

Mean field games
with partial information and
relative performance criteria

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Main problem

Study MFG where agents interact because of relative performance concerns in a common market with partial information about the stock's drift.

Joint work with P. Souganidis

Fundamental difficulties

1. Controls appear in both the drift and the volatility
2. Controls are not in general compat
3. Degeneracies ; lack of smoothness
4. General utilities
5. General couplings

The mean-field model

1. Agents invest in a market with a stock and a bond
2. The drift of the stock is not directly observable
3. The objective incorporates the representative agent's wealth as well as the wealth (e.g. average, low, running maxima,...) of the competitors
4. Both the utility and the couplings are general

MFG in portfolio management
with
relative performance concerns

- Finite game (Espinoza-Touzi, 2015)
- MFG (Lacker-Z., 2019)



So far, only complete information portfolio models

1. MFG for CRRA and multiplicative couplings
2. MFG for CARA and additive couplings
3. MFG for general utilities and couplings
4. MFG under forward criteria

MFG under partial information; quadratic criteria, uncontrolled volatility,

The solution roadmap

1. Start with the N-player game
2. Incorporate filtering techniques to identify the "projected" market
3. Pass (formally) to the limit as $N \uparrow \infty$ to deduce a "master system" consisting of the candidate master equation and an optimality condition for the MFG optimal controls
4. Solve the "master system"
5. Produce the optimal policy processes.

The market

- Trading horizon $[0, T]$; $r=0$ (bond)
- The stock

$$dS_t = \ominus S_t dt + S_t dW_t \quad (\sigma=1)$$

$$(\Omega, \mathcal{F}, \mathbb{P}) ; W_t \rightarrow \mathcal{F}_t^W$$

$$\ominus \rightarrow \text{distribution } \nu, \int_{\mathbb{R}} |\ominus| d\nu < \infty$$

$\ominus$ and W are independent

$$\gamma_t := \ominus t + W_t \quad 0 \leq t < T$$

$$\mathcal{F}_t^\gamma := \sigma(\gamma_s; 0 \leq s \leq t) \subseteq \sigma(\ominus) \vee \mathcal{F}_t^W$$

Complete information case

- Espinoza-Touzi (N-game)
- Lacker-Z (N-game and MFG)

↓ Probabilistic dfn

extensive literature but only for MFG under the special cases:

- $$U\left(X_T^i, \frac{1}{N-1} \sum_{j \neq i} X_T^j\right) = \exp\left(-\left(X_T^i - \frac{\theta_i}{N-1} \sum_{j=1}^N X_T^j\right)\right)$$
- $$U\left(X_T^i, \frac{1}{N-1} \sum_{j \neq i} X_T^j\right) = \frac{1}{\gamma_i} (X_T^i)^{\gamma_i} \left[\prod_{j \neq i} X_T^j\right]^{-\theta_i/(N-1)}$$

These cases give static (random type) slns!

General case in log-normal markets
in unbounded domain (Souranidis-Z.)
2024

$$U\left(x_T^i, \frac{1}{N-1} \sum_{j \neq i} x_T^j\right)$$

- (1) Analyze the N -game $(HJB)_i$ $i=1, \dots, N$ eqn
- (2) Find the optimality condition
- (3) Pass to the limit in both (1) and (2)

Derive a master system composed
by the master equation and the optimality
condition in the limit

General stns for separable
payoffs of form:

$$U(X_T^i, \frac{1}{N-1} \sum_{\hat{j} \neq i} X_T^{\hat{j}})$$

$$= J \left(X_T^i - C \left(\frac{1}{N-1} \sum_{\substack{\hat{j} \neq i \\ \hat{j}=1}}^N X_T^{\hat{j}} \right) \right)$$

Pair: (J, C) general (mild condns)

The partial information
case.

