Adiabatic charge pumps and Galilei covariance

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Mathematical Aspects of 2D Quantum Materials and Meta-materials
IMSI - Institute for Mathematical and Statistical Innovation
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Based on joint work with Tilman Esslinger, Filippo Santi

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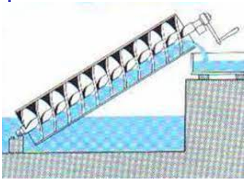
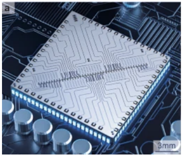
Galilei covariance

Other views

Quantum pumps





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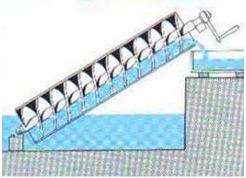

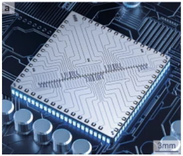

Infinitely extended 1-dimensional system

$$H(t) = p^2 + V(t,x)\,, \quad (p = -\mathrm{i}d/dx)$$

depending on time t. Potential $V = V(t,x)$ doubly periodic

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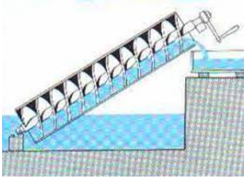

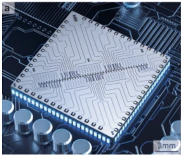

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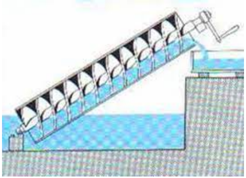

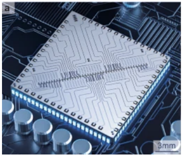

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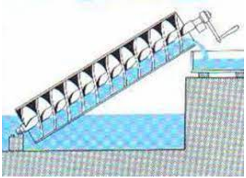

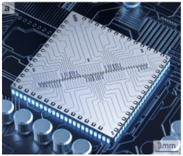

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- Its base space is a 2D torus, somehow reflecting space and time periodicity (1+1=2)
- The pump is a synthetic 2D quantum meta-material!

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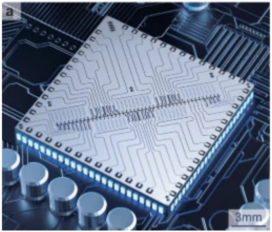
- Experimental realization (Liu et al., 26 authors, 2024: "We demonstrate Thouless pumping in the presence of disorder . . .")



The *Chuang-tzu* 1D superconducting processor

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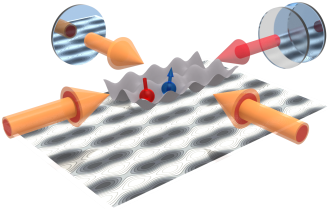

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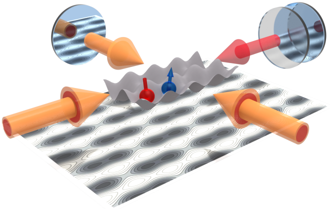

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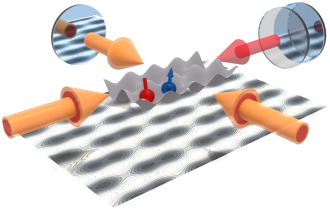

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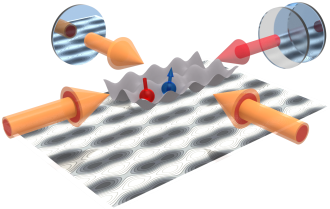

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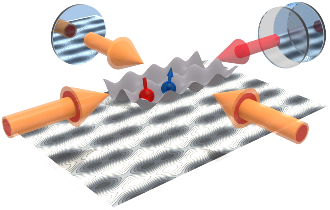

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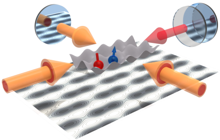

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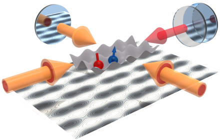

