_x\Big(m(t,x)\big(-\tfrac{b_\alpha^2}{c_\alpha}\partial_x u(t,x) + b_x x\big)\Big),$$

$$u_T = 0, \qquad m(0,x) = \mu_0(x),$$

where $\bar{x}_t = \int x m(t,x) dx$.

Learning Altruism Levels from Data: Identifiability & Experiments

Theorem: Given $\boldsymbol{X}$ and $\boldsymbol{\alpha}$ observations, λ (or the couple (λ, c_α)) can be uniquely identified.

We simulate M observations of $N = 100$ agents' state and control flows at discrete time. By using this simulated data, we learn the unknown parameters.

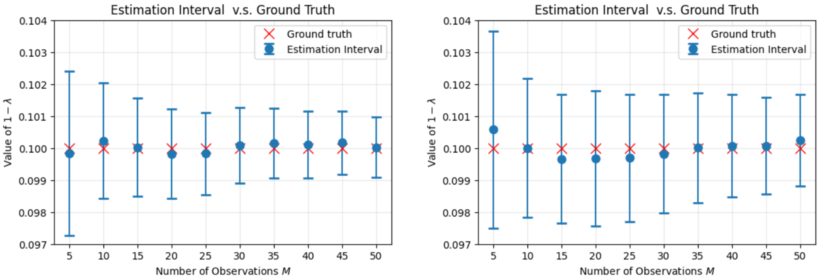


Figure 9: Visualization of the learned $1 - \lambda$ from observed data vs. the ground truth values when only λ is unknown (left) and when both λ and c_α are unknown (right).

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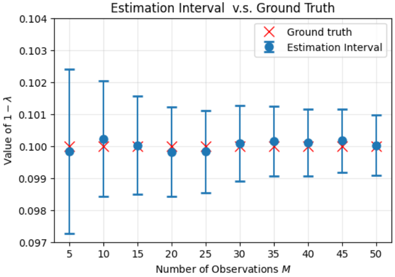

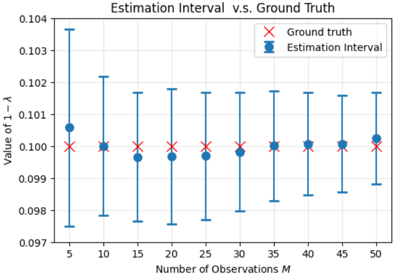

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Ongoing: General model form, introducing the optimization of the regulator.

Conclusion

→ We defined and analyzed different equilibrium notions when there are both cooperative and competitive tendencies in the population.

- Mixed Individual Models
- Mixed Population Models
- We also modeled Common Pool Resources in these models

→ We discussed equilibrium definitions, characterizations, special cases.

→ We introduced, analyzed and solved a motivating example of fisher and fish stock.

→ We discussed the inverse learning problem for learning the altruism levels and presented initial results.

Thank you!

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