# Continuum Floquet Materials

## A dynamical Perspective

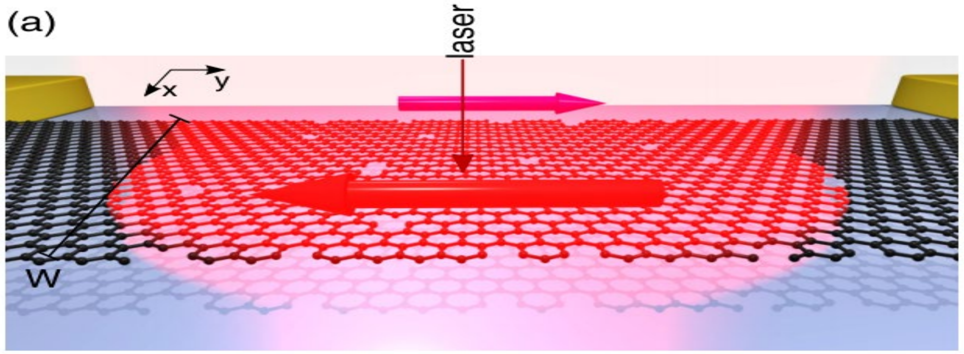
Amir Sagiv // New Jersey Institute of Technology

support by NSF, BSF

# Floquet Materials

Time-periodic changes in materials

Electron in Graphene **driven by a laser beam**




Source: Perez-Piskunow et al
Physical Review B ‘14

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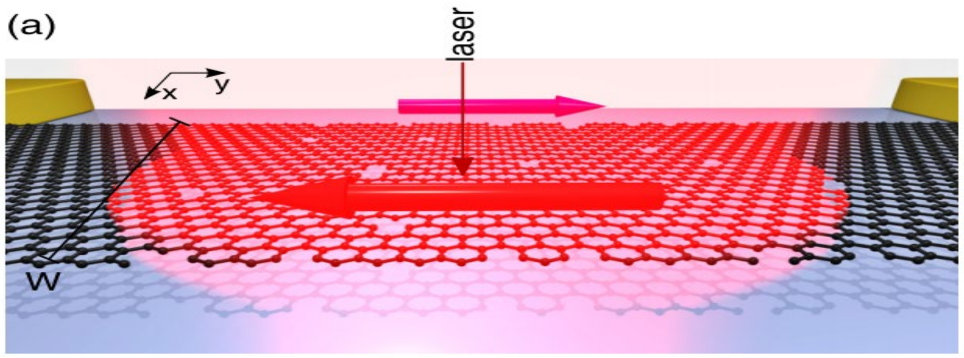


dynamics = lattice + laser

$\boxed{\text{constant in time}}$ $\boxed{\text{periodic in time} \sim e^{-i\omega t}}$

Source: Perez-Piskunow et al
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# Floquet Materials

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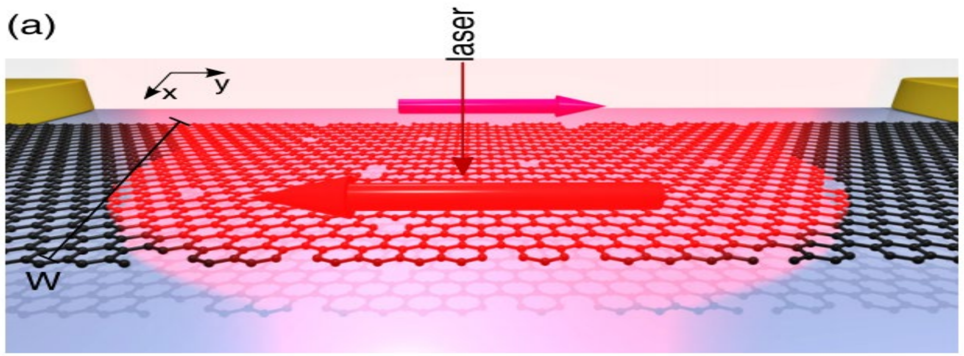
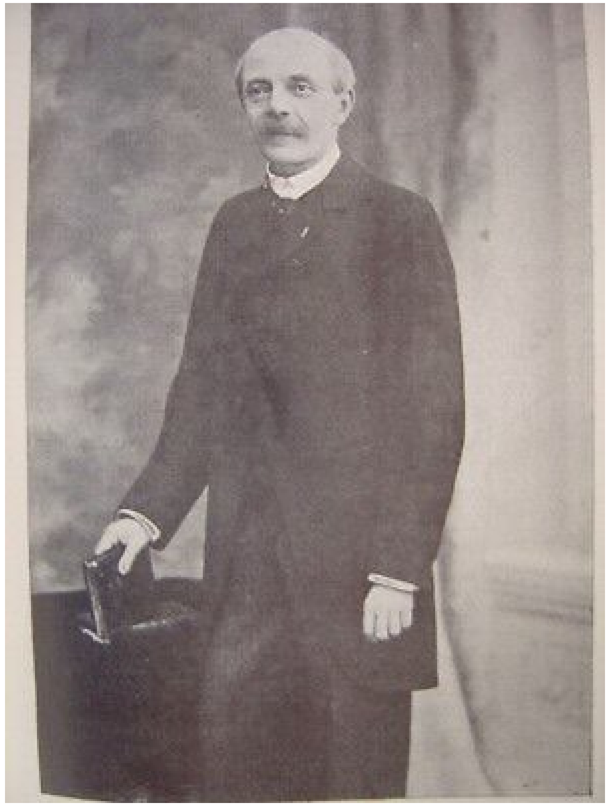


dynamics = lattice + laser

**“Floquet” media (Gaston Floquet)**

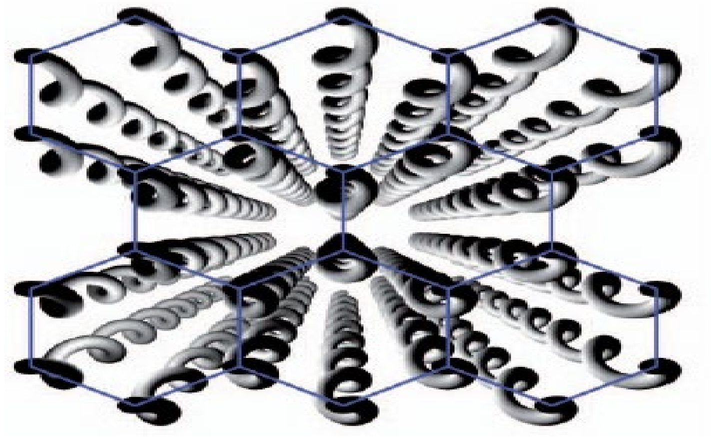
1879- theory of linear, time-periodic, *ordinary* differential equations (ODEs)

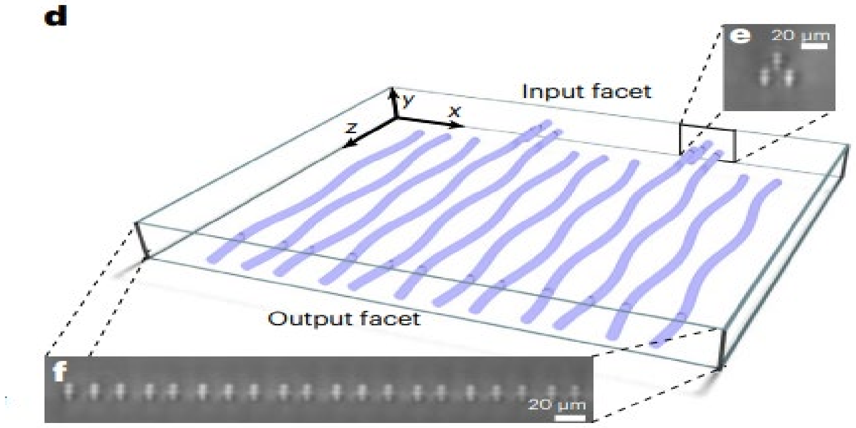
# Floquet Materials

Laser propagation in **twisted** waveguide arrays
"**Floquet** photonic graphene"

Rechtmsan et al Nature '13

[Jurgensen et al, Nature Physics '23]






$$i\psi_z(z,x) = (\underbrace{-\Delta + V(x)} + \underbrace{i\varepsilon A(\varepsilon z) \cdot \nabla})\psi$$

- unforced problem
- lattice structure

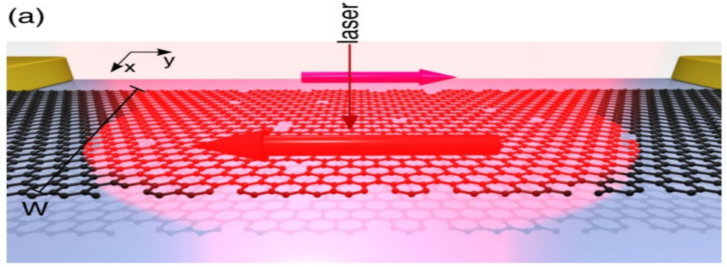
Coiling

$$x \mapsto x - h(z)$$

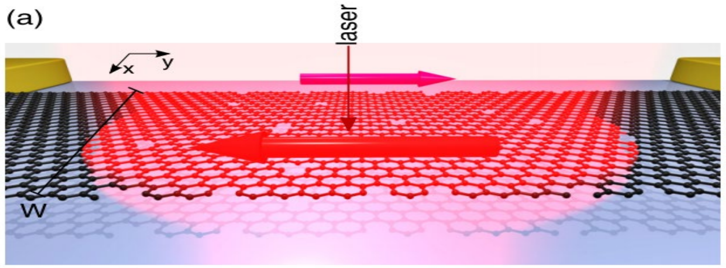
Change of variables

$$A = -h'(z)$$

# Phenomena in Floquet Materials



# Phenomena in Floquet Materials



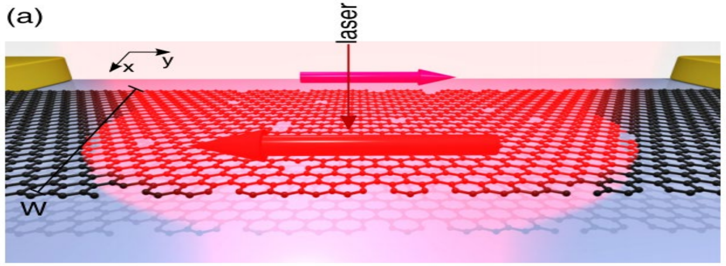

1) Spectral gaps:

Transforming a conductor to an insulator

<u>conducting</u> system


<u>insulating</u> system

# Phenomena in Floquet Materials



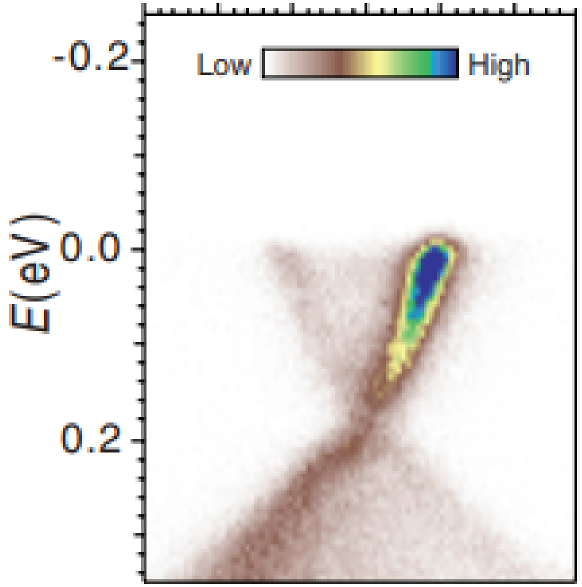

Experimental Data

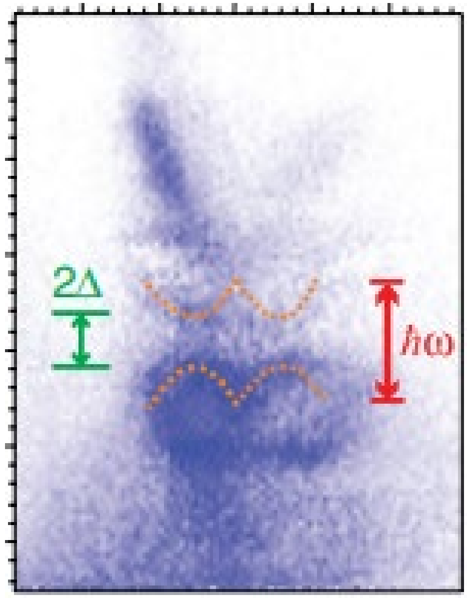
1) Spectral gaps:

Transforming a conductor to an insulator

no light

with light (Floquet)



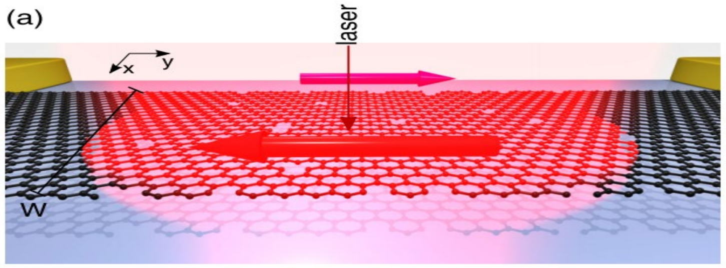

energy gap

Laser-driven Bismuth Selenide

[Wang et al, Science '13]

see also:
- He et al, Nat Com '19 (photonics)
- McIver et al, Nat Phys '19
- Merboldt et al, Nat Phys '25
- Choi et al, Nat Phys '25

# Phenomena in Floquet Materials

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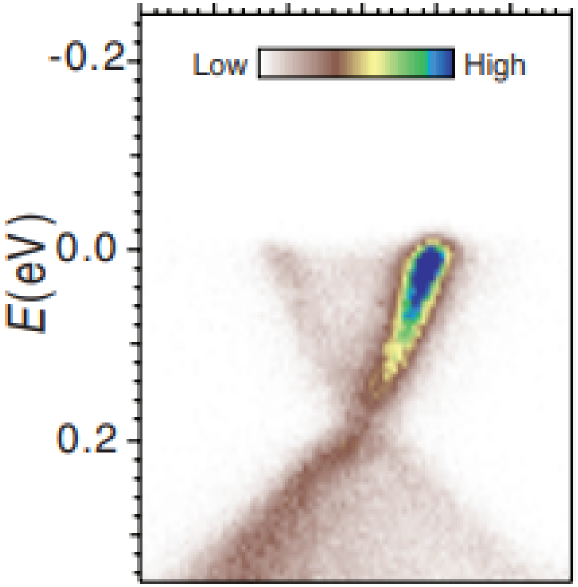


Laser-driven Bismuth Selenide

with light
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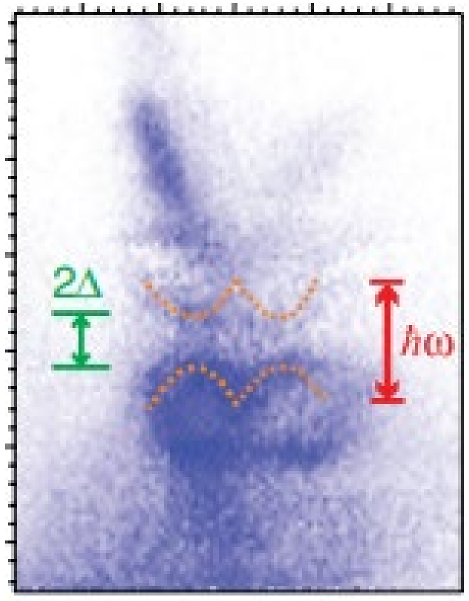


[Wang et al, Science '13]

**No gaps in PDE models**

**Interesting** (I think)
**But not today**

Sagiv, Weinstein,
SIMA 2022,
JST 2026

# Phenomena in Floquet Materials

2) Edge modes – waves (currents) that “stick” to the edge

# Phenomena in Floquet Materials

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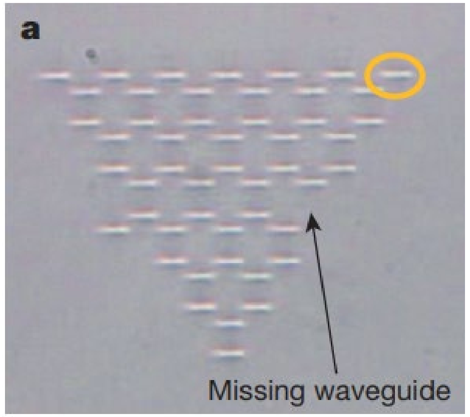


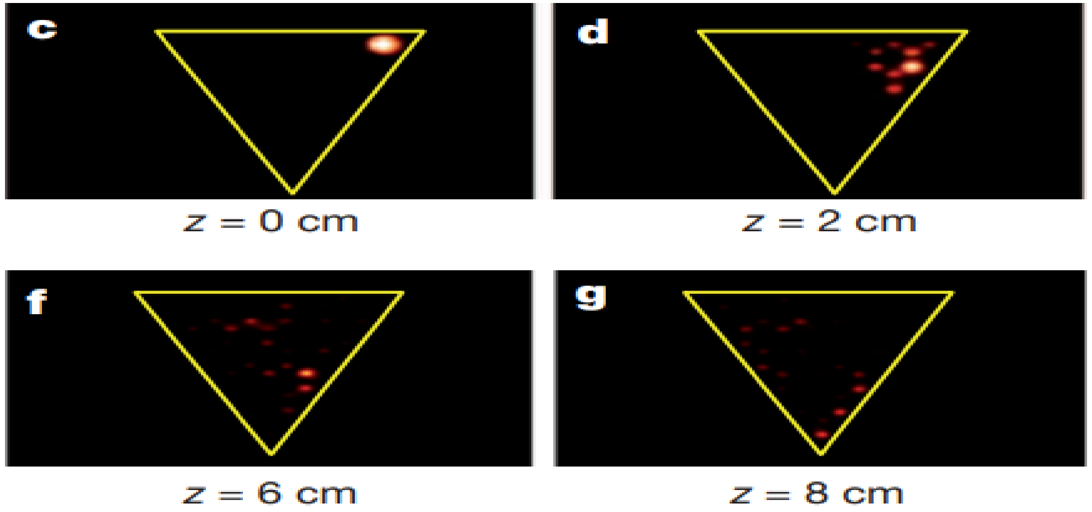

Persistence to noise

(topological protectedness)

- Hameedi, Sagiv, Weinstein,
  **(SIAM MMS 2023)**

see also Bal & Massatt MMS 2022

uni-directionality (chirality)




[Rechtsman et al, Nature `13]