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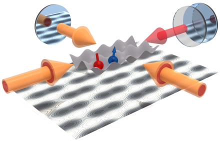

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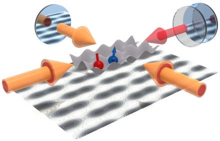

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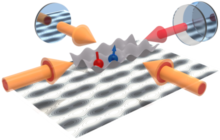
A same landscape, experienced by different observers



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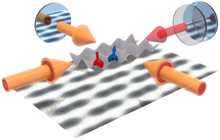

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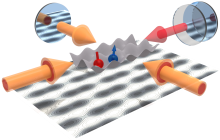

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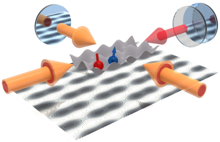

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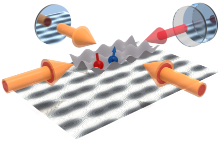

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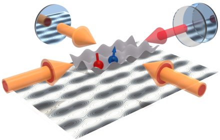

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- Theory has got to be Galilei covariant, not Q invariant

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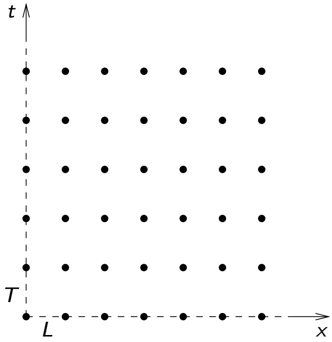

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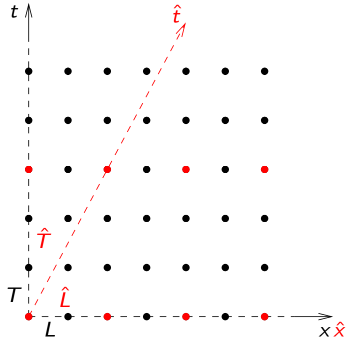

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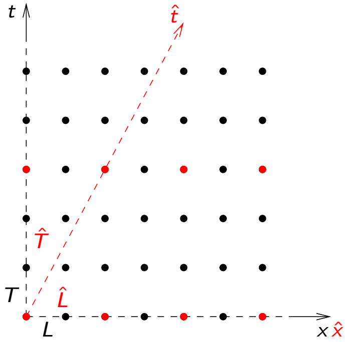

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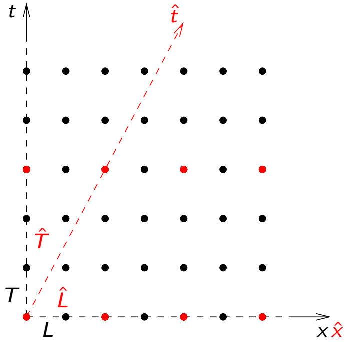

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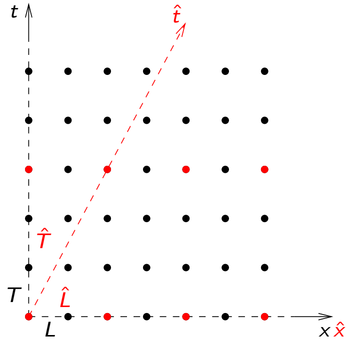

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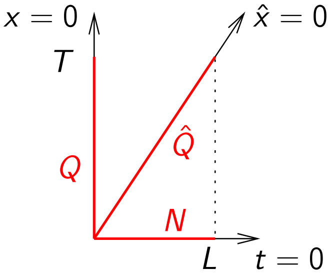

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2-current (ρ, j)
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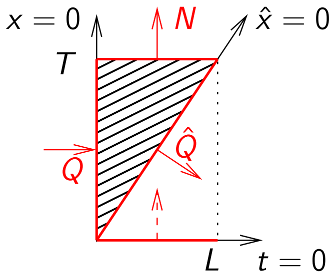