Review of the single-player game

Wealth process:

$$dX_s = \ominus \pi_s ds + \pi_s dW_s \quad ; \quad t \leq s \leq T$$
$$X_t = x$$

Strategies are only $\mathcal{F}_t^Y$ -adapted

$$dX_s = \pi_s dY_s = \pi_s (\ominus dt + dW_s)$$

Criterion

$$V(x, t; \ominus) = \sup_{\pi} E [U(X_T) / X_t = x]$$

Review of the single-player game

The key quantities are:

$$\hat{\Theta}_t := E[\Theta / \mathcal{F}_t^Y]$$

$$\gamma_t := \mathbb{P}[\Theta / \mathcal{F}_t^Y]$$

Classical filtering results

$$\exists g: \mathbb{R} \times [0, T] \rightarrow \mathbb{R} \text{ s.t. } \hat{\Theta}_t = g(\gamma_t, t)$$

Innovations process : $(\{\mathbb{F}_t^Y\}, \mathbb{P})$ BM

$$N_t := Y_t - \int_0^t \hat{\Theta}_s ds = Y_t - \int_0^t g(Y_s, s) ds$$

"Projected market"

$$\begin{cases} dX_s = g(Y_s, s) \pi_s ds + \pi_s dN_s \\ dY_s = g(Y_s, s) ds + dN_s \end{cases}$$

Complete-information reduced market $\longleftrightarrow$ local factor stock price model.

The N-player game
in the reduced market

The N-player game

$$dX_s^i = g(Y_s, s) \pi_s^i ds + \pi_s^i dN_s \quad ; \quad i=1, \dots, N$$

$$dY_s = g(Y_s, s) ds + dN_s$$

- $u^i(x_1, \dots, x_N, y, t) = \sup_{\pi_{i,N}} E \left[J(x_T^i - C(\frac{1}{N-1} \sum_{\substack{j \neq i \\ j=1}}^N x_T^j)) \mid X_t = (x_1, \dots, x_N), Y_t = y \right]$

Best-response case

- Nash-equilibrium

Key model input : (J, C)

HJB-eqn for the i^{th} player, $i=1, \dots, N$
 (first-response problem)

Assume feedback optimal policies of competitors

$$\pi_{j,N}^* = \pi_{j,N}^*(x_1, \dots, x_N, y, t)$$

$$\begin{aligned} u_t^i + \max_{\pi_{i,N}} & \left(\frac{1}{2} \pi_{i,N}^2 u_{x_i x_i}^i + \pi_{i,N} \sum_{\substack{j=1 \\ j \neq i}}^N \pi_{j,N}^* u_{x_i x_j}^i \right. \\ & \left. + \pi_{i,N} u_{x_i y}^i + g(y, t) \pi_{i,N} u_{x_i}^i \right) + \frac{1}{2} u_{y_i y_i}^i + g(y, t) u_{y_i}^i \\ & + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{\substack{k=1 \\ k \neq i}}^N \pi_{j,N}^* \pi_{k,N}^* u_{x_j x_k}^i + \sum_{\substack{j=1 \\ j \neq i}}^{N-1} g(y, t) \pi_{j,N}^* u_{x_j}^i \\ & + \sum_{j \neq i} \pi_{j,N}^* u_{x_j y}^i = 0 \end{aligned}$$

Passing to the limit
as $N \uparrow \infty$

Passing to the limit $N \uparrow \infty$

$$\pi_{i,N}^*(x_1, \dots, x_N, y, t) \simeq \pi_N^*(x_i, y, \mu^{N,i}, t) \quad N \text{ large}$$

$$u_i^*(x_1, \dots, x_N, y, t) \simeq u_N(x_i, y, \mu^{N,i}, t) \quad N \text{ large}$$

$$\pi_N^*(x_i, y, \mu^{N,i}, t) \longrightarrow \pi^*(x, y, m, t)$$

$$u_N(x_i, y, \mu^{N,i}, t) \longrightarrow U(x, y, m, t)$$

Task 1: pass to the limit at the i th optimality con

Task 2: pass to the limit at the (i^{th}) -HJB eqn
player

Where do the various terms "converge"?