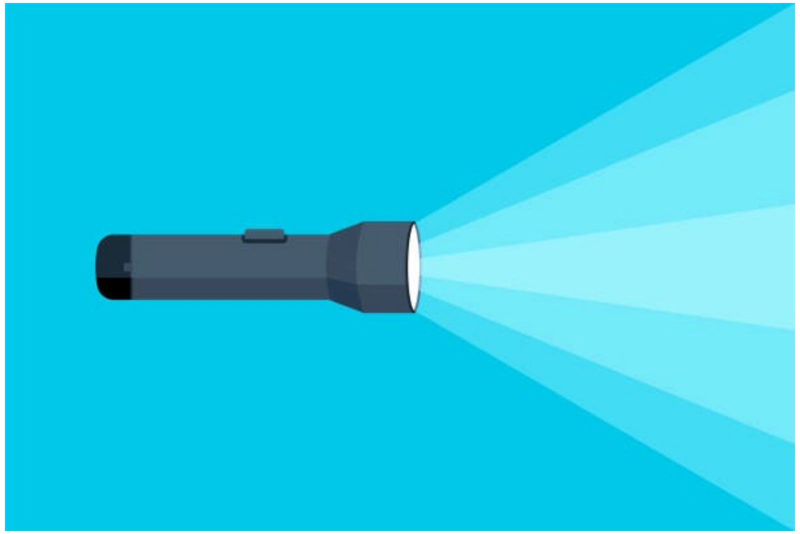
**Towards the end of the talk,**

**This motivates topic 3…**

# Phenomena in Floquet Materials

3) Anomalously Slow Dispersion

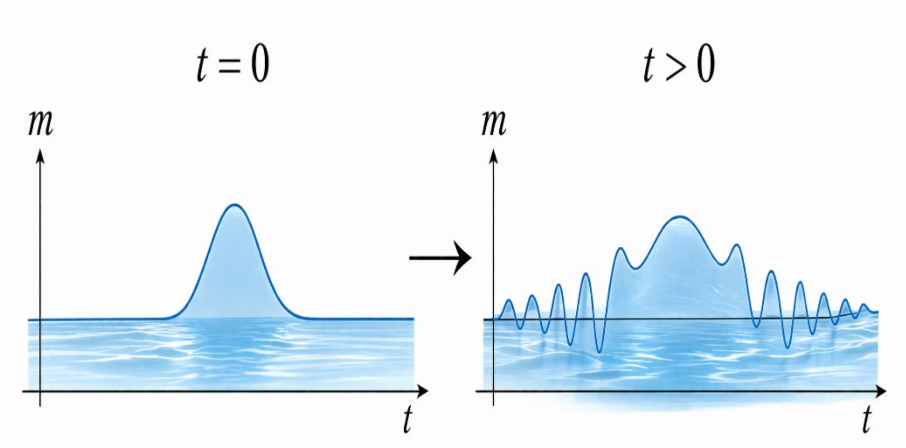
$$\begin{cases} i\partial_t \psi(t) = H(t)\psi \\ \psi(0) = f \end{cases}$$



A fundamental question:

Does $\psi(t)$ disperses as $t \to \infty$?

How fast? In what sense?




**Today's main result –**
**in non-autonomous systems, dispersion can be very slow!**

# Agenda

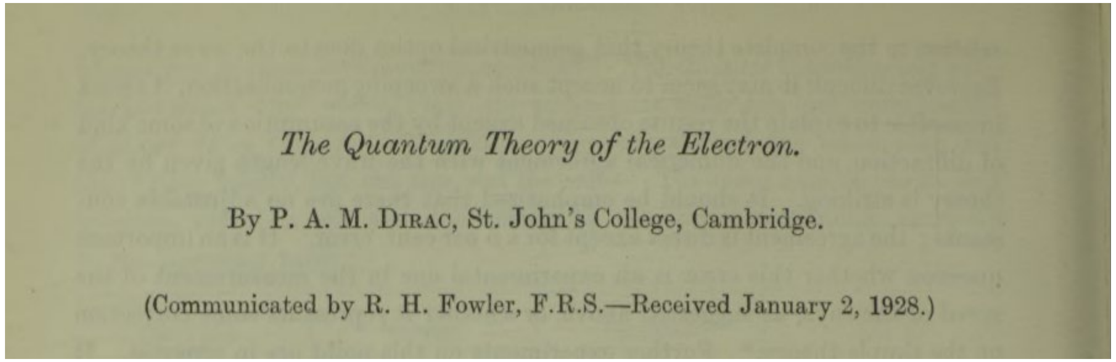
- The time-periodic Dirac equation
- How to get very slow dispersion
- Application to Floquet edge modes
- A discrete Analog

# 1D Dirac Equation

$$\begin{cases} i\partial_t \alpha(t,x) = D\alpha \\ \alpha(0,\cdot) = \alpha_0 \end{cases}$$

$$D = \begin{pmatrix} i\partial_x & m \\ m & -i\partial_x \end{pmatrix} = i\partial_x \sigma_3 + m\sigma_1$$

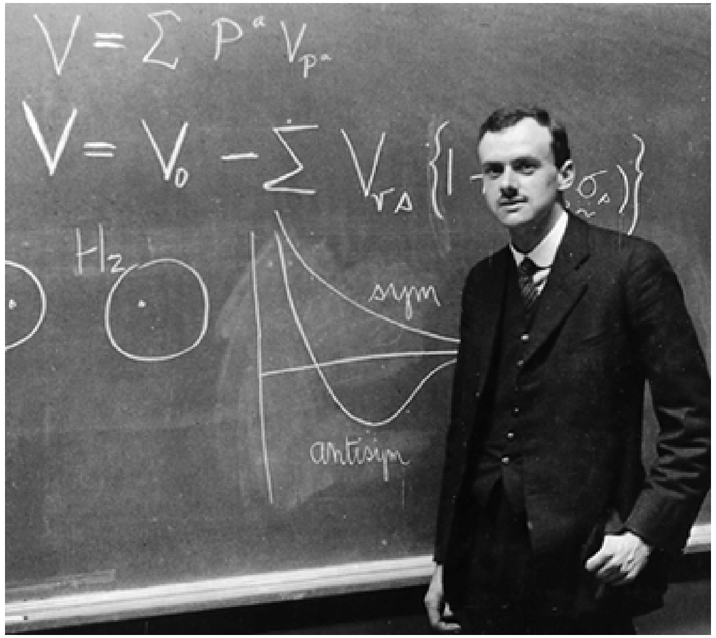
or with a localized potential



*The Quantum Theory of the Electron.*

By P. A. M. Dirac, St. John's College, Cambridge.

(Communicated by R. H. Fowler, F.R.S.—Received January 2, 1928.)




Originally – relativistic quantum electrons

# 1D Dirac Equation

$$\begin{cases} i\partial_t \alpha(t,x) = D\alpha \\ \alpha(0,\cdot) = \alpha_0 \end{cases}$$

$$D = \begin{pmatrix} i\partial_x & m \\ m & -i\partial_x \end{pmatrix} = i\partial_x \sigma_3 + m\sigma_1$$

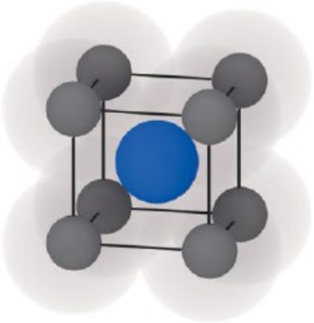
… but today, **time**-dependent

$$D = \begin{pmatrix} i\partial_x & m(t) \\ m(t) & -i\partial_x \end{pmatrix}$$

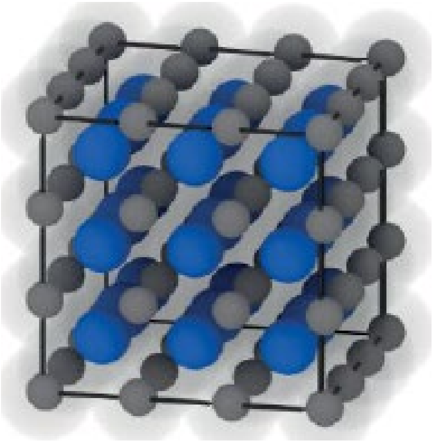
What motivates a time-dependent Dirac equation?

# Homogenization to a Dirac Equation

Non-autonomous Dirac equations are the simplified (homogenized) models for Floquet materials



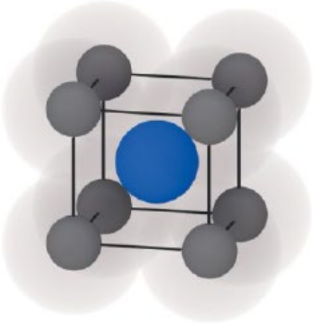
Micro-structure



multi-scale,
periodic

# Homogenization to a Dirac Equation

Non-autonomous Dirac equations are the simplified (homogenized) models for Floquet materials



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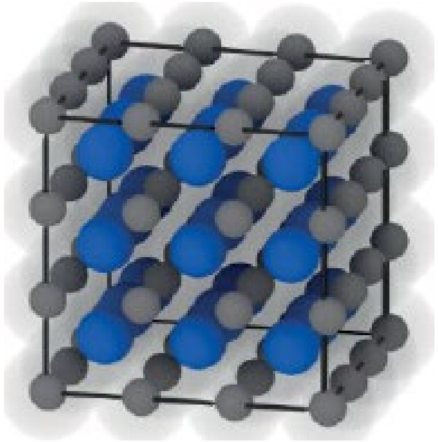

multi-scale,
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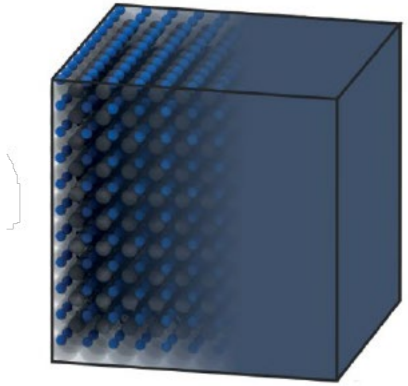


Macro-structure
Homogenized

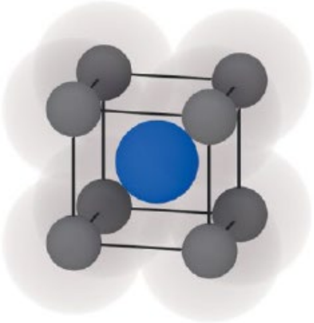
[image: Kadic et al, Nature Physics 2019]

# Homogenization to a Dirac Equation

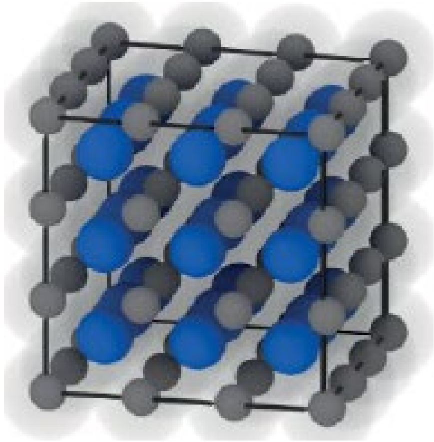
**Non-autonomous Dirac equations are the simplified (homogenized) models for Floquet materials**

Multi-scale structure

$$\psi(t,x) \approx c\boldsymbol{\alpha}(T,X)\Phi\,(x;\pi) \qquad T = \varepsilon t\,, \quad X = \varepsilon x$$



Micro-structure



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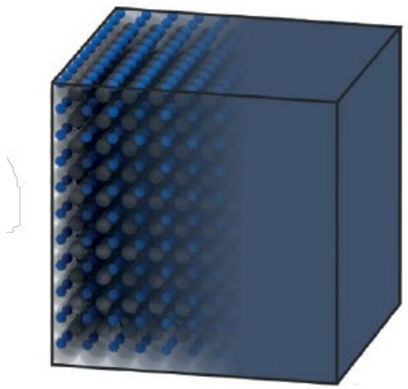


Macro-structure
Homogenized

The goal:

**Describe $\boldsymbol{\alpha}(T,X)$** without the micro-scale

[image: Kadic et al, Nature Physics 2019]

# Homogenization

$$i\partial_t\psi(t,x) = H^{\varepsilon}(t)\psi$$

$$\psi(t=0,x) = \varepsilon\vec{\alpha}_0(\varepsilon x)^{\top}\,\vec{\boldsymbol{\Phi}}(\boldsymbol{x})$$

Slow-fast ansatz

$$\psi(t,x) \approx \vec{\alpha}(\varepsilon x, \varepsilon t)^{\top}\,\vec{\Phi}(x)e^{-iE_D t}$$

$$i\partial_T\vec{\alpha}(T,X) = \mathfrak{D}(T)\vec{\alpha}$$

$$\mathfrak{D}(T) = \mathrm{v_D}\big(\mathrm{i}\nabla_X + A(T)\big)\cdot(\sigma_1,\sigma_2)$$

Homogenized model - only involves slow scales $X, T$

$v_D > 0$ depends on the potential $V(x)$

$\sigma_1, \sigma_2$ - Pauli matrices

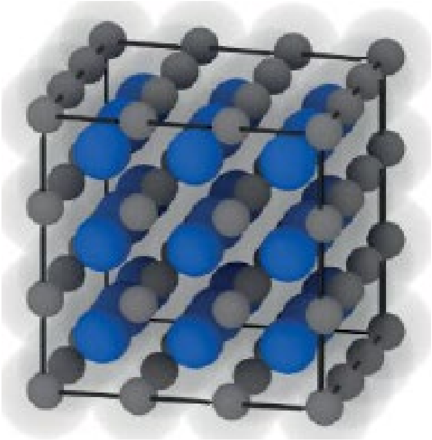
# A moment of rigor

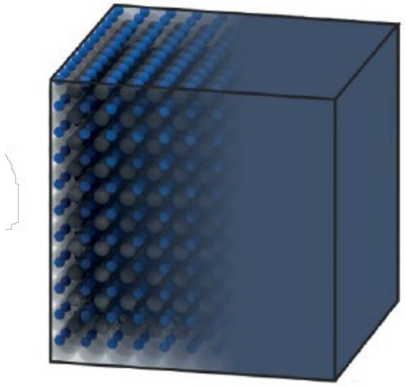
$$i\partial_t \psi^\varepsilon(t,x) = (H + i\varepsilon \mathrm{A}(\varepsilon \mathrm{t})\partial_x)\psi^\varepsilon$$

$$i\partial_T \vec{\alpha}(T,X) = \mathfrak{D}(T)\vec{\alpha}$$

Schrodinger <u>near a Dirac point</u> (like graphene)
Multiscale

Dirac equation
Homogenized





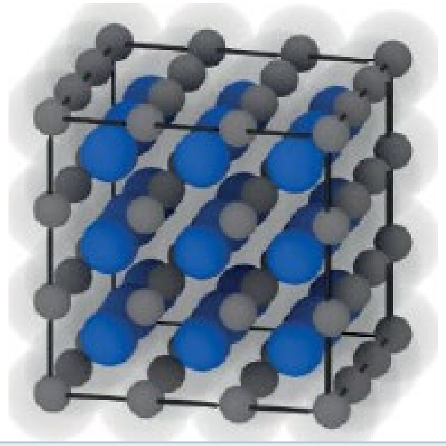
# A moment of rigor

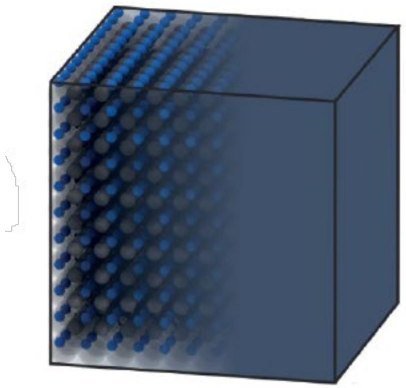
$i\partial_t \psi^\varepsilon(t,x) = (H + i\varepsilon A(\varepsilon t)\partial_x)\psi^\varepsilon$

Schrodinger near a Dirac point (like graphene)
Multiscale

$i\partial_T \vec{\alpha}(T,X) = \mathfrak{D}(T)\vec{\alpha}$

Dirac equation
Homogenized





**Theorems** (AS & Weinstein, SIMA '22)

$\exists C, T_0 > 0$, for all $\boldsymbol{\rho} \in [\mathbf{0}, \mathbf{1})$

$$||\psi^\varepsilon(t,\cdot) - \psi_{effective}(t,\cdot)||_{L^2} \lesssim C(||\vec{\alpha}_0||_{H^4})\boldsymbol{\varepsilon}^{\mathbf{1}-\boldsymbol{\rho}}, \qquad 0 \le t \le T_0 \boldsymbol{\varepsilon}^{-\mathbf{1}-\boldsymbol{\rho}}$$

Autonomous version ($A(T) \equiv 0$): Fefferman Weinstein CMP '14

# Another moment of rigor

$i\partial_t \psi^\varepsilon(t,x) = (H + i\varepsilon A(\varepsilon t)\partial_x)\psi^\varepsilon$

$i\partial_T \vec{\alpha}(T,X) = \mathfrak{D}(T)\vec{\alpha}$

Does regularity matter?