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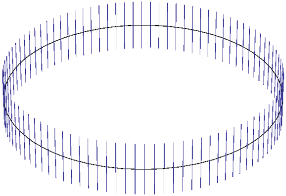

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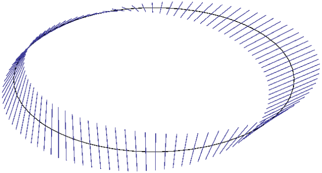

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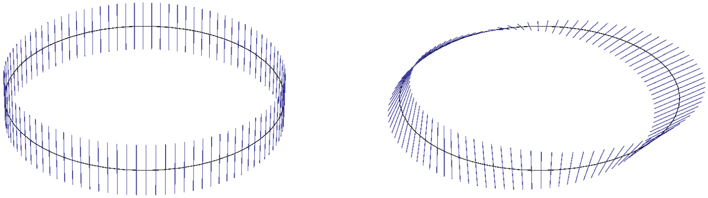

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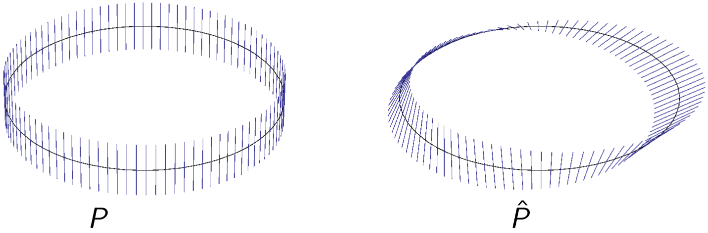

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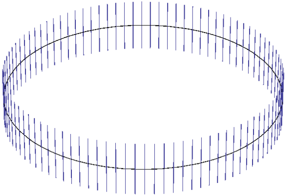

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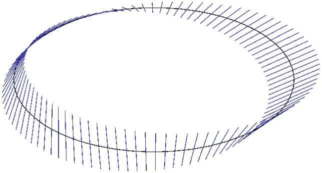

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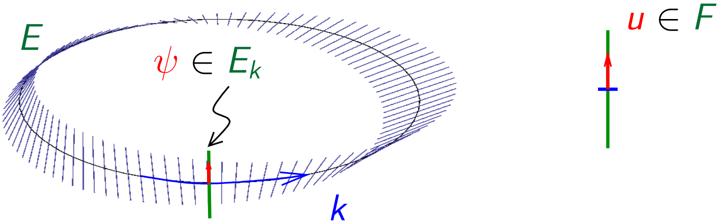

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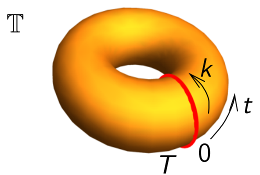

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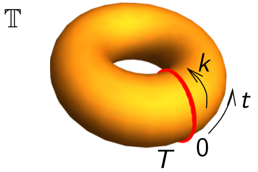

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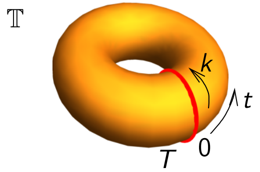

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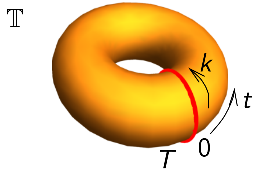

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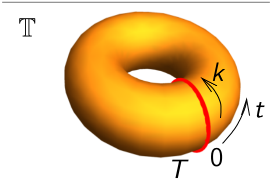

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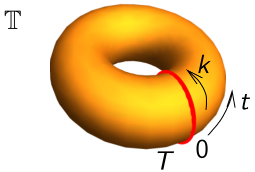

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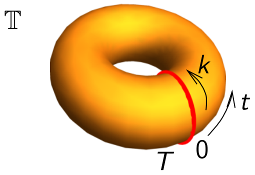

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- all rk P of those Bloch states are being translated by a period

Remark: Strong and weak index

Theorem ($n = m = 1$, i.e. $vL = T$)

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- (Altland-Zirnbauer/Kitaev table). The model belongs to class A (no symmetry).