OC : optimality condition

$$\max_{\pi_{i,N}} \left(\frac{1}{2} \pi_{i,N}^2 u_{x_i x_i}^i + \pi_{i,N} \left(\sum_{j \neq i} \pi_{j,N}^* u_{x_i x_j}^i + u_{x_i y} + g(y,t) u_{x_i}^i \right) \right)$$

↓ at the optimum

$$\pi_{i,N} u_{x_i x_i}^i + \sum_{j \neq i} \pi_{j,N}^* u_{x_i x_j}^i + u_{x_i y} + g(y,t) u_{x_i}^i = 0$$

$$\pi^*(x,y,m,t) \downarrow U(x,y,m,t)$$

$$(1) \quad \pi^* U_{xx} + \int \pi^* U_{xm} dm + U_{xy} + g(y,t) U_x = 0$$

MFG - OC

Passing to the limit
 "term-by-term" in the HJB eqn.

- $u_t^i(x_1, \dots, x_N, \gamma, t) \longrightarrow U_t(x, \gamma, m, t)$

- at the maximum:

$$\frac{1}{2}(\pi_{i,N}^*)^2 u_{x_i x_i}^i + \pi_{i,N}^* \sum_{j \neq i} \pi_{j,N}^* u_{x_i x_j}^i + \pi_{i,N}^* u_{x_i \gamma}^i + g(\gamma, t) \pi_{i,N}^* u_{x_i}^i$$



$$\frac{1}{2}(\pi^*)^2 U_{xx} + \pi^* \int \pi^* U_{xm} dm + \pi^* U_{xy}$$

$$+ g(\gamma, t) \pi^* U_x ; \quad \pi^* = \pi^*(x, \gamma, m, t)$$

Passing to the limit "term-by-term" (cont'd)

$$\bullet \quad \frac{1}{2} u_{yy}^i + g(y,t) u_y^i \rightarrow \frac{1}{2} U_{yy} + g(y,t) U_y$$

$$\bullet \quad \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{\substack{k=1 \\ k \neq i}}^N \pi_{j,N}^* \pi_{k,N}^* u_{x_j x_k}^i + \sum_{\substack{j,k \neq i \\ j,k=1}}^N g(y,t) \pi_{j,N}^* u_{x_j}^i$$

$$+ \sum_{\substack{j \neq i \\ j=1}}^N \pi_{j,N}^* u_{x_j}^i \rightarrow \frac{1}{2} \int (\pi^*)^2 U_{mm}(x,y,z,m,t) dm(z)$$

$$+ \frac{1}{2} \int \int \pi^* \pi^* U_{mm} dm dm + g(y,t) \int \pi^* U_m dm + \int \pi^* U_{my} dm$$

Passing to the limit
from the " i^{th} HJB" eqn
to a candidate master eqn

Master equation

$$\begin{aligned} & U_t + \frac{1}{2} U_{\gamma\gamma} + g(\gamma, t) U_{\gamma} \\ & + \frac{1}{2} (\pi^*)^2 U_{xx} + \pi^* \int \pi^* U_{xm} dm + \pi^* U_{xy} \\ & + g(\gamma, t) \pi^* U_x + \frac{1}{2} \int (\pi^*)^2 U_{mz}(x, y, z, m, t) dm(z) \\ & + \frac{1}{2} \int \int \pi^* \pi^* U_{mm} dm dm + \int \pi^* U_{my} dm \\ & + g(y, t) \int \pi^* U_m dm = 0 \end{aligned}$$

with

$$U(x, y, m, T) = J(x, m)$$

and the second equation (optimality condition)

$$\pi^* U_{xx} + \int \pi^* U_{xm} dm = -g(y, t) U_x - U_{xy} \quad ; \quad (x, y, m, t) \in \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2 \times [0, T]$$

Qn: Find a solution pair (U, π^*)

Therefore, the object of study becomes a "master system"



$$\left\{ \begin{array}{l} \text{Master equation : } U \\ + \\ \text{Master optimality condition : } \pi^* \end{array} \right.$$

Construct a solution
to the "master system"

"Separable payoffs" : $J(x, m) = J(x - C(m))$

Then, the problem to consider takes the form:

$$U(x, y, m, t) \\ = \sup_{\pi} E \left[J(X_T - C(m_T)) / X_t = x, Y_t = y, m_t = m \right]$$