**Forthcoming #1** (GQ Chen, Hameedi, AS, Weinstein)

For $\alpha_0 \in H^1$, **rough** ($L^\infty$) potential, and any $T > 0$, then

$$\lim_{\varepsilon\to 0^+} \sup_{t\in[0,T]} ||\psi(t,\cdot) - \psi_{effective}(t,\cdot)||_{L^2} = 0 \ .$$

**Theorems** (AS & Weinstein, SIMA '22)

For $\alpha_0 \in H^2$ and $|\widehat{\alpha_0}(\xi)| \lesssim \langle\xi\rangle^{-3}$, **rough** ($L^\infty$) potential, $\boldsymbol{\rho \in [0,1]}$

$$||\psi^\varepsilon(t,\cdot) - \psi_{effective}(t,\cdot)||_{L^2} \lesssim C(||\vec{\alpha}_0||_{H^4})\boldsymbol{\varepsilon^{1-\rho}}, \qquad 0 \le t \le T_0\boldsymbol{\varepsilon^{-1-\rho}}$$

# Agenda

- The time-periodic Dirac equation
- How to get very slow dispersion
- Application to Floquet edge modes
- A discrete Analog

With:

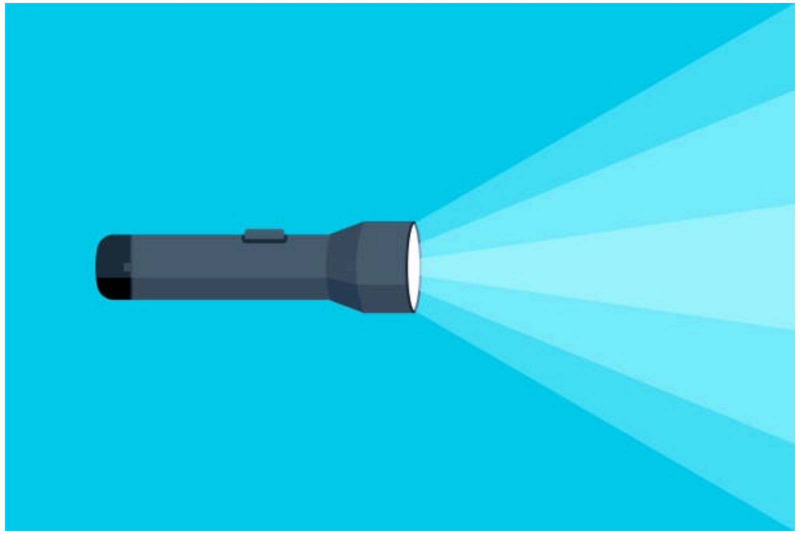
Tony Bloch (Michigan)
Stefan Steinerberger (U Washington)
Joe Kraisler (Amherst)
Michael Weinstein (Columbia)

# Dispersive Decay Estimates

$$i\partial_t \alpha(t,x) = D(t)\alpha \qquad\qquad D = \begin{pmatrix} i\partial_x & m(t) \\ m(t) & -i\partial_x \end{pmatrix}$$

This is a simplified model, a steppingstone to:

- 2D graphene (previous slides)
- 1D with a domain wall (more later)



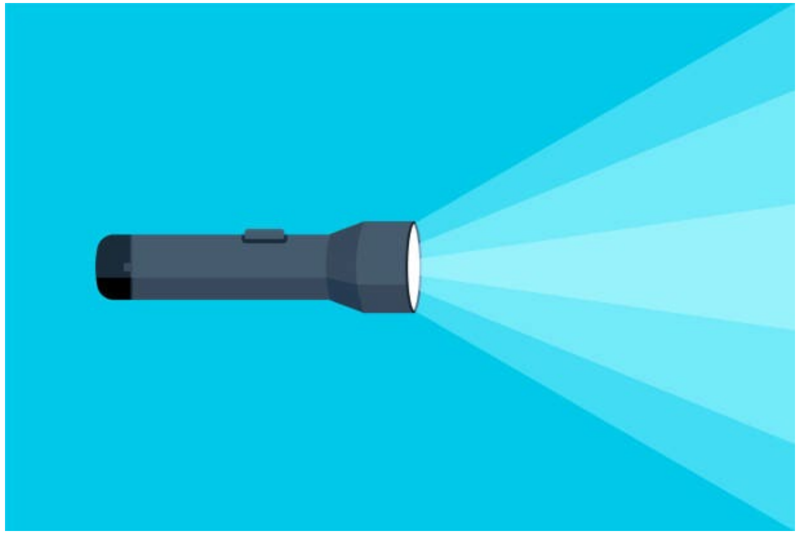
# Dispersive Decay Estimates

$$i\partial_t \alpha(t,x) = D(t)\alpha \qquad D = \begin{pmatrix} i\partial_x & m(t) \\ m(t) & -i\partial_x \end{pmatrix}$$

A fundamental question:

$\alpha(t)$ * disperses as $t \to \infty$,**

but how fast?



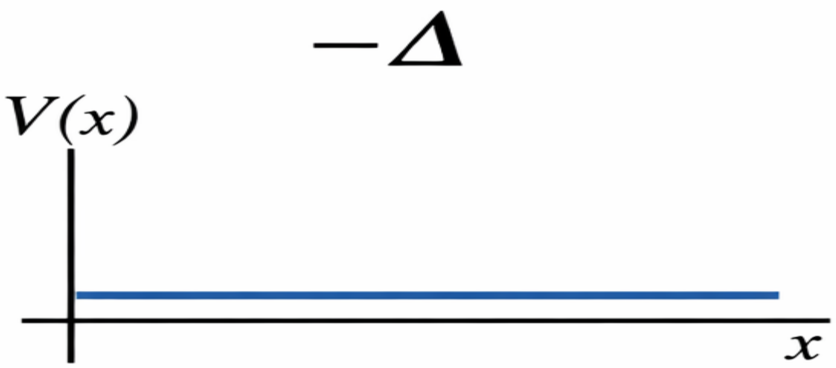
* If $\alpha(0)$ is in the continuous spectrum
** due to RAGE Theorem

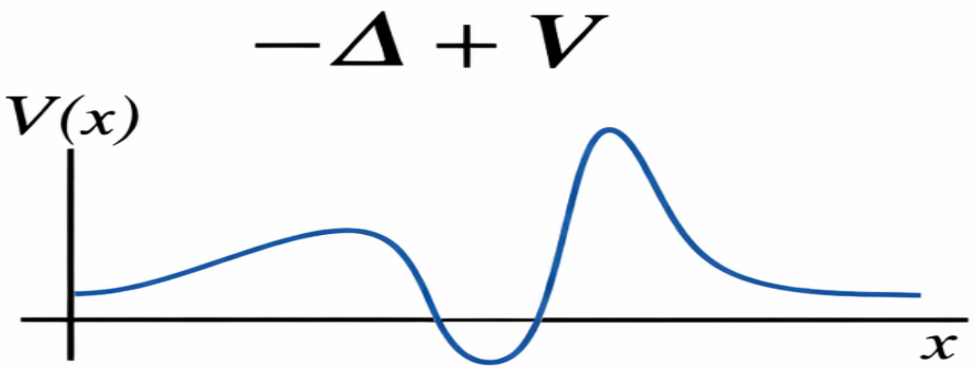
# The Potential Well Paradigm

$$\begin{cases} i\partial_t \psi(t) = \left(-\boldsymbol{\Delta} + \boldsymbol{V(x)}\right)\psi \\ \psi(0) = f \in L^2(\mathbb{R}^d) \end{cases}$$

- "Free particle" ($V = 0$) $\quad ||\psi(t)||_{L^\infty} \le C\boldsymbol{t^{-d/2}}||f||_{L^1}$
- $V \neq 0$ : a "race" of <u>decades</u> to characterize the best assumptions and best rates on $V$
- For example, in 1D: $\quad V \in L^1_1(\mathbb{R})$ has $\boldsymbol{t^{-1/2}}$ rate

[Goldberg Schlag CMP 2004, Egorova, Kopylova Marchenko and Teschl 2016]




- Similar paradigm for Dirac, Klein Gordon, Discrete Laplacian, Wave equation, and many more

# The Potential Well Paradigm

$$\begin{cases} i\partial_t \psi(t) = \color{red}{(-\boldsymbol{\Delta} + \boldsymbol{V}(\boldsymbol{x}))}\psi \\ \psi(0) = f \in L^2(\mathbb{R}^d) \end{cases}$$

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- $V \neq 0$ : a "race" of decades to characterize the best assumptions and best rates on $V$
- For example, in 1D: $V \in L^1_1(\mathbb{R})$ has $\boldsymbol{t}^{-\boldsymbol{1}/\boldsymbol{2}}$ rate

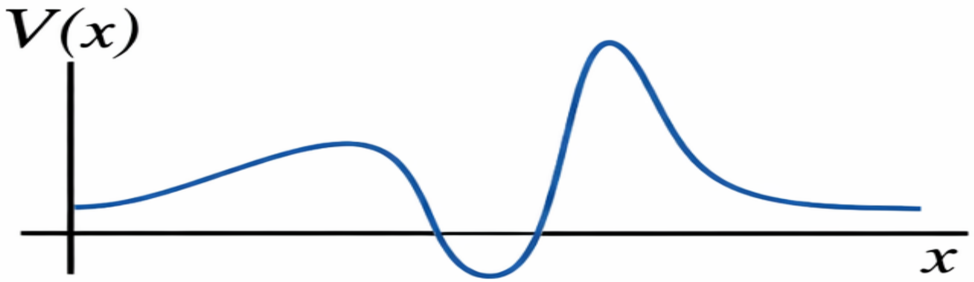
…l, 2016]

**This is not** the typical settings found in condensed matter and meta-materials

- Similar paradigm for Dirac, Klein Gordon, Discrete Laplacian, Wave equation, and many more

$-\boldsymbol{\Delta}$

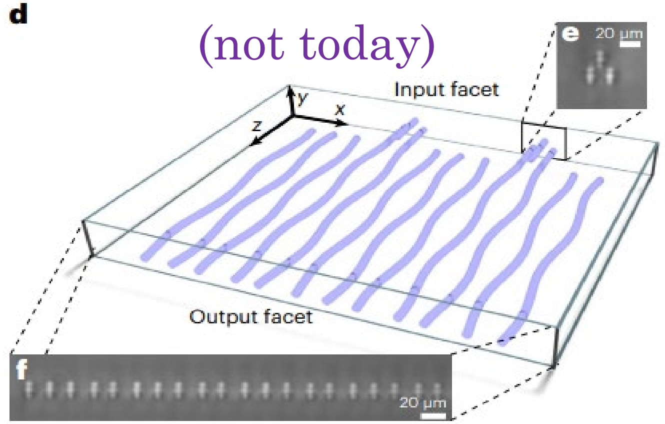

$-\boldsymbol{\Delta} + \boldsymbol{V}$



# Types of “Other Backgrounds”

## Discrete and Periodic

(not today)

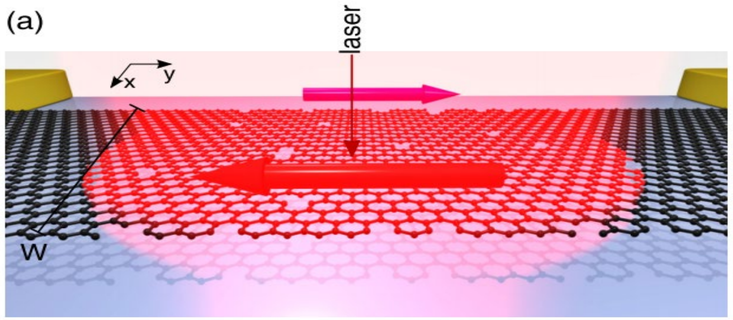



[Jurgensen et al, Nature Physics ‘23]

$$J = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & a_3 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}$$

- Kassem, AS, Weinstein, JMAA ‘26
- Kassem, AS, Weinstein, forthcoming
- Goldberg, AS, forthcoming

## Time Periodic

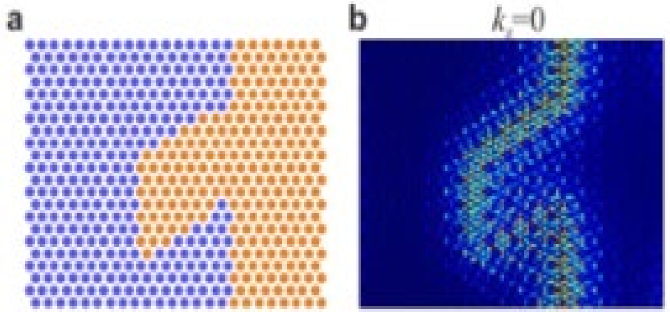



$$D = \begin{pmatrix} i\partial_x & m(t) \\ m(t) & -i\partial_x \end{pmatrix}$$

- Kraisler, AS, Weinstein, JDE ‘25
- Bloch, AS, Steinerberger, ‘26

## Domain Wall

(not today)



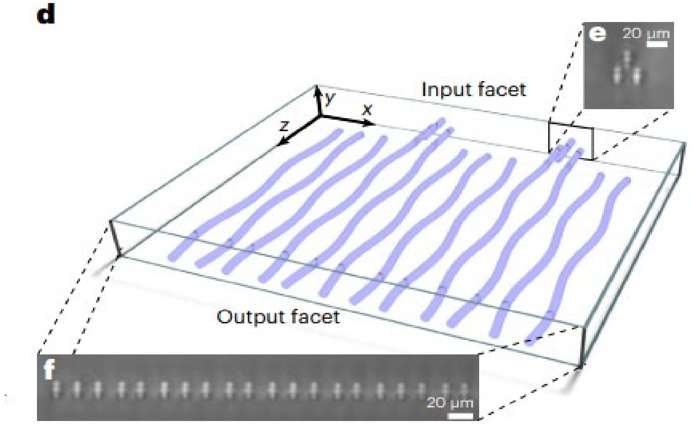

[NASA/JPL]

$$D = \begin{pmatrix} \tanh(y) & i\partial_x - \partial_y \\ i\partial_x + \partial_y & -\tanh(y) \end{pmatrix}$$

- Kraisler, AS, Weinstein, SIMA ‘24
- Erdogan, Kraisler, AS, forthcoming

# Types of “Other Backgrounds”

## Discrete and Periodic




[Jurgensen et al, Nature Physics ‘23]

$$J = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & a_3 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}$$

- Kassem, AS, Weinstein, JMAA ‘26
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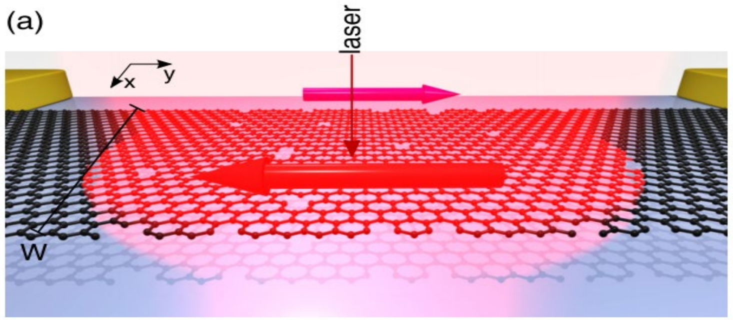
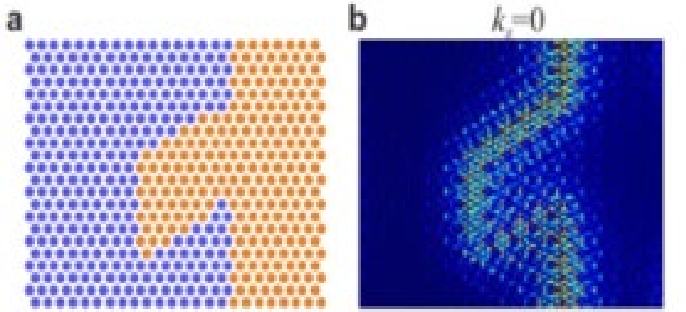
## Time Periodic




$$D = \begin{pmatrix} i\partial_x & m(t) \\ m(t) & -i\partial_x \end{pmatrix}$$

- Kraisler, AS, Weinstein, JDE ‘25
- Bloch, AS, Steinerberger, ‘26

## Domain Wall




[NASA/JPL]

$$D = \begin{pmatrix} \tanh(y) & i\partial_x - \partial_y \\ i\partial_x + \partial_y & -\tanh(y) \end{pmatrix}$$

- Kraisler, AS, Weinstein, SIMA ‘24
- Erdogan, Kraisler, AS, forthcoming

# Dispersive Decay Estimates

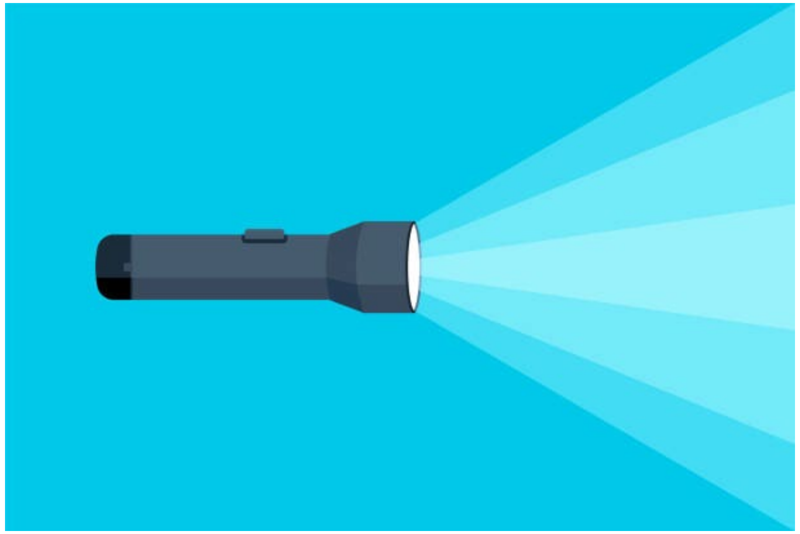
$$i\partial_t \alpha(t,x) = D(t)\alpha \qquad D = \begin{pmatrix} i\partial_x & m(t) \\ m(t) & -i\partial_x \end{pmatrix}$$

$$\text{“}||\alpha(t,\cdot)||_{L^\infty} \leq C\boldsymbol{t}^{-r}||\alpha(0)||_{L^1}\text{”}$$

A fundamental question:

$\alpha(t)$ * disperses as $t \to \infty$,**

but how fast?



* If $\alpha(0)$ is in the continuous spectrum
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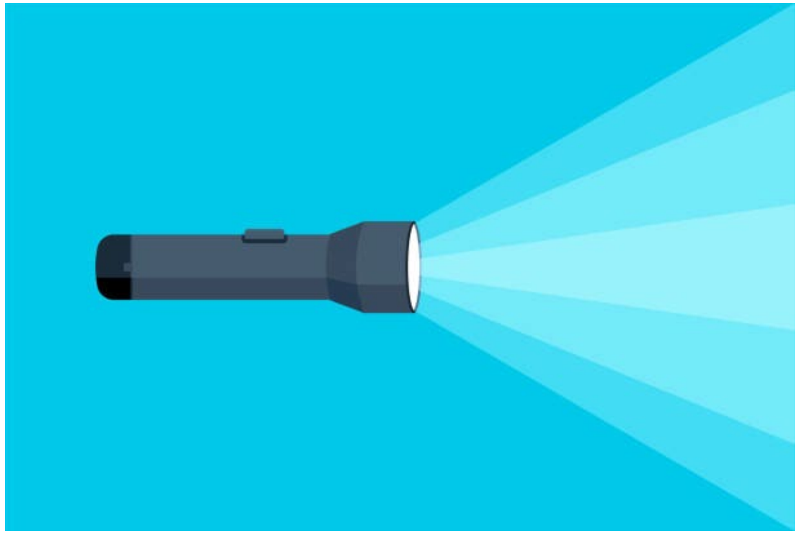

Today – in non-autonomous Dirac, dispersion can be **very slow**

* If $\alpha(0)$ is in the continuous spectrum

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# Autonomous flow

- Classical settings. What is known?
- Schrodinger:

$$i\partial_t \psi(t,\vec{x}) = H_0 \psi \qquad \vec{x} \in \mathbb{R}^d$$

$$\psi(t,\cdot) = e^{-iH_0 t}\psi_0 \qquad H_0 = -\Delta + V(\vec{x})$$

Extensive literature

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Extensive literature

- For Dirac – more recent, but by now, well understood

[D'Ancona & Fanelli CPAM '07,
Pelinovsky & Stefanov, '19
Erdogan, Goldberg & Green CMP '19, JST '21,
Erdogan, Green & Toprak, JDE '19, Am J. Math '19
Erdogan & Green, JMPA '21
Kovaric, Adv Math '21
Green et al, JDE '24]

# Non-Autonomous flows – dispersive decay

- What is known?