Symmetry				d							
Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
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- Galilei transformations mix strong and weak indices.

Adiabatic charge pumps

Galilei covariance

Other views

Other views I: No trivializations

Keep the Bloch bundle, but forgo trivializations.

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(trivialization: $[P, x] \rightsquigarrow -\mathrm{i}\partial_k P$)

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Difference between frames $\mathrm{ch}\, \hat{P} - \mathrm{ch}\, P$ evaluated not by CS form, but by the Gauss-Codazzi (GC) equation.

GC equation relates the curvatures of (flat) bundle E and subbundle P.

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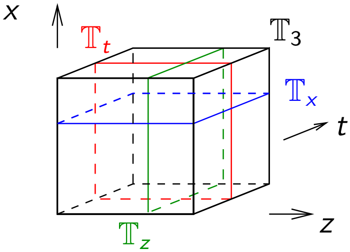

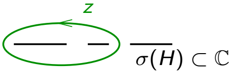

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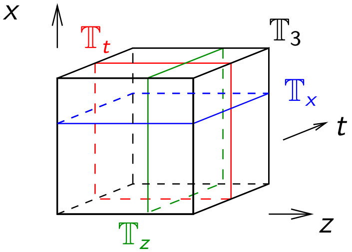

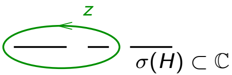

There is vector bundle P on $\mathbb{T}_3$

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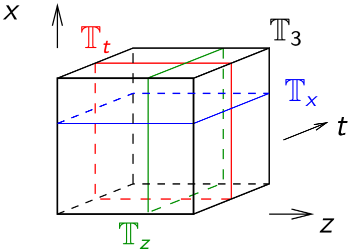

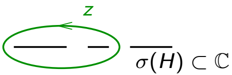

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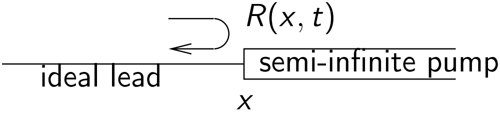

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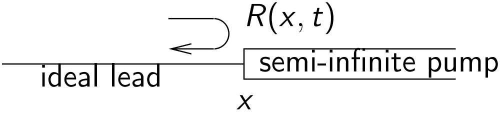

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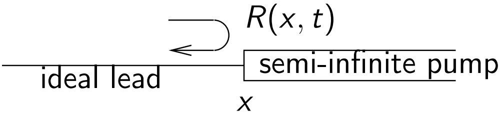

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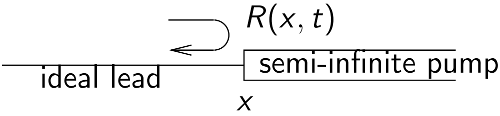

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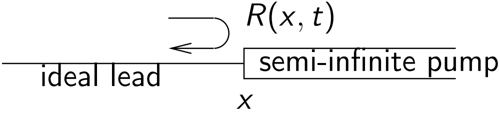

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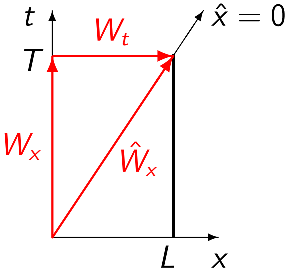

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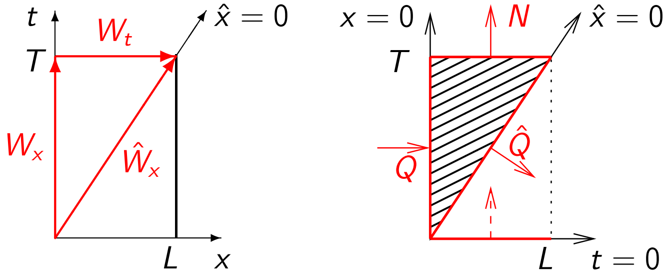

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Thank you all for your attention!