Conjecture

Find a pair of functions (u, f) such that

$$U(x, y, m, t) = u(x - f(y, m, t), y, t)$$

u : value function of the single player
(no coupling)

f : dynamic (indifference) price of
claim $C_T = C(m_T^*)$

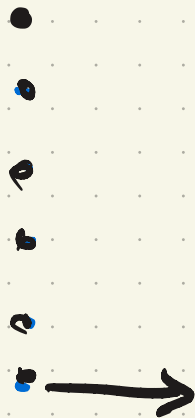
Recall the master equation

$$\begin{aligned} & U_t + \frac{1}{2} U_{yy} + g(y, t) U_y \\ & + \frac{1}{2} (\pi^*)^2 U_{xx} + \pi^* U_{xy} + \pi^* \int \pi^* U_{xy} dm \\ & + g(y, t) \pi^* U_x + \frac{1}{2} \int (\pi^*)^2 U_{mz}(x, y, z, m, t) dm(z) \\ & + \frac{1}{2} \iint \pi^* \pi^* U_{mm} dm dm + \int \pi^* U_{my} dm \\ & + g(y, t) \int \pi^* U_m dm = 0 \end{aligned}$$

Taking derivatives in

$$U(x, y, m, t) = u(x - f(y, m, t), y, t)$$

- differentiate and use the master eqn
- use indifference valuation intuition
for guidance how to group terms



Putting terms together

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Putting terms together

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We arrive at

$$\begin{aligned}
 & u_x(x-f, y, t) \left[f_t + \frac{1}{2} \iint \pi^* \pi^* f_{mm} dm dm \right. \\
 & \left. \frac{1}{2} \int (\pi^*)^2 f_{mz} du + \int \pi^* f_{my} dm + \frac{1}{2} f_{yy} = 0 \right] \\
 & + \left[u_t(x-f, y, t) - \frac{1}{2} \frac{(g(y, t) u_x(x-f, y, t) + u_{xy}(x-f, y, t))^2}{u_{xx}(x-f, y, t)} \right. \\
 & \left. + \frac{1}{2} u_{yy}(x-f, y, t) + g(y, t) u_y(x-f, y, t) \right] = 0
 \end{aligned}$$

The expression for u looks like

$$u_t(z, \gamma, t) - \frac{1}{2} \frac{[g(\gamma, t)u_z(z, \gamma, t) + u_{zy}(z, \gamma, t)]^2}{u_{zz}}$$

$$+ \frac{1}{2} u_{\gamma\gamma} + g(\gamma, t)u_\gamma = 0$$

$$\hat{z} = x - f(\gamma, m, t)$$

this looks $\downarrow$ like a HJB eqn with
no coupling.

$$\text{HJB} \left\{ \begin{aligned} &u_t + \max_{\alpha} \left[\frac{1}{2} \alpha^2 u_{zz} + \alpha u_{zy} + g(\gamma, t) \alpha u_z \right] \\ &+ \frac{1}{2} u_{\gamma\gamma} + g(\gamma, t) u_\gamma = 0 \quad ; \quad u(z, \gamma, T) = J(z) \end{aligned} \right.$$

So, we put together the value fn
of a single agent in the reduced
market and a price but of a security
we don't know (yet)

$$\left\{ \begin{array}{l} U(x, y, m, t) = u(x - f(y, m, t), y, t) \\ + \\ \text{A linear "pricing" equation for } f(y, m, t) \\ \text{of a claim to be determined at} \\ \text{equilibrium.} \end{array} \right.$$

Difficulties

While the auxiliary function u is the value fn of a well-studied complete market problem, we don't have regularity results for general utilities.

$$u(x, y, t) = \sup_{\alpha} E [J(X_T) / X_t = x, Y_t = y]$$

$$dX_s = g(Y_s, s) \alpha_s ds + \alpha_s dN_s, \quad X_t = x \in \mathbb{R}$$

$$dY_s = g(Y_s, s) ds + dN_s, \quad Y_t = y \in \mathbb{R}$$

Results to date

General utilities

duality

HJB and Fenchel
↓

linear pde but not
uniformly elliptic

Homothetic utilities

dual multiplier known

Reduction to 1-d. HJB eqn
Easy to solve

New results : i) regularity
(SZ 2026) ii) optimal processes
in closed form

Regularity

Let $F(\gamma, t) := \int e^{zy - \frac{1}{2}y^2 t} \nu(dy)$; ν Θ 's prior

Define $w(x, \gamma, t) := u(x, \gamma, t) F(\gamma, t)$

Then,

$$\begin{cases} w_t - \frac{1}{2} \frac{w_{xy}^2}{w_{xx}} + \frac{1}{2} w_{yy} = 0 \\ w(x, \gamma, T) = J(x) F(\gamma, T) \end{cases}$$