- Schrodinger, also Dirac for dimension n $\geq 2$

$$i\partial_t \psi(t, \vec{x}) = H(t)\psi \qquad\qquad H(t) = H_0 + W(t, x)$$

*[Rodnianski & Schlag Invent Math 04, Galtbayar, Jensen, & Yajima, J Stat Phy '04, Marzuola, Metcalfe, Tataru JFA '08, Goldberg JFA '09, Beceanu, Duke '11, Becaenu & Soffer Comm PDE '12, Herr & Hong '25]*

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$$i\partial_t\psi(t,\vec{x}) = H(t)\psi \qquad\qquad H(t) = H_0 + W(t,x)$$

- If $W(t,x)$ decays– **autonomous-like dispersive** (+Strichartz) estimates

$$\text{"}\ \|\boldsymbol{\psi}(\boldsymbol{t},\cdot)\|_{L^{p_1}} \lesssim \left\|\boldsymbol{e}^{-\boldsymbol{i}\boldsymbol{t}\boldsymbol{H}_0}\boldsymbol{\psi}(\mathbf{0},\cdot)\right\|_{L^{p_2}}\ \text{"}$$

- Motivation:
  - **Technical**: resolvent comparison between $H_0$ and $K = i\partial_t - H(t)$

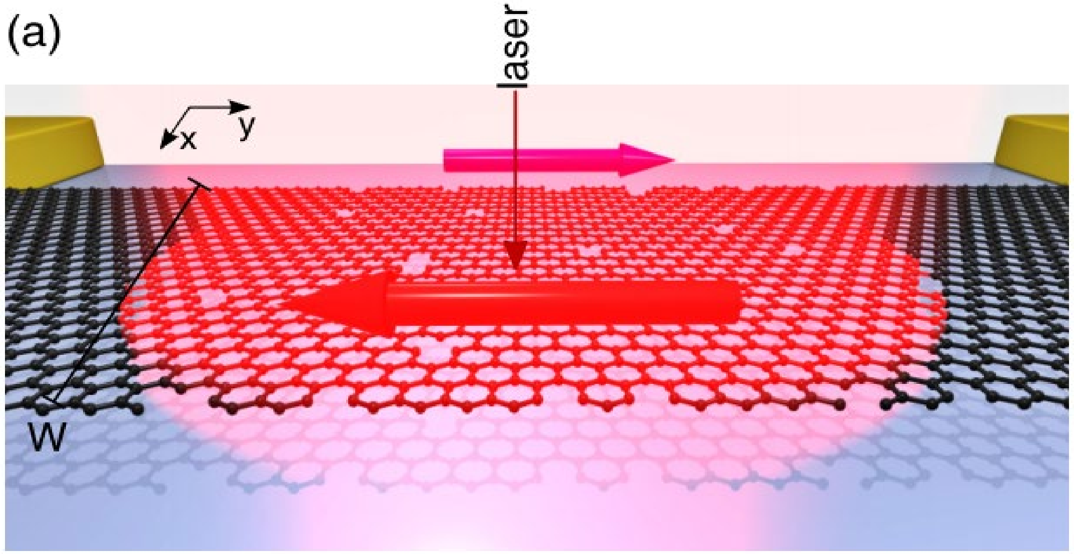
  - **Nonlinear problems**, $W(t,x)$ is "induced" by a solitary wave

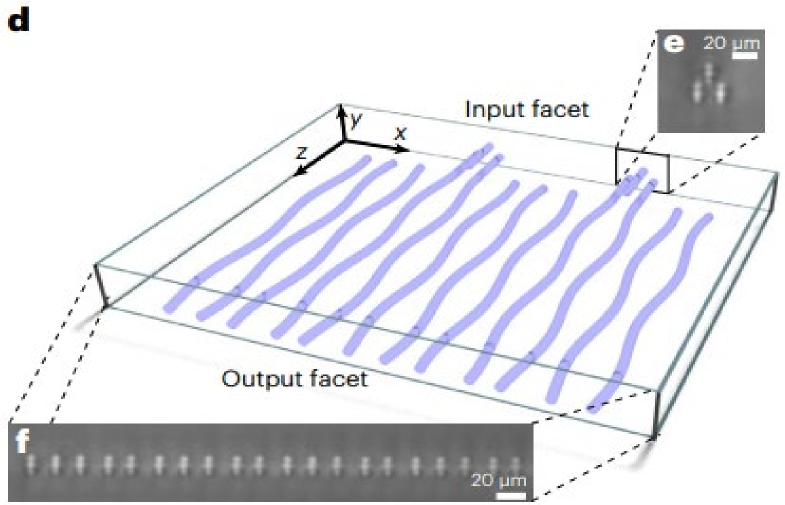
*[Rodnianski & Schlag Invent Math 04, Galtbayar, Jensen, & Yajima, J Stat Phy '04, Marzuola, Metcalfe, Tataru JFA '08, Goldberg JFA '09, Beceanu, Duke '11, Becaenu & Soffer Comm PDE '12, Herr & Hong '25]*

# Localized driving?

**Non-localized driving** is

- Physically relevant

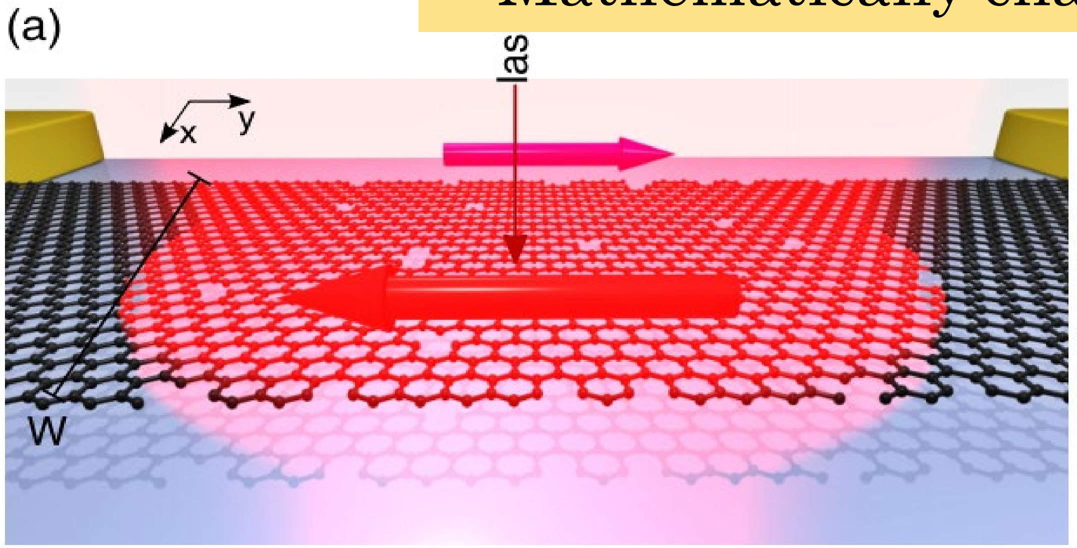


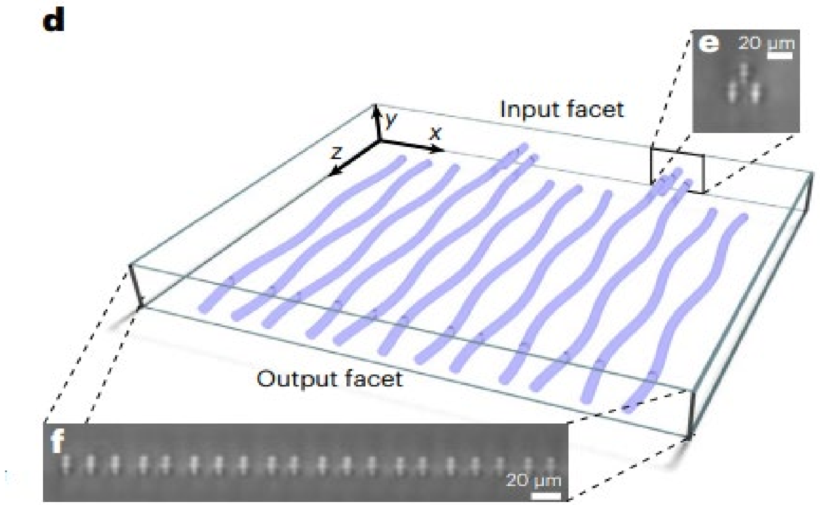



# Localized driving?

**Non-localized driving** is

- Physically relevant
- Mathematically challenging




**Question:**

localization – technical requirement or essential?

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# Non-autonomous Dirac

By analogy: Dirac 1D

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with decay of $D_1$, we expect *at least**

$$\|\alpha(t,\cdot)\|_{L^\infty} \lesssim \left\| e^{-iD_0 t} P_c \alpha_0 \right\|_{L^\infty} \lesssim t^{-\frac{1}{2}} \|\alpha_0\|_{L^1}$$

*[Erdogan & Green JMPA `21]*

*for now, we ignore smoothing, weighting, and specifics…

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# Switching mass model

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$$\begin{cases} i\partial_t \alpha(t,x) = D_m(t)\alpha \\ \alpha(0,x) = \alpha_0(x) \in L^2(\mathbb{R};\mathbb{C}^2) \end{cases}$$

$$D_m(t) = i\partial_x \sigma_3 + m\nu(t)\sigma_1 = \begin{pmatrix} i\partial_x & \boldsymbol{m\nu(t)} \\ \boldsymbol{m\nu(t)} & -i\partial_x \end{pmatrix}$$

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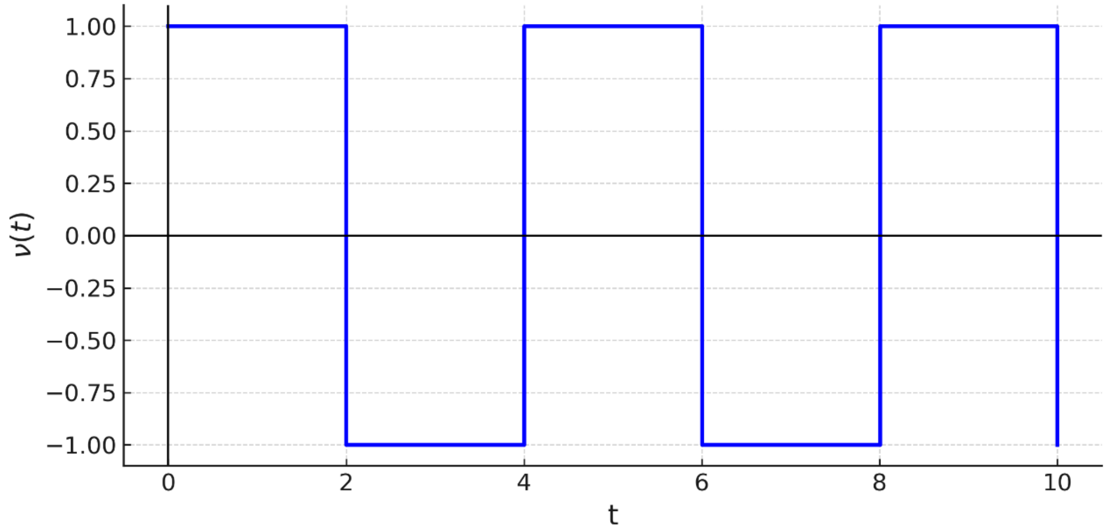

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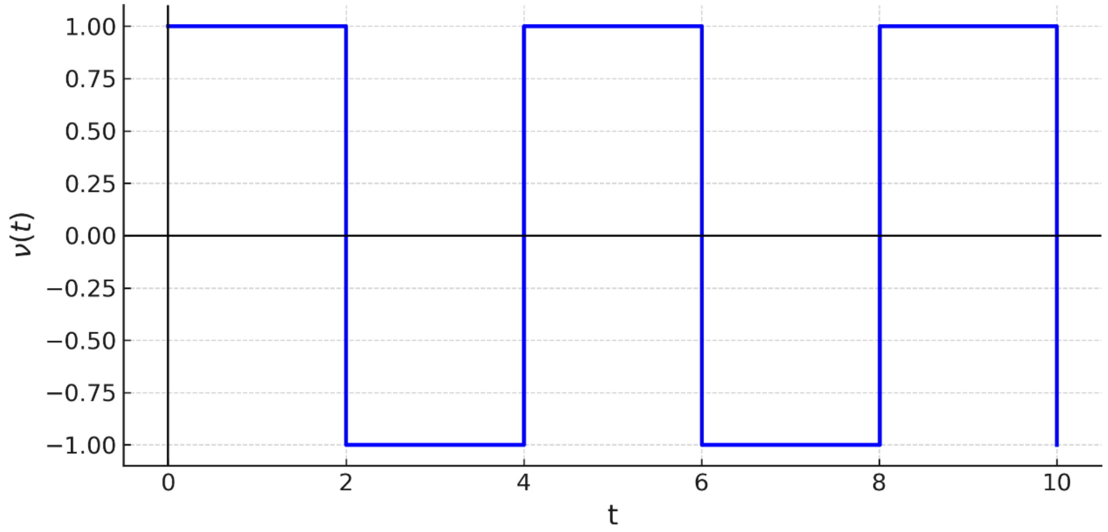


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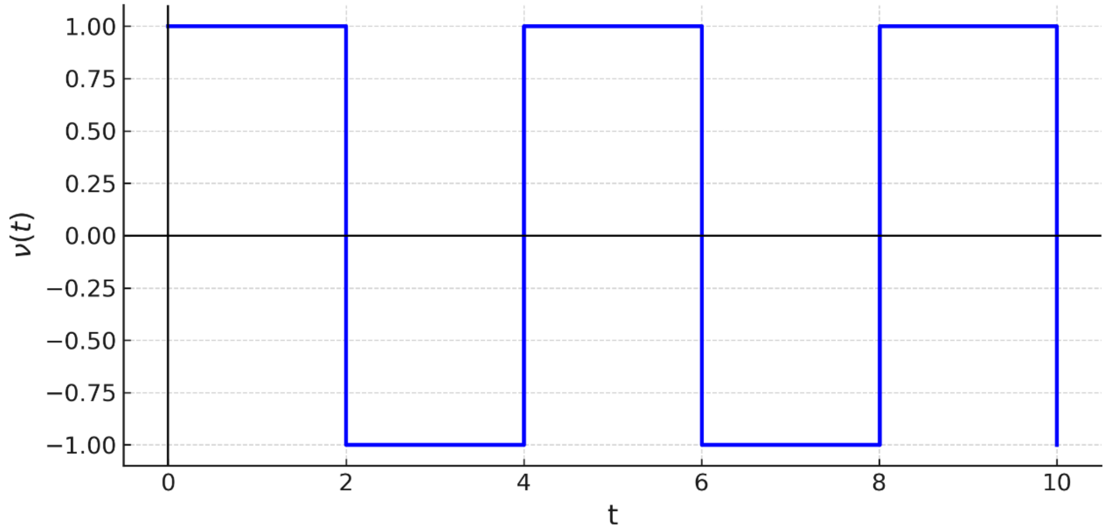


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- Dispersion estimate: $\left\| e^{-iD_\pm t}\langle D_{\pm m}\rangle^{-\frac{3}{2}-\varepsilon} \right\|_{L^1\to L^\infty} \lesssim t^{-1/2}$ [Erdogan, Green, JMPA '21]

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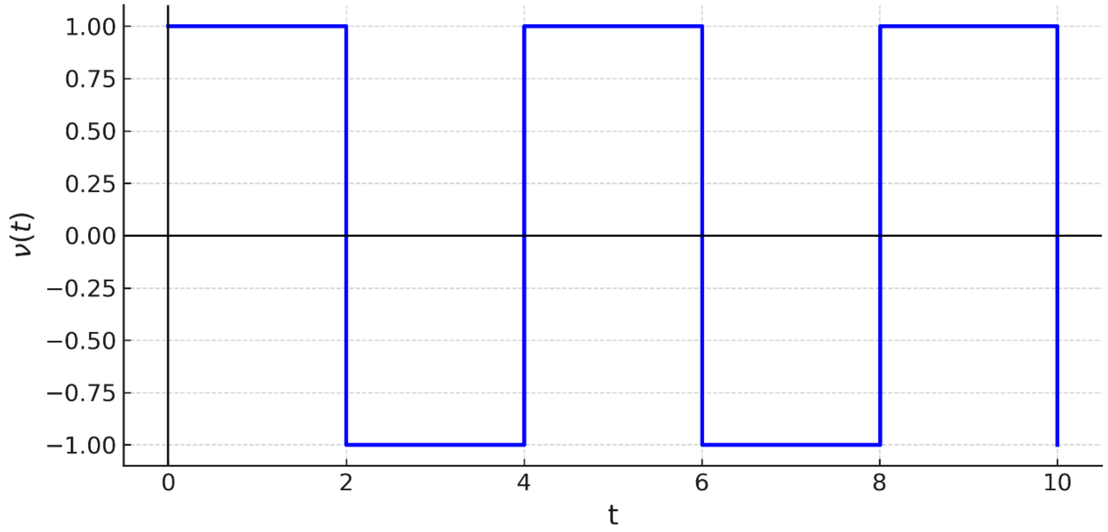


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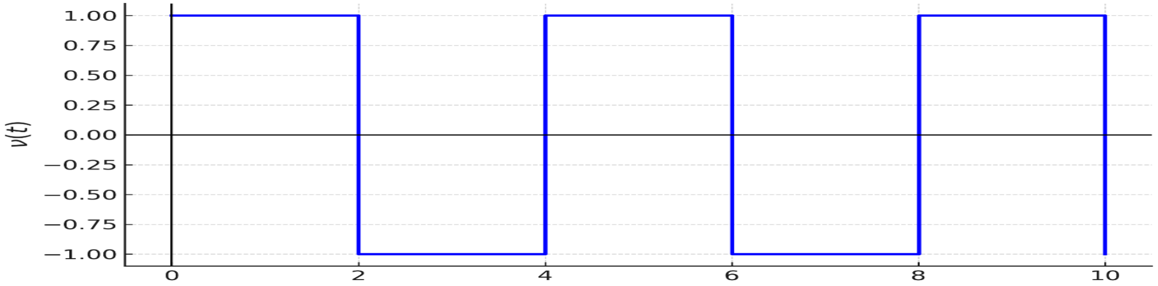
- *But – switching "counteracts" dispersion + is not localized in space.*

# Flipping mass model – results

$$D_m(t) = \begin{pmatrix} i\partial_x & mv(t) \\ mv(t) & -i\partial_x \end{pmatrix}$$




**Theorem** (Kraisler, AS, Weinstein, JDE 2025)

For every $m > 0$ there exists $c > 0$ and an orthogonal matrix P such that

for data $\alpha_0$ with supp $\hat{\alpha}_0 \subseteq [-c, c]$ then

1. ***Generic*** *case:* a.e. $m > 0$ ,

$$\|\alpha(t,\cdot)\|_{L^\infty} \sim \boldsymbol{t^{-1/3}} \int_{\mathbb{R}} P\alpha_0(x)dx$$

2. ***Exceptional*** *case:* for a discrete set of $m > 0$,

$$\|\alpha(t,\cdot)\|_{L^\infty} \sim \boldsymbol{t^{-1/5}} \int_{\mathbb{R}} \boldsymbol{P\alpha_0(x)dx}$$

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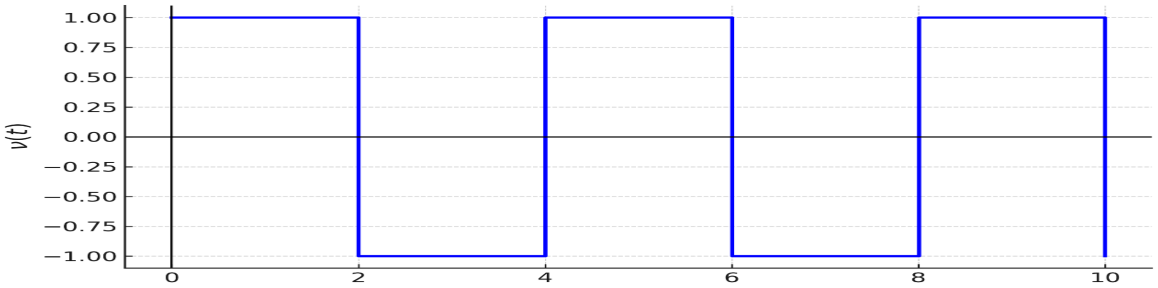



**Corollary** (Kraisler, AS, Weinstein, JDE 2025)

Consider a dispersive estimate with any amount of smoothing

$$\|\alpha(t,\cdot)\|_{L^\infty} \lesssim \boldsymbol{t^{-r}} \left\| \langle i\partial_x \rangle^{(-\sigma)} \alpha_0 \right\|_{L^1}$$

1. ***Generic*** *case:* a.e. $m > 0$ , $\boldsymbol{r \in (0, 1/3)}$
2. ***Exceptional*** *case:* for a discrete set of $m > 0$, $\boldsymbol{r \in (0, 1/5)}$

What does this mean?