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Identities

$$\begin{aligned}
&\text{(a)} \qquad P[[P, x], [P, p]]P = [PxP, PpP] - P[x, p]P \\
&\text{(b)} \qquad \operatorname{tr}_{E_{t,k}} [PxP, PpP] = \mathrm{i} \frac{\partial}{\partial k} \operatorname{tr}_{E_{t,k}} PpP \\
&\text{(c)} \qquad \operatorname{tr}_{E_{t,k}} P[x, p]P = \mathrm{i} \operatorname{tr}_{E_{t,k}} P = \mathrm{i} \operatorname{rk} P
\end{aligned}$$

- (b) is total derivative, does not contribute
- (c) Heisenberg, yields result, by $(v/2\pi)T(2\pi/L) = 1$
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- (a) is Gauss-Codazzi equation relating curvatures of (flat) bundle E and subbundle P. Indeed Gauss-Codazzi says

$$PRP = R^{(P)} - A^* \wedge A\,.$$

(R ambient curvature (here $= 0$), $R^{(P)}$ induced curvature, A 2nd fundamental form, $\nabla_X = \nabla_X^{(P)} + A(X)$). Identity (a) is (literally)

$$A^* \wedge A = R^{(P)}$$

with $X = x$, $Y = p$ and

$$A(X) = [X,P]P\,, \quad R^{(P)}(X,Y) = [\nabla_X^{(P)}, \nabla_Y^{(P)}] - \nabla_{[X,Y]}^{(P)}$$

Other views II: Improper solutions

Setup

As before,

$$H = -\frac{d^2}{dx^2} + V(x) \qquad \text{on } L^2(\mathbb{R}_x, \mathbb{C}^n)$$

with $V(x) = V^*(x) \in M_n(\mathbb{C})$.

Schrödinger equation (as 2nd order ODE) and its "real adjoint":

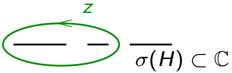
$$(H - z)\psi(x) = 0\,, \qquad \tilde{\psi}(x)(H - z) = 0$$

with $\psi(x)$ (resp. $\tilde{\psi}(x)$) column (row) vectors in $\mathbb{C}^n$

The solutions ψ of the 2nd order ODE are determined by their initial values (ψ_0, ψ_0') at any point $x = x_0$;

A circle and a bundle

- Let $\mathbb{C} \ni z$ be the (complex) energy plane. Spectrum $\sigma(H)$ is real subset
- Let $\gamma \subset \mathbb{C}$ be a loop enclosing all spectra below μ



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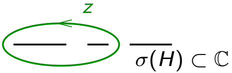

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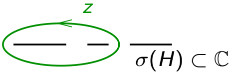

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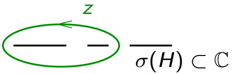

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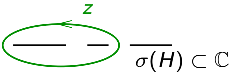

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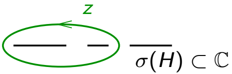

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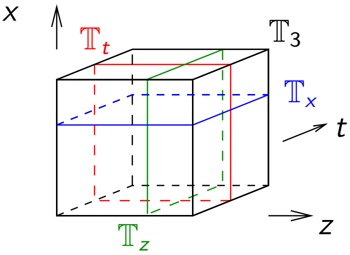

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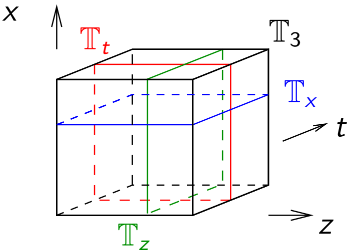

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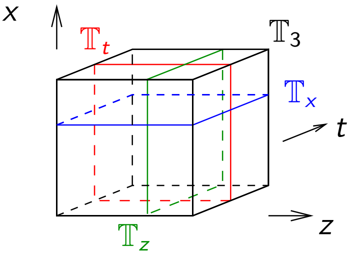

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Theorem [Implementation of Galilei transformations]

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