Function w as a value function

Dynamics

$$d\tilde{X}_s = \tilde{\alpha}_s d\tilde{W}_s, \quad dY_s = d\tilde{W}_s$$

$$\tilde{X}_t = x$$

$$Y_t = y$$

$$w(x, y, t) = \sup_{\tilde{\alpha}} \tilde{E} [J(\tilde{X}_T) F(Y_T) / \tilde{X}_t = x, Y_t = y]$$

HJB eqn

$$\begin{cases} w_t + \max_{\tilde{\alpha}} \left(\frac{1}{2} \tilde{\alpha}^2 w_{xx} + \tilde{\alpha} w_{xy} \right) + \frac{1}{2} w_{yy} = 0 \\ w(x, y, T) = J(x) F(y, T) \end{cases}$$

2. Let

$$h_t + \frac{1}{2} h_{yy} = 0, \quad h(z, y, \tau) = I \left(\frac{e^{-z}}{F(y, \tau)} \right)$$
$$I = (I')^{(-1)}$$

3. Then, the optimal processes $\tilde{X}^*$ and $\tilde{\alpha}^*$ are:

$$\tilde{X}_s^* = h(h^{(1)}(x, y, t), Y_s, s)$$

$$\tilde{\alpha}_s^* = h_z(h^{(1)}(x, y, t), Y_s, s)$$

4. The value function

$$w(x, y, t) = \tilde{\mathbb{E}} \left[h(h^{(1)}(x, y, t), Y_\tau, \tau) / \tilde{X}_t = x, Y_t = y \right]$$

w smooth, unique viscosity soln (in appropriate class)

Back to the original problem

1. The value for u is the unique viscosity soln (in suitable class) of the HJB eqn

2. Let $\hat{u}(x, \gamma, t) = \frac{w(x, \gamma, t)}{F(\gamma, t)}$

Then $\hat{u}$ is smooth and solves the HJB $\rightarrow \dots$

$$u \equiv \hat{u}$$

3. The new soln approach elements

$$\left. \begin{aligned} w_x(h, \gamma, t) &= e^{-x} & (h, 0) \\ u_x(H, \gamma, t) &= e^{-x+k(\gamma, t)} & (H, k) \end{aligned} \right\} \begin{array}{l} \text{pairs} \\ \text{to} \\ \text{find!} \end{array}$$

4.

Pair $(H(z, \gamma, t), k(\gamma, t))$

$$k_t + \frac{1}{2} k_{\gamma\gamma} = \frac{1}{2} g^2(\gamma, t), \quad k(\gamma, T) = 0$$

$$H_t + \frac{1}{2} c^2(\gamma, t) H_{zz} + c(\gamma, t) H_{zy} + \frac{1}{2} H_{\gamma\gamma} = 0$$

$$H(z, \gamma, T) = (J')^{(\epsilon_1)}(e^{-z})$$

$$c_t + \frac{1}{2} c_{\gamma\gamma} = 0, \quad c(\gamma, T) = g(\gamma, T)$$

$$g(\gamma, t) : \quad g(\gamma_s, s) = E[\Theta / \mathcal{F}_s]$$

$$c_t + \frac{1}{2} c_{\gamma\gamma} = 0 ; \quad c(\gamma, T) = g(\gamma, T)$$

5. Optimal wealth and portfolio processes

Let the "market" process

$$L_{t,s} = \int_t^s g(\gamma_{p,p}) c(\gamma_{p,p}) ds + \int_t^s c(\gamma_{p,p}) dN_p$$

Then,

$$X_s^{x,*} = H(H^{(-1)}(x, \gamma, t) + L_{t,s}, \gamma_{s,s})$$

$$\begin{aligned} \alpha_s^{x,*} &= c(\gamma_{s,s}) H_z(H^{(-1)}(x, \gamma, t) + L_{t,s}, \gamma_{s,s}) \\ &\quad + H_\gamma(H^{(-1)}(x, \gamma, t) + L_{t,s}, \gamma_{s,s}) \end{aligned}$$

6. Value function of the single agent problem

$$u(x, y, t)$$

$$= E \left[J \left(H^{(-1)}(H(x, y, t) + L_{t,T}, Y_T, T) \mid Y_t = y \right) \right]$$

These representations play a fundamental role in constructing slns and establishing regularity in the upcoming MFG problem.