How slow can you go?

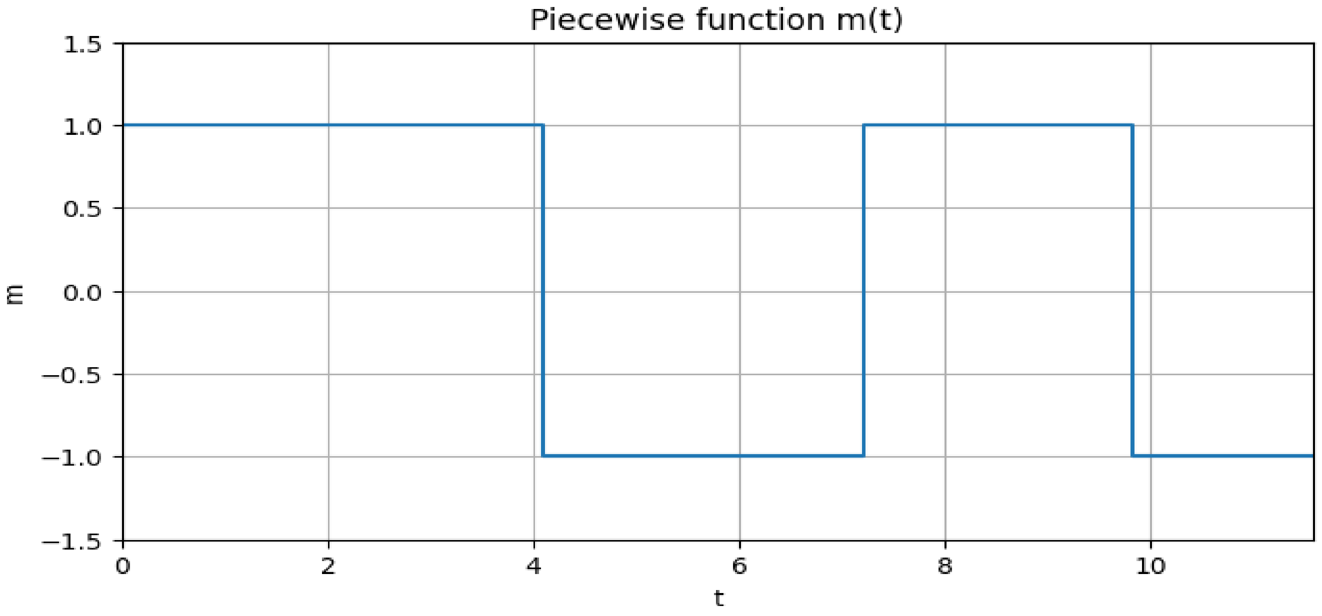
Both estimates are sharp

**Far slower** than the analogous *autonomous* rates

Exceptional masses $m_k \sim \pi(k + k_0 + 1/2)$

# How slow can you go?

$$D_m(t) = \begin{pmatrix} i\partial_x & m\nu(t) \\ m\nu(t) & -i\partial_x \end{pmatrix}$$




**Theorem** (Bloch, AS, Steinerberger 2026)

For every $m = 1$ there exists $c > 0$ such that

for data $\alpha_0$ with supp $\hat{\alpha}_0 \subseteq [-c, c]$ then

***There exists exceptional forcing term*** *for which*

$$\|\alpha(t,\cdot)\|_{L^\infty} \lesssim \boldsymbol{t}^{-r} \|\alpha_0\|_{L^1} \qquad \textit{then } \boldsymbol{r} \in \mathbf{(0, 1/10)}$$

This construction is robust - let us understand it better

# The analysis in a nutshell (1)

$$i\partial_t \alpha(t,x) = \left[i\partial_x \sigma_3 + m\nu(t)\sigma_1\right]\alpha$$

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With Fourier transform, a parametric family of 2x2 Floquet ODEs

$$i\frac{d}{dt}\hat{\alpha}\,(t;\xi) = [\xi\sigma_3 + m\nu(t)\sigma_1]\,\hat{\alpha}$$

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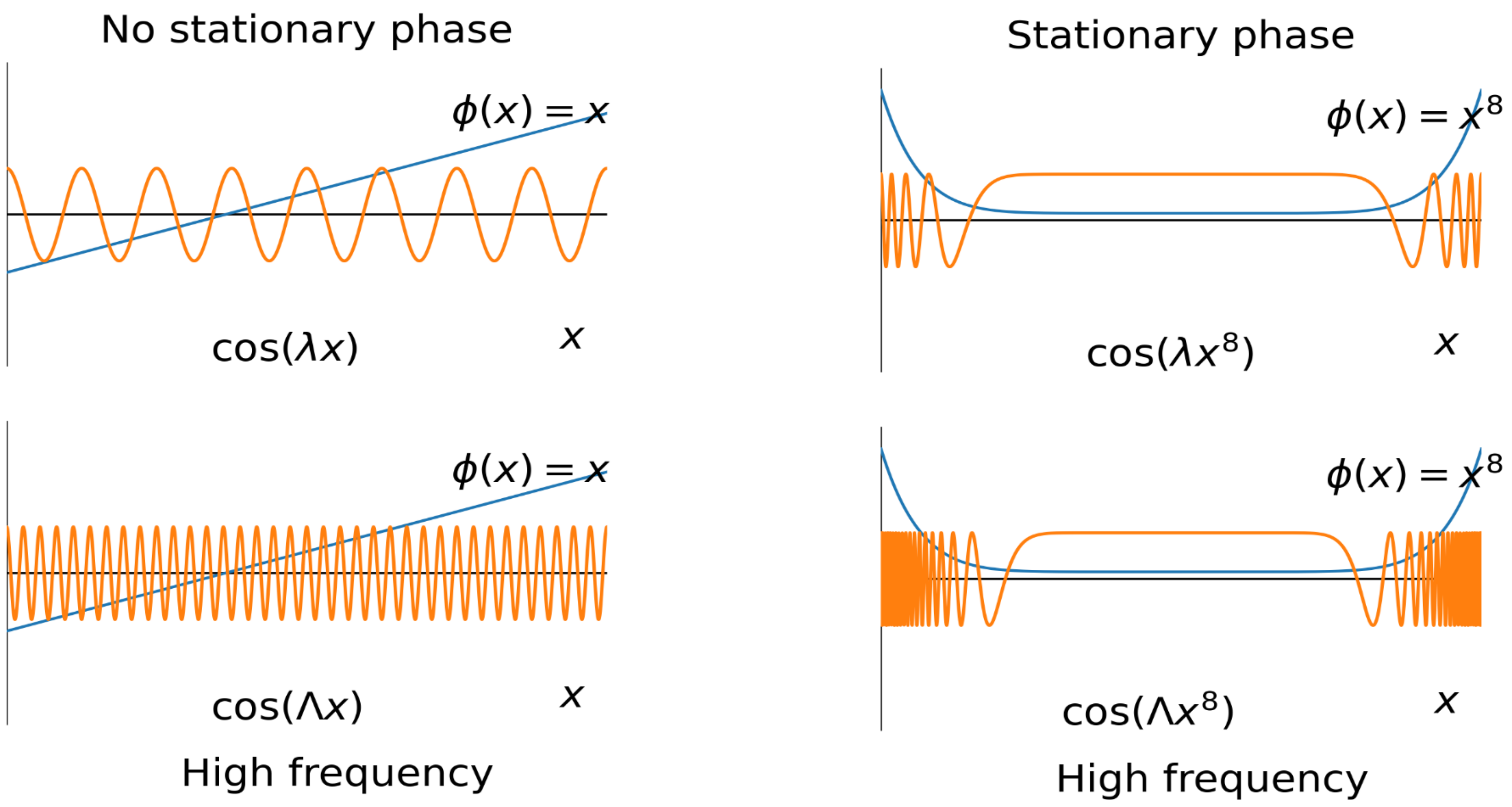
Inverse Foruier transform:

$$\alpha(2n,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\;\hat{P}(\xi)\begin{pmatrix}\exp[-in\theta(\xi)] & 0 \\ 0 & \exp[in\theta(\xi)]\end{pmatrix}\hat{P}^*(\xi)\hat{f}(\xi)d\xi$$

# Stationary Phase

- As t increases, more oscillations, more cancellations
- But <u>**how much more**</u>?

$$\left| \int_a^b e^{i\phi(x)\lambda}\, dx \right| \sim ? \quad \text{as } \lambda \to +\infty$$



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Asymptotic Expansion (see e.g., Hormander)
Under general conditions, if $|\boldsymbol{\lambda}^{(k)}(\boldsymbol{z})| > \boldsymbol{\lambda_0} > \mathbf{0}$, but $\boldsymbol{\lambda}^{(j)}(\boldsymbol{z}) = \mathbf{0}$ for $1 \le j < k$, and $f \in C_c^\infty$, then

$$\int_{-\infty}^{\infty} f(z)\boldsymbol{e}^{\boldsymbol{i\lambda(z)t}}dz \sim \frac{C_k}{(\boldsymbol{t\lambda_0})^{\boldsymbol{1/k}}}\|f\|_{L^1}$$

# The analysis in a nutshell (2)

Find points where the first nonzero derivative is $\mathbf{0 \neq \theta^{10}\,(\xi_0)}$, to obtain

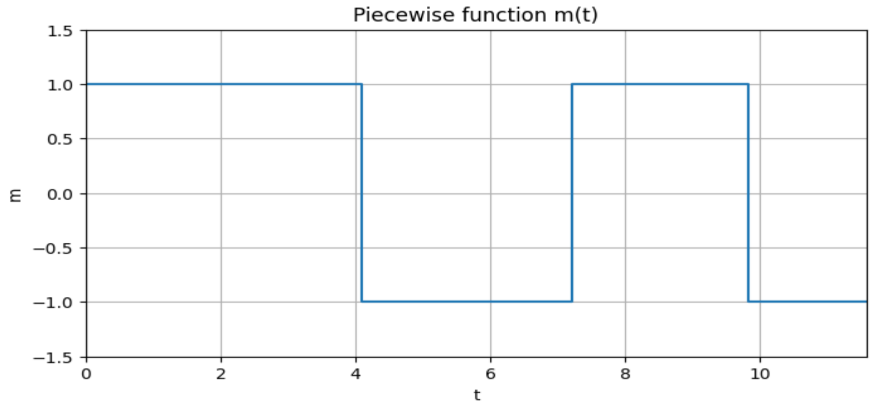
$$\|\alpha(nT,x)\|_{L^\infty_x} \leq C\boldsymbol{n^{-\frac{1}{10}}}\|\alpha(0,\cdot)\|_{L^1}$$

Generally, <u>still</u> an open problem (finding exponents $\theta(\xi)$ for Floquet ODEs)

Even harder –

<u>Designing</u> m(t) such that the Floquet exponents are "degenerate"

# The algebraic construction




On every time interval

$$\frac{d}{dt}\hat{\alpha}(t;\xi) = g_m(\xi)\hat{\alpha} \qquad g_m = -i\,(\xi\sigma_3 + m\sigma_1) \in \mathfrak{su}(2) \quad m = \pm 1$$

And so, the period-flow operator is

$$\widehat{M}(\xi) = \prod_{j=1\ldots4} \exp\left[g_{m_j}(\xi)t_j\right] \in \boldsymbol{SU(2)}$$

The problem in the JDE paper:

diagonalizing $\widehat{M}(\xi)$ is daunting!

**Proposition** (Bloch, AS, Steinerberger 2026)

Using properties of the Lie algebra/group $\mathfrak{su}(2)$ & $\boldsymbol{SU(2)}$ , the Floquet Exponent $\theta(\xi)$ satisfies

$$\cos\big(\theta(\xi)\big) = F(\xi) \qquad F(\xi) = \frac{1}{2}Tr\,(\widehat{M}(\xi))$$

<u>Goal</u> – get *algebraic* equations for $\theta^{(j)}(0) = F^{(j)}(0) = 0 \quad$ for $1 \le j \le 9$

<u>Tools</u> - commutation relations, $Tr(\sigma_j) = 0$, symmetries, and Taylor expansions of cos, sin …

# The algebraic equations

$$\widehat{M}(\xi) = \prod_{j=1\ldots4} \exp\left[g_{m_j}(\xi)\boldsymbol{t_j}\right] \in SU(2)$$

**Goal**: make the derivatives of $F(\xi) = \frac{1}{2} Tr\,(\widehat{M}(\xi))$ vanish at $\xi = 0$

By symmetries – only even derivatives matter $\boldsymbol{F^{(j)}(0) = 0}$ for $j = 2,4,6,8$

Four equations, four variables $(t_1, t_2, t_3, t_4)$ – what can go wrong?

2nd derivative

$$\begin{aligned} a_2(t_1,t_2,t_3,t_4) = {} & \frac{-t_1+t_2-t_3+t_4}{2}\sin(t_1-t_2+t_3-t_4) + \cos(t_1-t_2-t_3-t_4) \\ & - \cos(t_1+t_2-t_3-t_4) - 2\cos(t_1-t_2+t_3-t_4) \\ & + \cos(t_1+t_2+t_3-t_4) - \cos(t_1-t_2-t_3+t_4) \\ & + \cos(t_1+t_2-t_3+t_4) + \cos(t_1-t_2+t_3+t_4). \end{aligned}$$

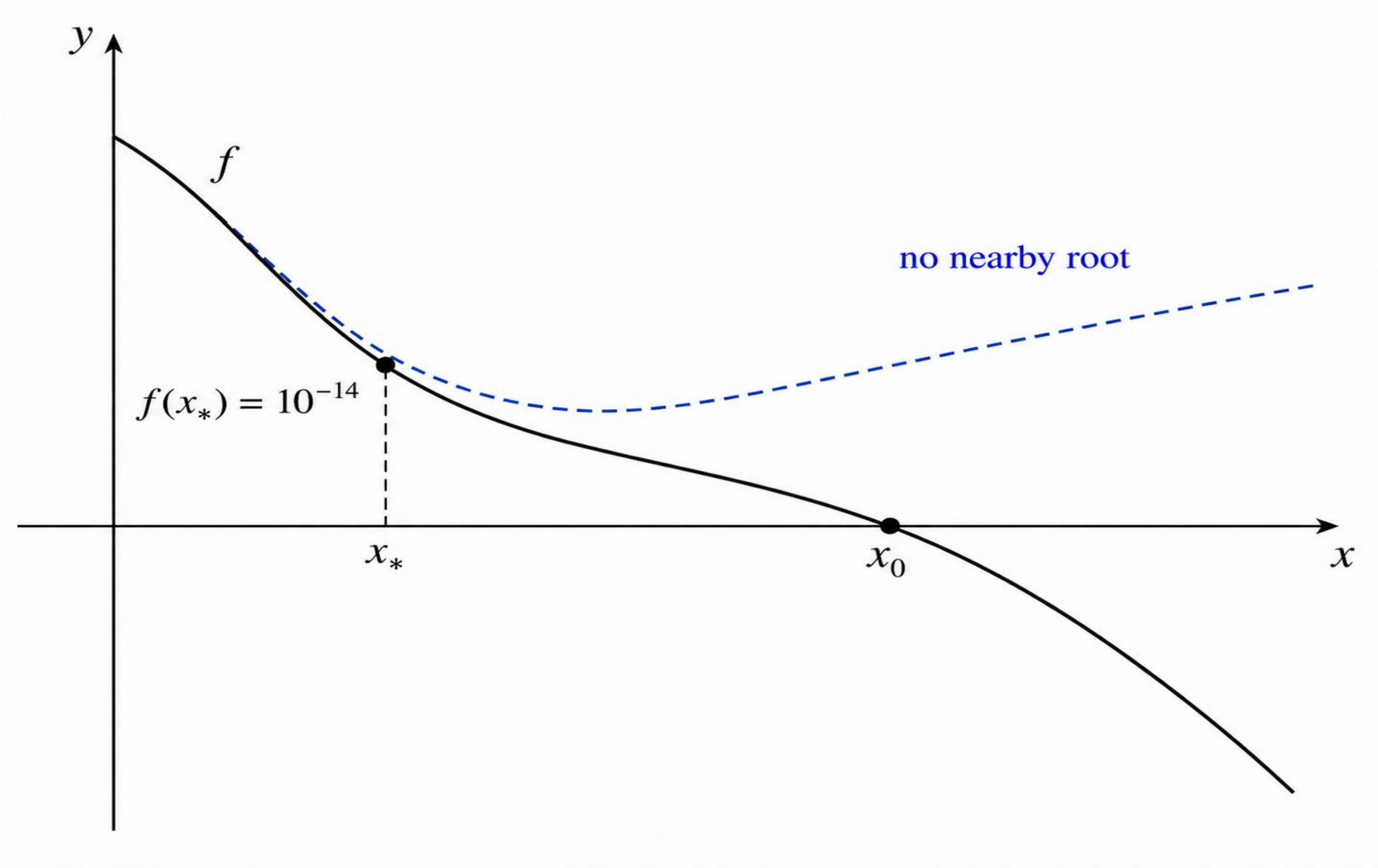
8th derivative has 295 terms with cubic powers of $t_j's$

**The solution:** a computer assisted proof

1. Find a numerically approximate solution
2. Show that a true solution lies nearby (using Newton-Kantorovich)

$$\begin{aligned} t_1 &= 4.088866559569492 \\ t_2 &= 3.117488248716022 \\ t_3 &= 2.615221023066265 \\ t_4 &= 1.762750988714514 \end{aligned}$$

# Newton Kantorovich Theorem



# How slow can you go? – a summary

**Theorem** (Bloch, AS, Steinerberger 2026)

***There exists 1D Floquet-Dirac equation*** *for which*

$$\|\alpha(t,\cdot)\|_{L^\infty} \lesssim \boldsymbol{t^{-r}} \|\alpha_0\|_{L^1}$$

Can only be true for r $\in \left(0, \frac{1}{10}\right)$

**Conjecture**

***There exists 1D Floquet-Dirac equation*** *for which*

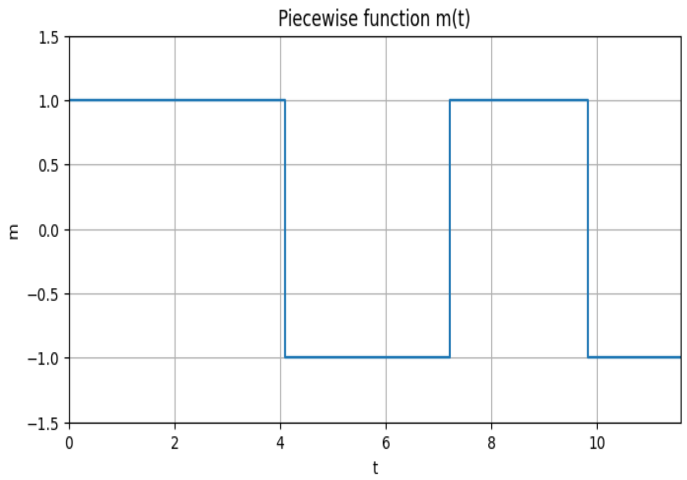
$$\|\alpha(t,\cdot)\|_{L^\infty} \lesssim \boldsymbol{t^{-r}} \|\alpha_0\|_{L^1}$$

Can only be true for r $\in (0, \epsilon)$ *for all* $\epsilon$

By the same algeb

**2D Radiated Graphene??**

$$i\partial_t \alpha(t, \boldsymbol{x}) = \left(\vec{\nabla} + A(T)\right) \cdot (\sigma_1, \sigma_2)\alpha$$




**Many open question** (even in 1D)

1. What is the generic behavior?
2. Smooth in time forcing (work in progress)
3. Is there always a bound?
4. Treating high wave-number $\xi$

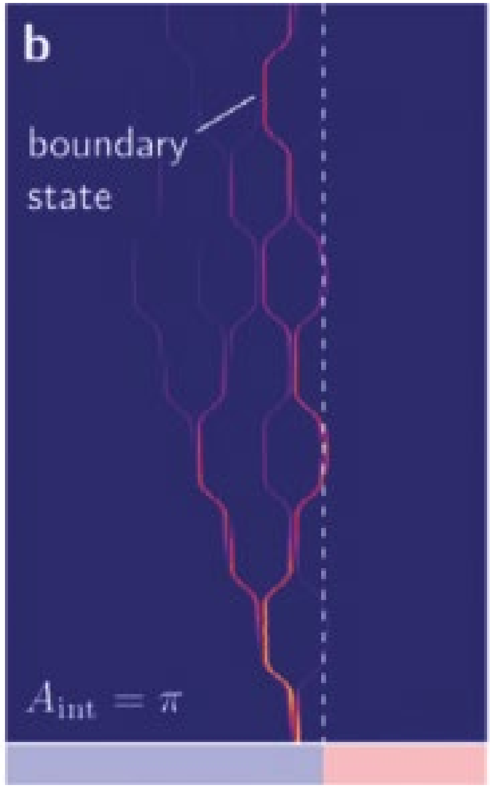
# Agenda

- The time-periodic Dirac equation
- How to get very slow dispersion
- Application to Floquet edge modes
- A discrete Analog

# One-dimensional Settings

1D + edge + Floquet = energy localizes at the edge*

[Bellec et al, EPL `17]

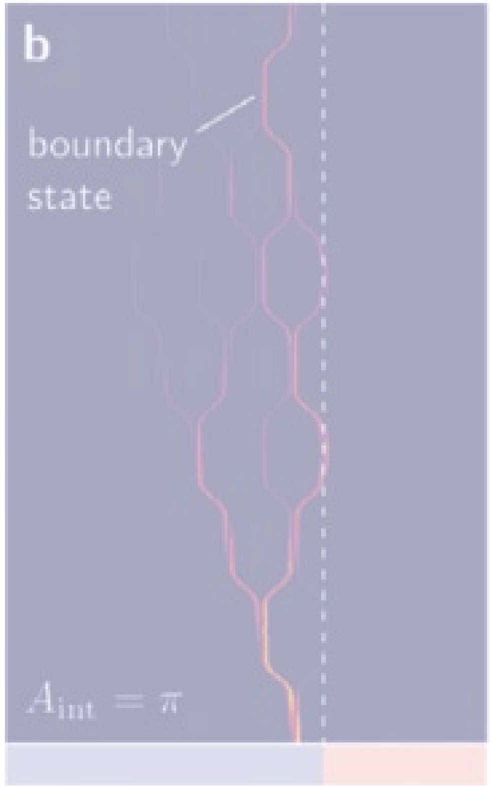


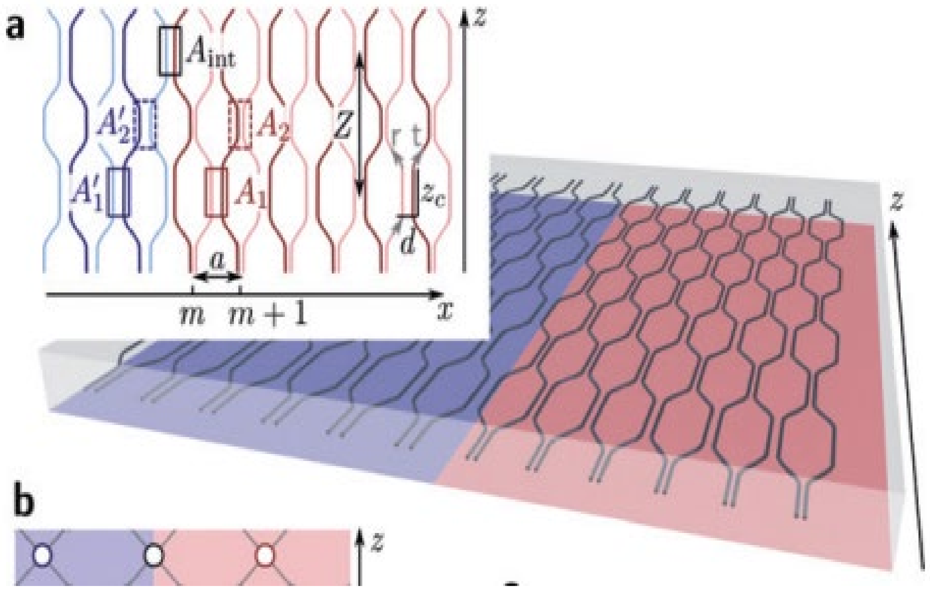

Experimental data

# One-dimensional Settings

1D + edge + Floquet = energy localizes at the edge

[Bellec et al, EPL `17]



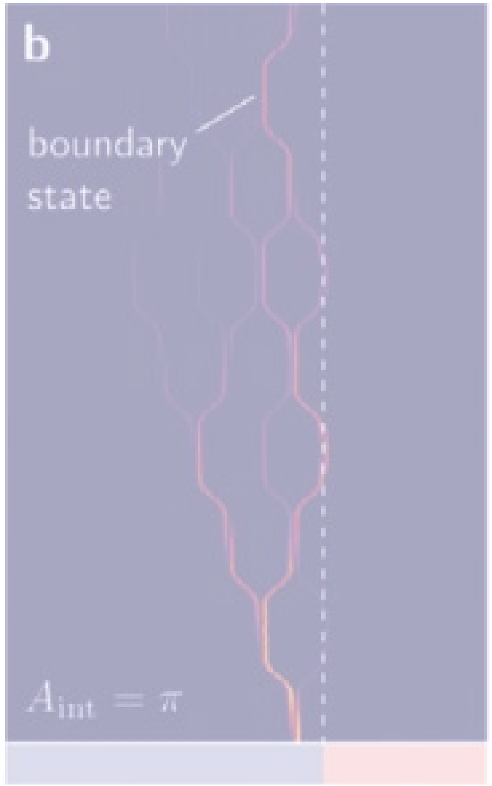

Discrete model -

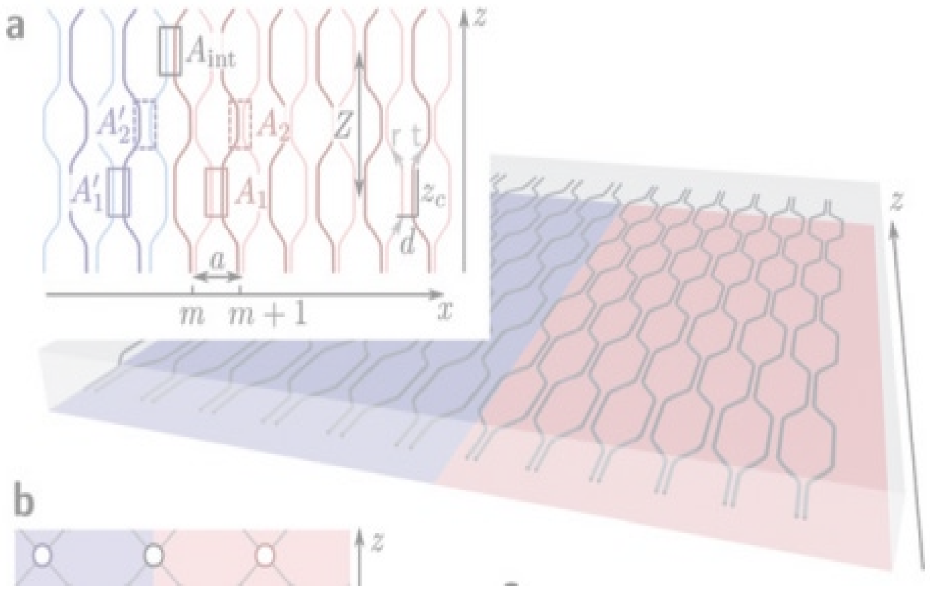
hopping between sites

located at $m = 0, \pm 1, \pm 2$ , …

# One-dimensional Settings

**Q:** Can **PDE Floquet** systems have edge modes?




Discrete model -
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# A Concrete Research Question

**Broad Question:** Can PDE Floquet systems have edge modes?

Model: Schrödinger equations

$$i\partial_t \psi(t,x) = \quad H_{dw}\psi \quad + \quad W(t)\psi$$

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**Concretely**: Is the edge mode of $H_{dw}$ stable under forcing?

$$i\partial_t \psi(t,x) = H_{dw}\psi + W(t)\psi$$

Static domain wall
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Periodic-in-time

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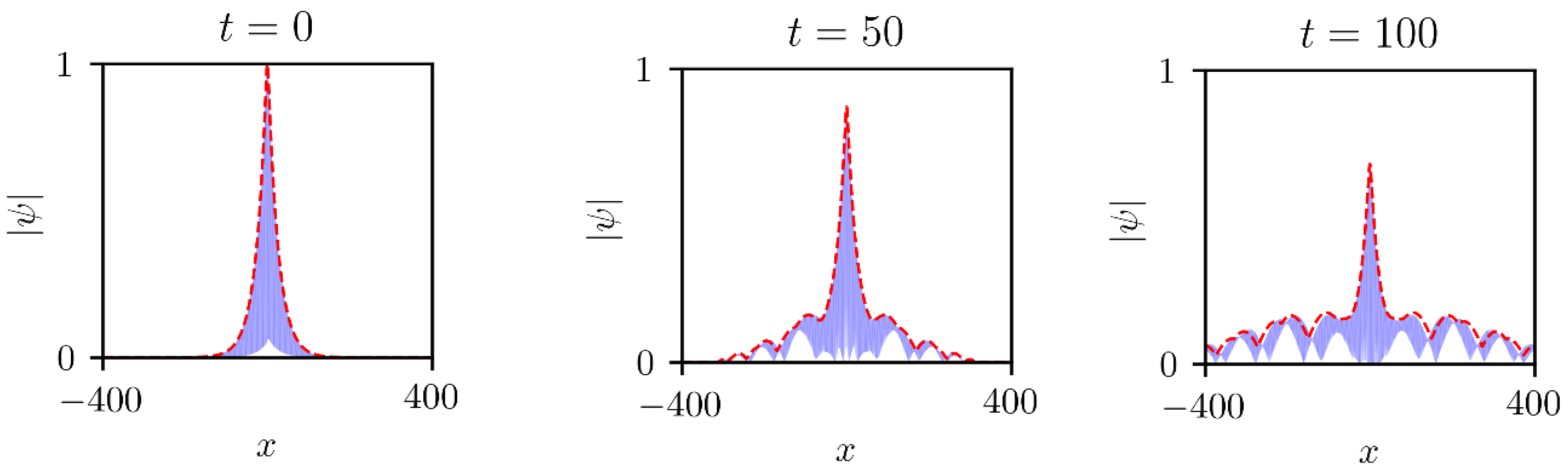
# A Very Short Answer

**Concretely**: Is the edge mode of $H_{dw}$ stable under forcing?

**No, due to resonance**

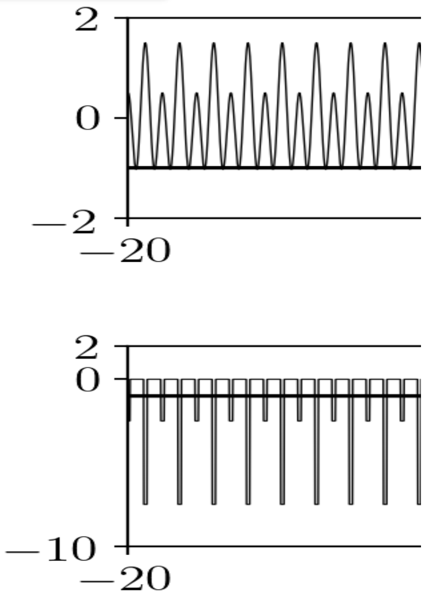



To prove this, we would need dispersive decay estimate of the full Floquet propagator (currently, still open)

# Domain-Wall Potentials

Plotting the potential vs. x

$\boxed{V - \varepsilon W \text{ as } x \to -\infty}$



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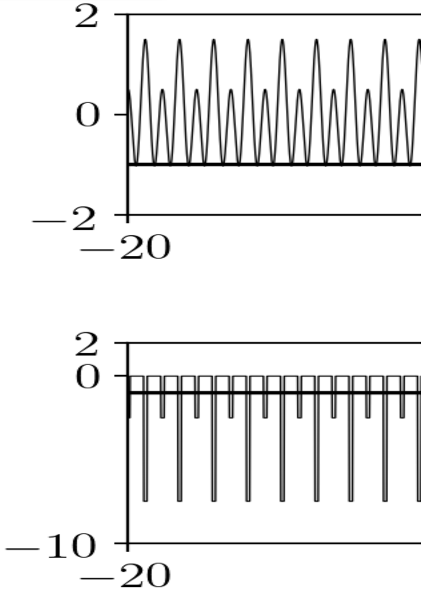
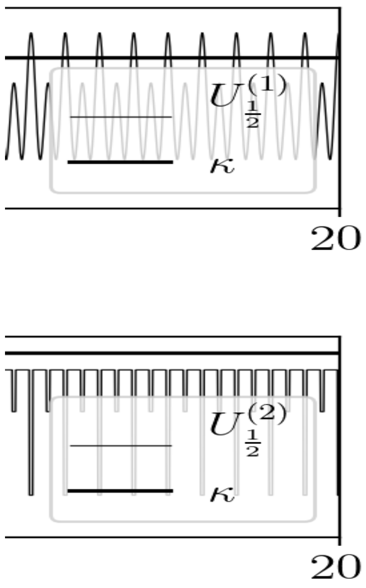