Back to the MFG

Representative cases

Case 1: General utility J

Linear coupling on average peer wealth

$$C(m) = C(\bar{m}) = \theta \bar{m}, \quad 0 < \theta < 1$$

Then,

$$f(y, m, t) = f(y, \bar{m}, t) = \theta \bar{m}$$

$$U(x, y, m, t) = u(x - \theta \bar{m}, y, t)$$

Mean field equilibrium wealth $X^{x,*}$

$$X_s^{x,*} = X_s^{x-\theta\bar{m},*} + \frac{\theta}{1-\theta} \int X_s^{x-\theta\bar{m},*} dm(x)$$

where $X_s^{x,*}$ is the wealth of the single agent problem with no competition. Thus,

$$X_s^{x,*} = H(H^{\epsilon_1})(x-\theta\bar{m}, y, t) + L_{t,s}, (\gamma_{s,s})$$

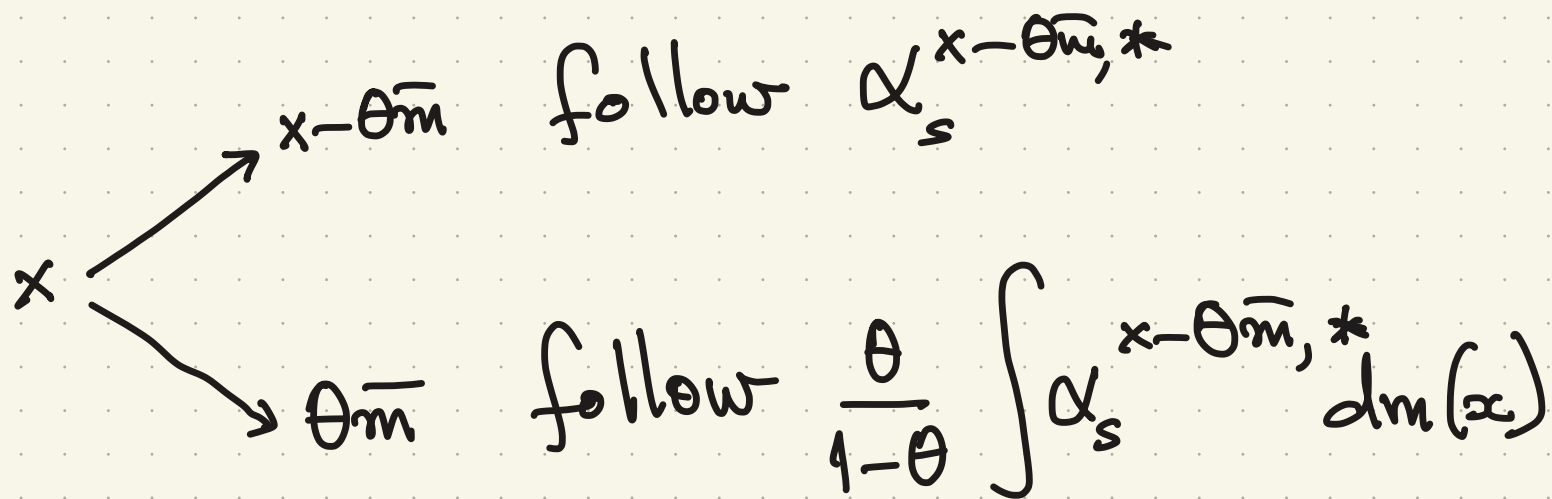
$$+ \frac{\theta}{1-\theta} \int H(H^{\epsilon_1})(x-\theta\bar{m}, y, t) + L_{t,s}, (\gamma_{s,s}) dm(x)$$