$V + \varepsilon W$ as $x \to \infty$



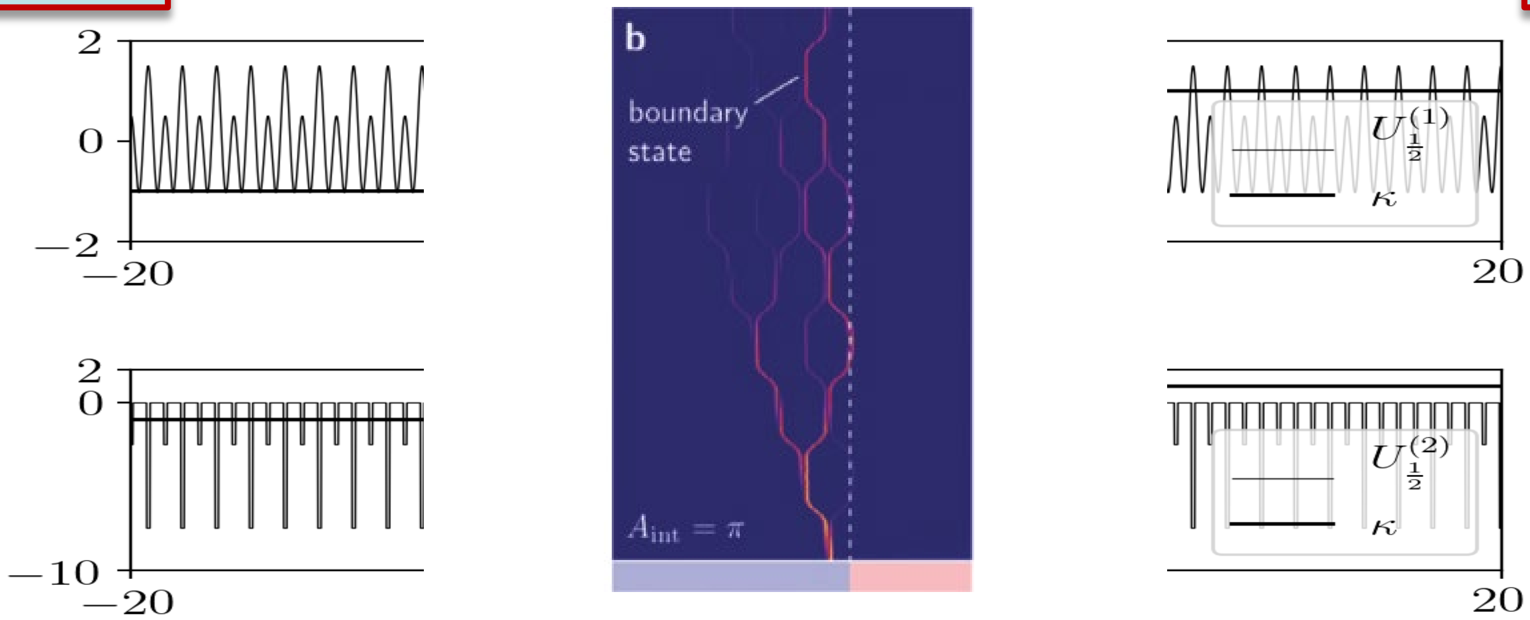

$V, W$ are periodic such that a phase shift is created between $\pm\infty$

# Domain-Wall Potentials

Plotting the potential vs. x

$V - \varepsilon W$ as $x \to -\infty$

$V + \varepsilon W$ as $x \to \infty$

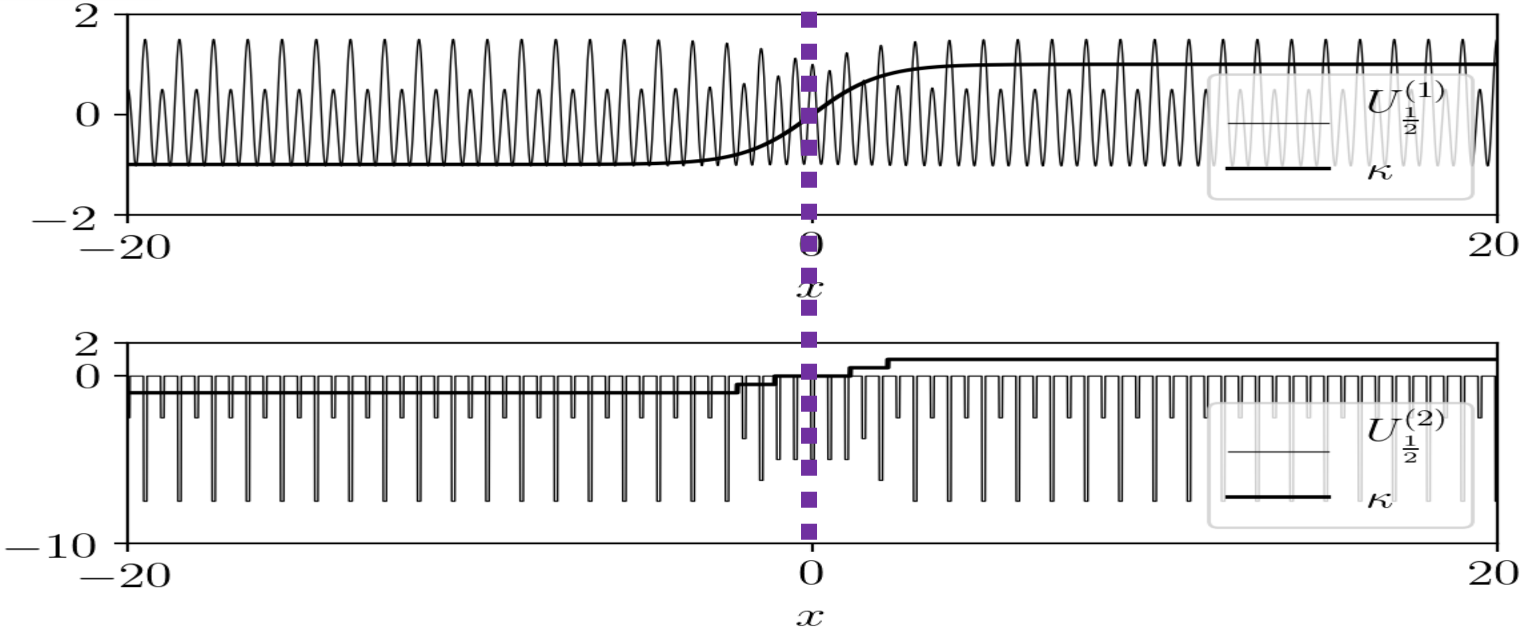


# Domain-Wall Potentials $U_\varepsilon$

$V - \varepsilon W$ as $x \to -\infty$

Slowly interpolating at $x = 0$

$V + \varepsilon W$ as $x \to \infty$

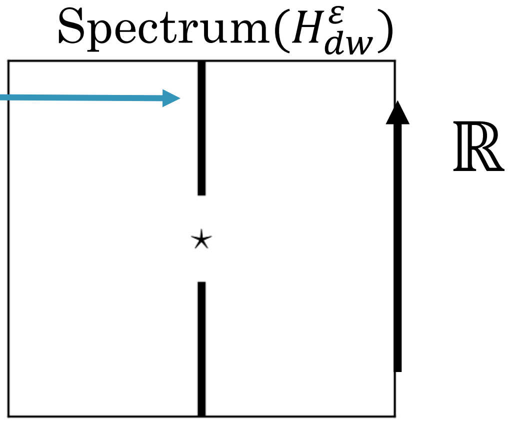


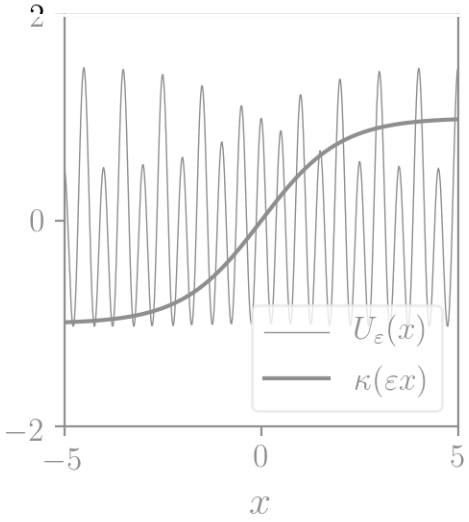

$$U_\varepsilon(x) = V(x) + \varepsilon\kappa(\varepsilon x)W(x)$$

# Spectrum of Domain-Wall Hamiltonians $H^{\varepsilon}_{dw}$

$$H^{\varepsilon}_{dw} \equiv -\partial_{xx} + V(x) + \varepsilon\kappa(\varepsilon x)W(x)$$

continuous spectrum



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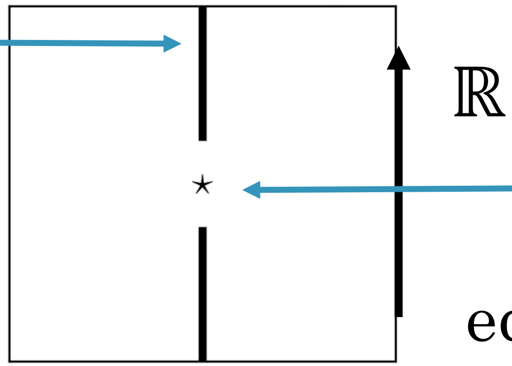
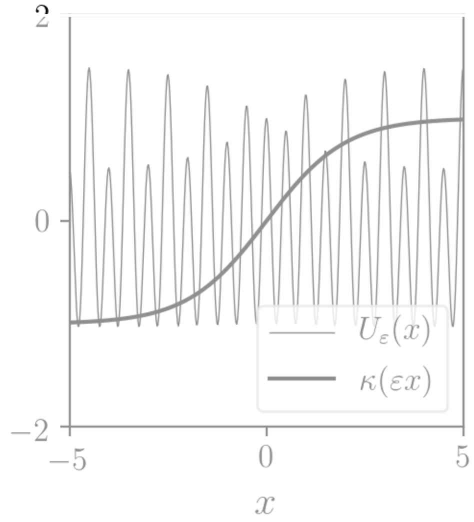
Spectrum($H^{\varepsilon}_{dw}$)

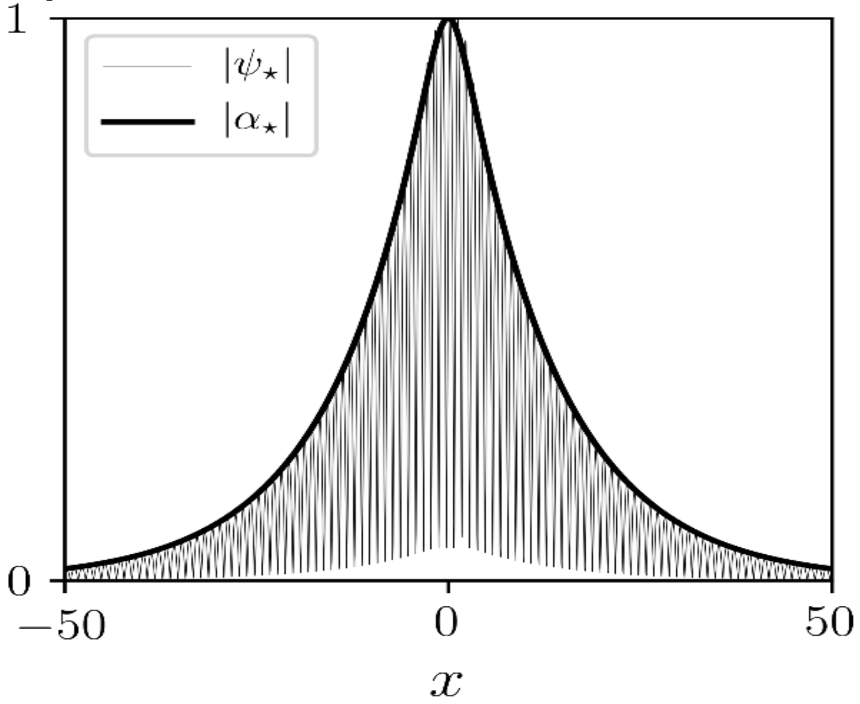



edge mode (point spectrum)

$$H_{dw}\psi_\star = E_\star\psi_\star$$

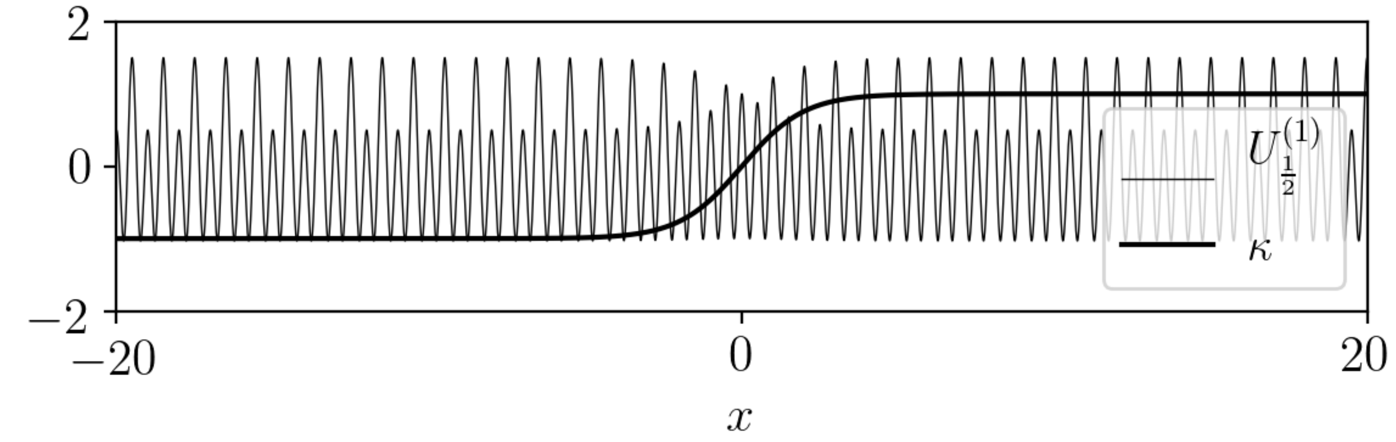
$\psi_\star$



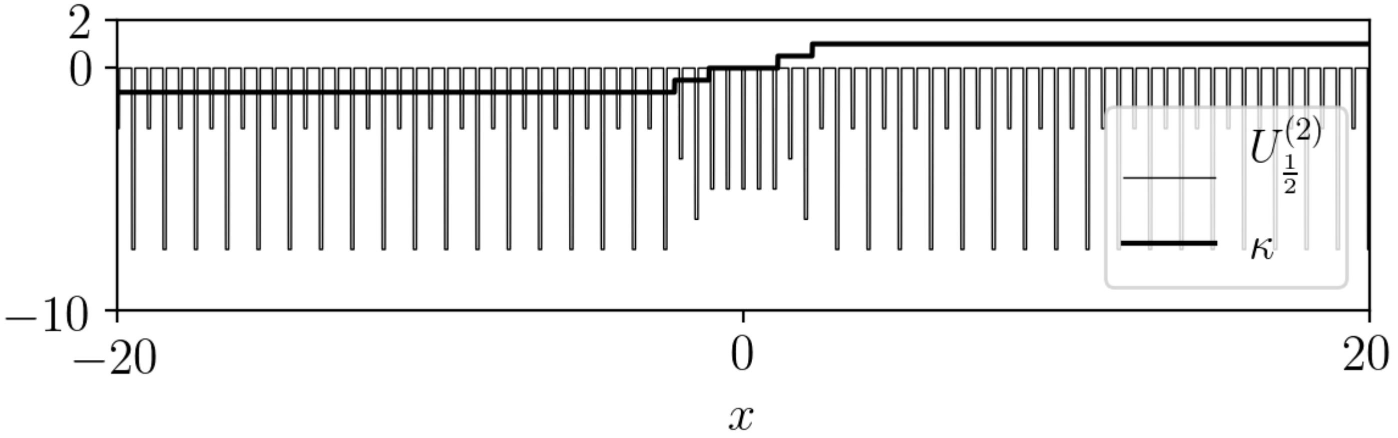

Fefferman, Lee-Thorp, Weinstein, PNAS ‘14
Fefferman, Lee-Thorp, Weinstein, Mem. AMS ‘17
Druout, Fefferman, Weinstein, CMP ‘20

# Numerical Simulations (Hameedi, AS, Weinstein, '23)

$$i\psi_t(t,x) = H^{\varepsilon}_{dw}\psi + i\varepsilon A(\varepsilon t)\partial_x\psi$$




Example #1 – cosines-based



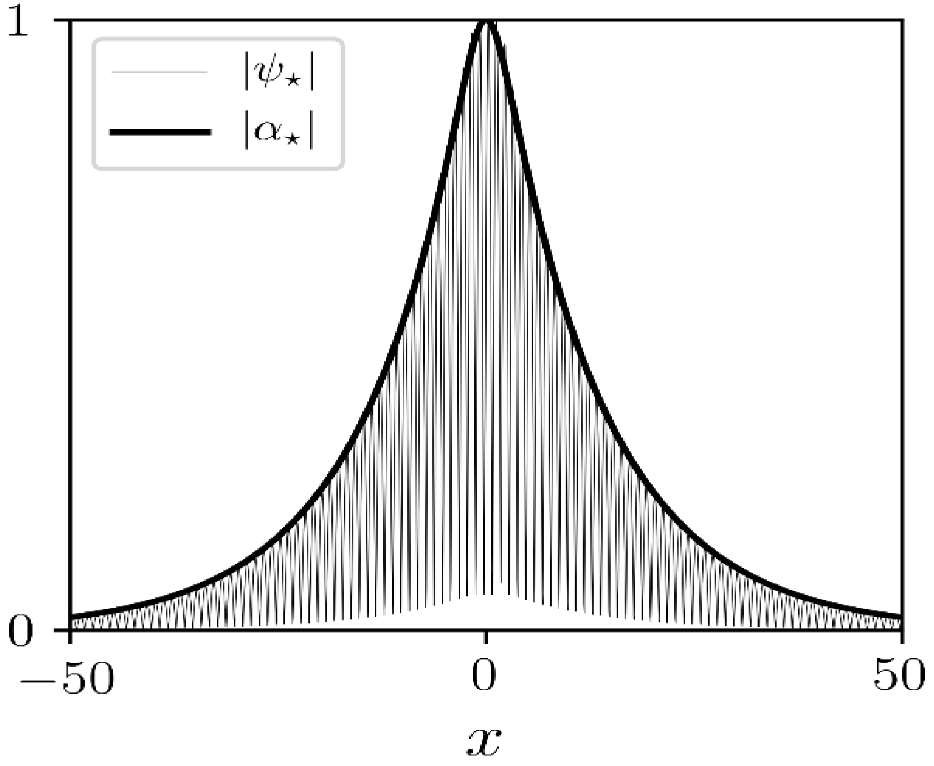

Example #2 – square wells

More in the paper…

# Numerical Simulations (Hameedi, AS, Weinstein, '23)

$$i\psi_t(t,x) = H_{dw}^{\varepsilon}\psi + i\varepsilon A(\varepsilon t)\partial_x\psi$$
$$\psi(0,x) = \boxed{\psi_\star(x)}$$

$A(T) = \beta cos(\omega T)$



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**Cosines potential**

**Square wells potential**

$|\psi(t,x)|^2$

**Static**
**($\boldsymbol{\beta = 0}$ )**

**Forced**
**($\boldsymbol{\beta \neq 0}$)**

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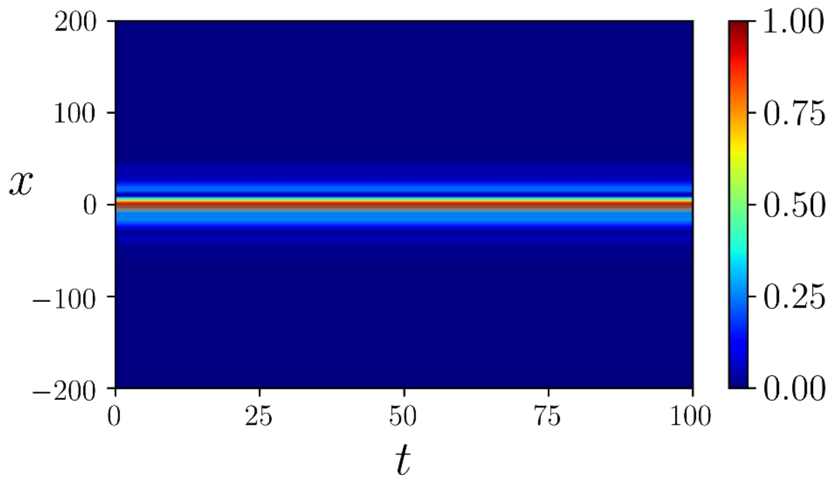