Mean field equilibrium policy process

$$\pi_s^{x,*} = \alpha_s^{x-\theta\bar{m},*} + \frac{\theta}{1-\theta} \int \alpha_s^{x-\theta\bar{m},*} dm(x)$$

where $\alpha^{x-\theta\bar{m},*}$ is the optimal policy of single agent problem starting at $x-\theta\bar{m}$

$$\alpha_s^{x-\theta\bar{m},*} = c(y_{s,s}) H_z(H^{(-1)}(x-\theta\bar{m}, y, t) + L_{t,s}, y_{s,s}) + H_y(H^{(-1)}(x-\theta\bar{m}, y, t) + L_{t,s}, y_{s,s})$$



Case 2: Exponential utility: $J(x) = -e^{-x}$
Coupling on the law of peer wealth: $C(m)$

Then, $u(x, y, t) = -e^{-x + k(y, t)}$

$$k_t + \frac{1}{2} k_{yy} = g^2(y, t) ; k(y, T) = 0$$

$$\begin{cases} \alpha^{x,*}(x, y, t) = c(y, t) \\ q_t + \frac{1}{2} c(y, t) = 0, c(y, T) = g(y, T) \end{cases}$$

Therefore,

$$U(x, y, m, t) = -e^{-x + f(y, m, t) + k(y, t)}$$

Optimal MFG $\pi^*(x, y, m, t)$ independent of wealth

$$\pi^*(x, y, m, t) = \pi^*(y, m, t)$$

$$\pi^*(y, m, t) = \frac{f_y(y, m, t) + c(y, t)}{1 - \int f_m(y, m, z, t) dm(z)}$$

Equation for $f(y, m, t)$ becomes autonomous:

$$f_t + \frac{1}{2} \left(\frac{f_y + c}{1 - \int f_m dm(z)} \right)^2 \left[\int \int f_{mm} dm dm + \int f_{mz} dm(z) \right]$$

$$+ \frac{f_y + c}{1 - \int f_m dm(z)} \int f_{my} dm + \frac{1}{2} f_{yy} = 0, \quad f(y, m, T) = c(m)$$

Case 3: Exponential utility: $J(x) = -e^{-x}$
Coupling depends only on the average peer wealth: $C(\bar{m})$

Value of the game

$$U(x, y, m, t) = U(x, y, \bar{m}, t) \\ = -e^{-x + f(y, \bar{m}, t) + k(y, t)}$$

Furthermore, equation for f reduces to:

$$f_t + \frac{1}{2} \frac{(f_y + c(y,t))^2}{(1-f_{\bar{m}})^2} f_{\bar{m}\bar{m}} + \frac{f_y + c(y,t)}{1-f_{\bar{m}}} f_{y\bar{m}} + \frac{1}{2} f_{yy} = 0$$

$$f(y, m, \tau) = f(y, \bar{m}, \tau) = C(\bar{m}),$$

and,

$$\pi^*(x, y, m, t) = \frac{f_y + c(y, t)}{1 - f_{\bar{m}}}$$

Mean field equilibrium processes

$$X_s^{x,*} = x - f(y, \bar{m}, t) + L_{t,s}$$

$$+ f(y_s, G(y_s, x - f(y, \bar{m}, t) + L_{t,s}, s))$$

$$= X^{x-f(y, \bar{m}, t),*}_s + G(y_s, X^{x-f(y, \bar{m}, t),*}_s, s)$$

$$G(y, \bar{m}, t) = (\cdot - f(y, \cdot, t))^{(-1)}(y, \bar{m}, t)$$

Regularity of f established for general $C(\cdot)_s$

Extensions (partial list)

1. Incorporate consumption (tedious but not impossible)

2. Crowding risk

3. Problem set in half-domain (e.g. $J: \mathbb{R}^+ \rightarrow \mathbb{R}$)

3a. Additive couplings

$$J(x - C(m)) \rightarrow u(x - f(y, m, t), y, t)$$

3b. Multiplicative couplings

$$J(x, m) = C(m)J(x)$$

Thank you

and

apologies if I went
overtime !