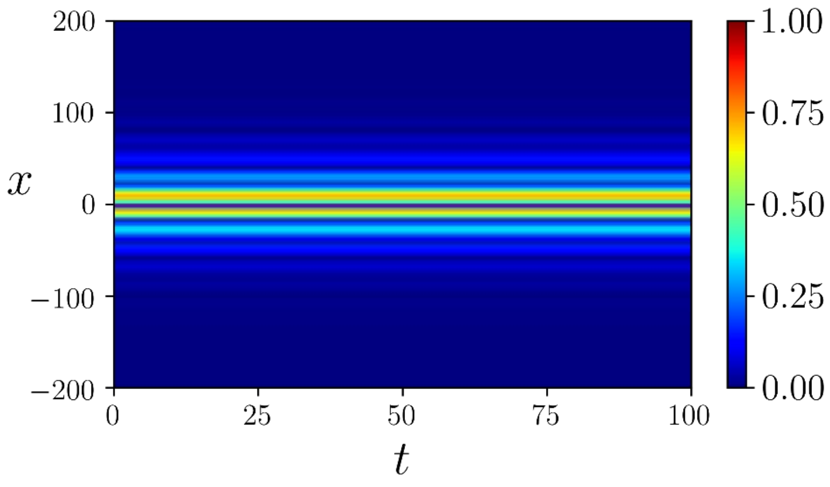


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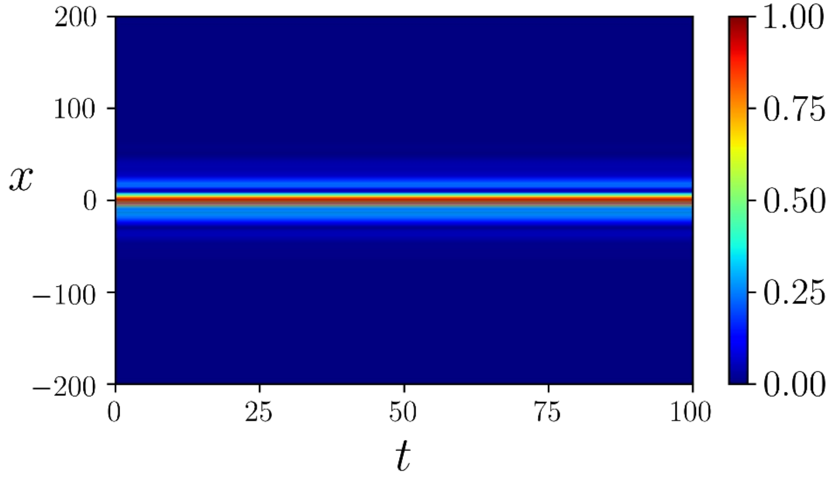


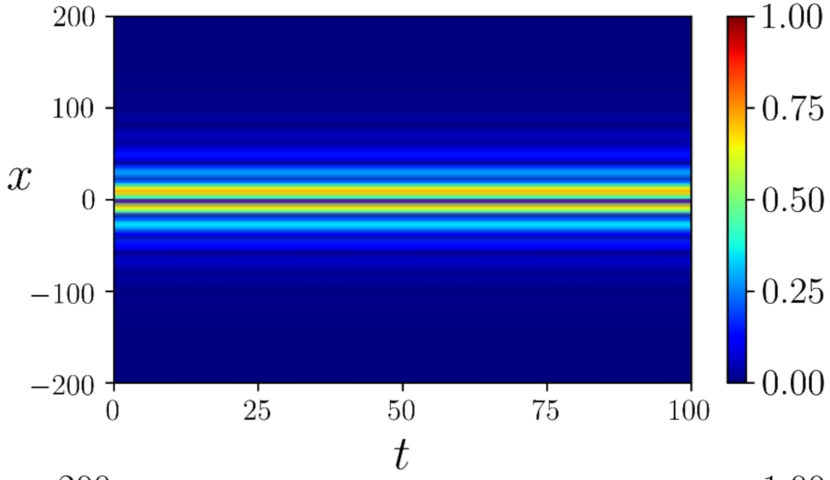


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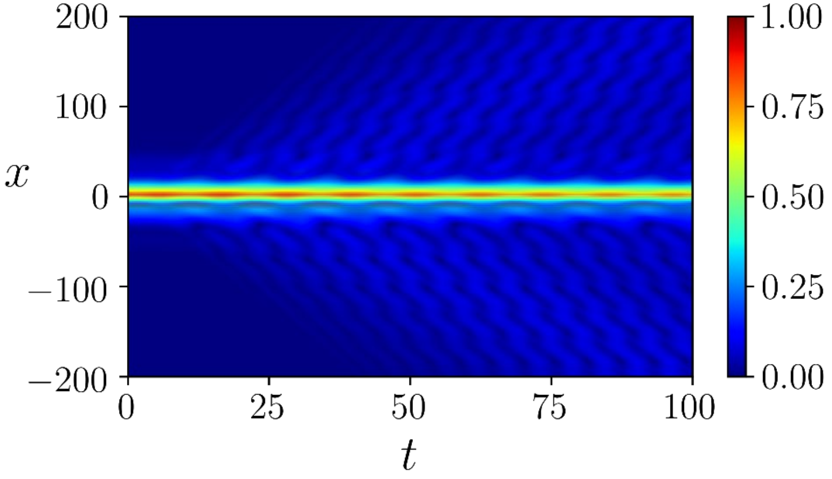


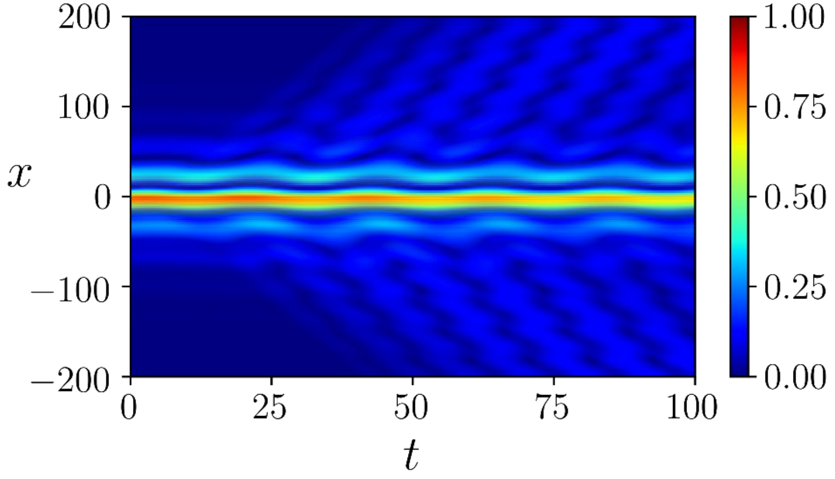

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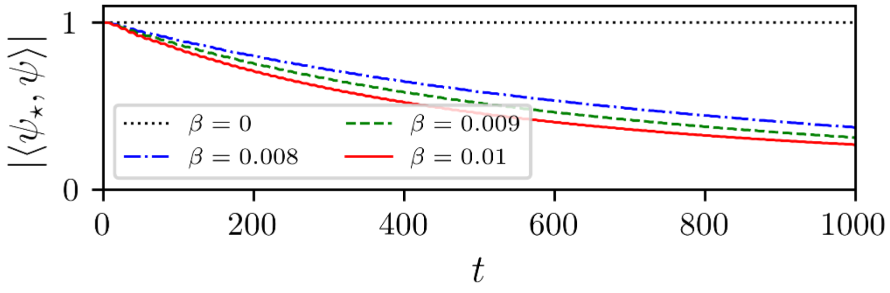


As $\beta$ increases, the decay increases

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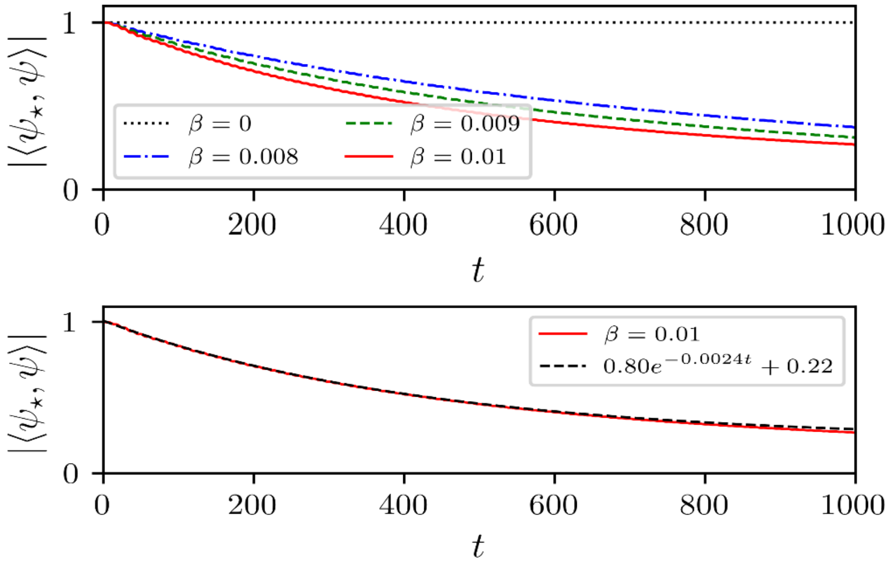


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The decay is exponential

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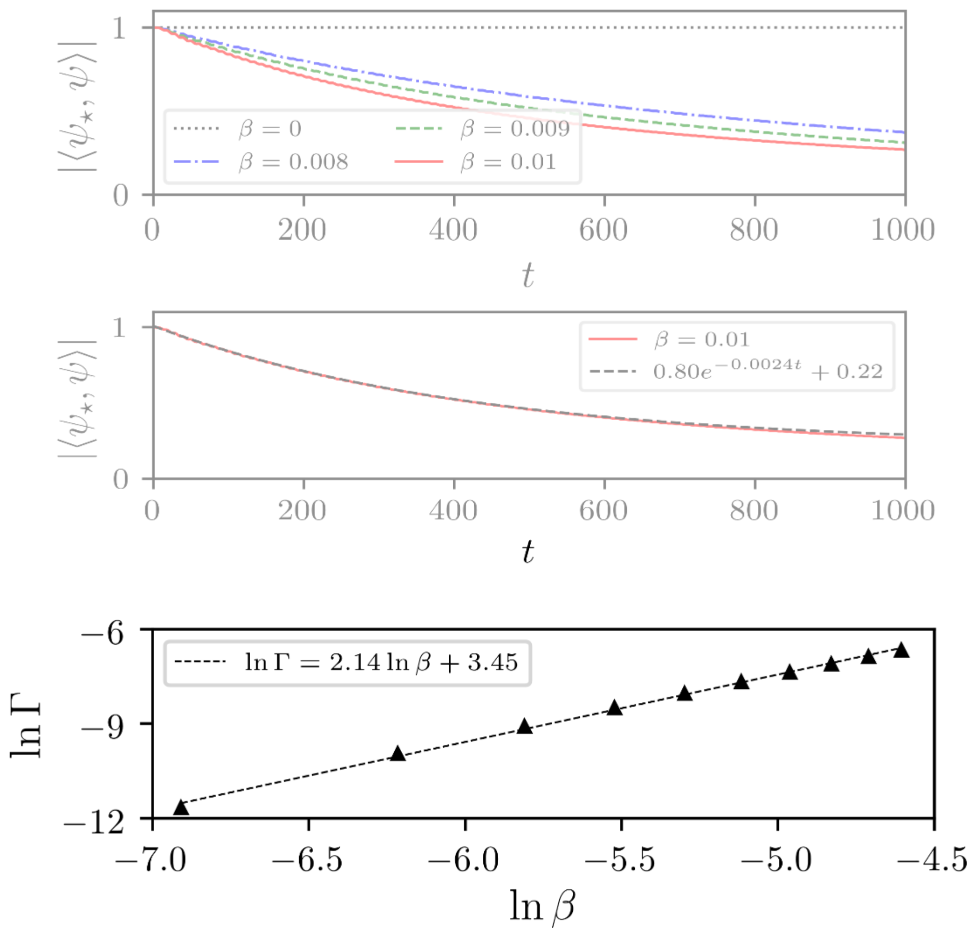


The decay $\exp(-\Gamma\varepsilon t)$ *with*

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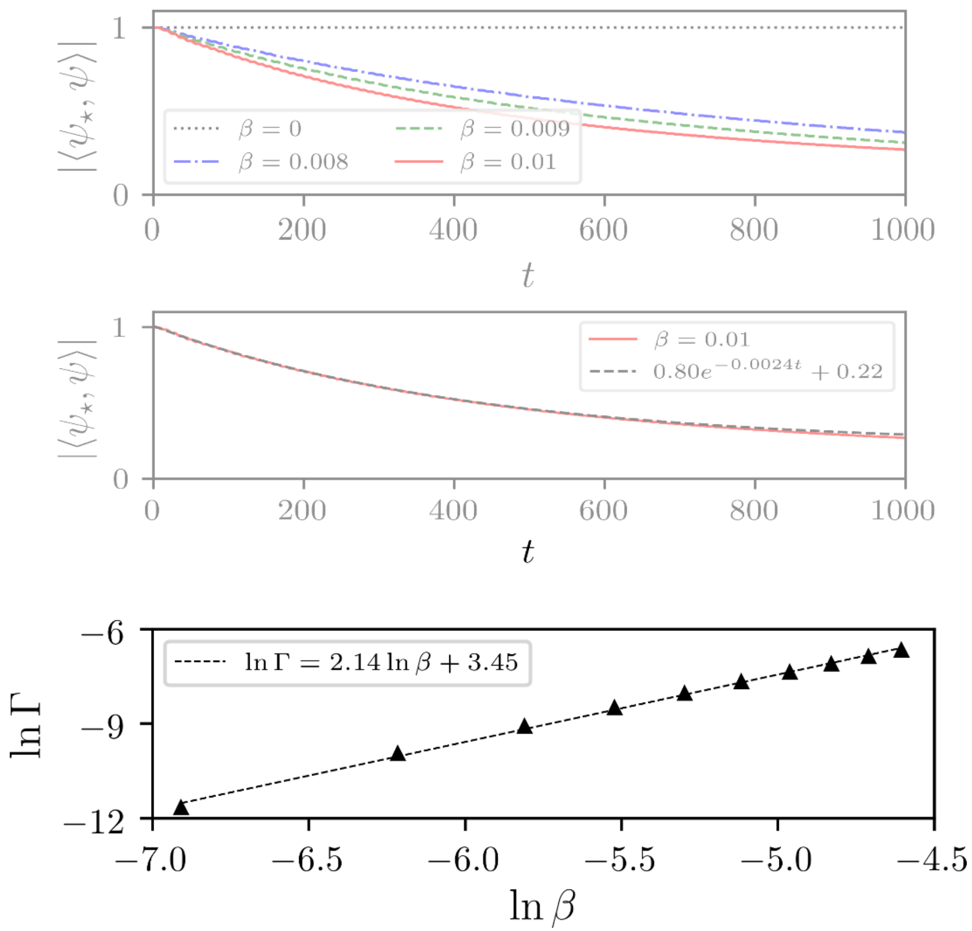


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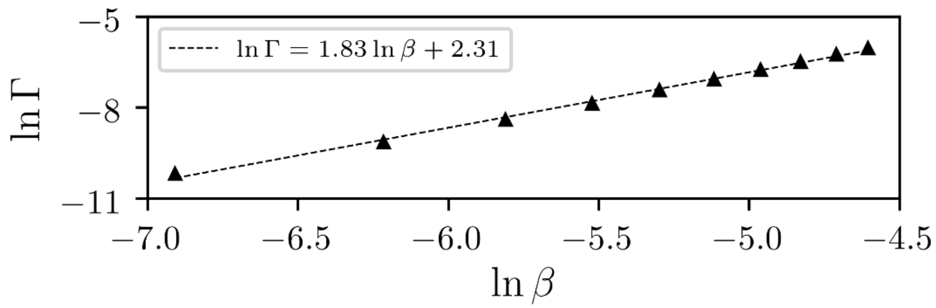
**Square wells potential**



# Dirac - Simulations

$$i\partial_T \vec{\alpha}(T,X) = \mathfrak{D}\,(T)\vec{\alpha}$$

Is the edge mode of $H_{dw}$ stable under forcing?

**Static**
$\beta = 0$

**Forced**
$\beta = 0.01$

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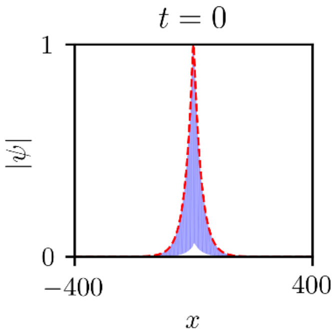


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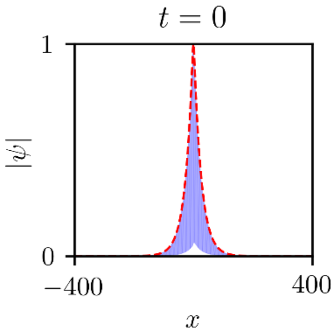

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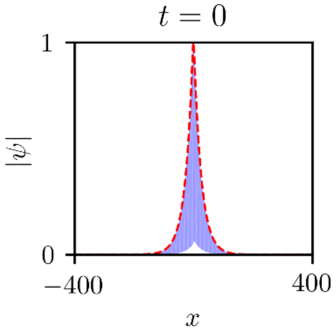


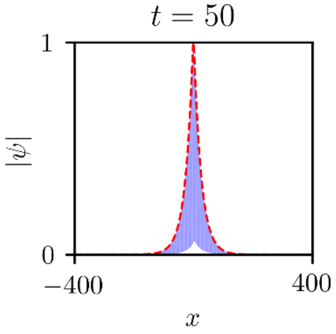


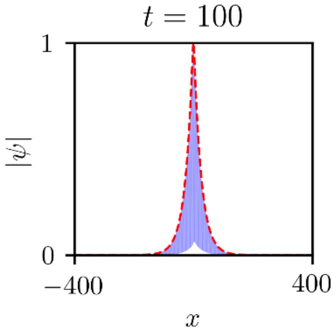


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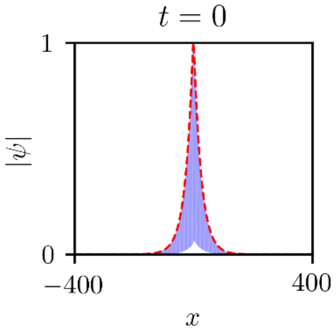

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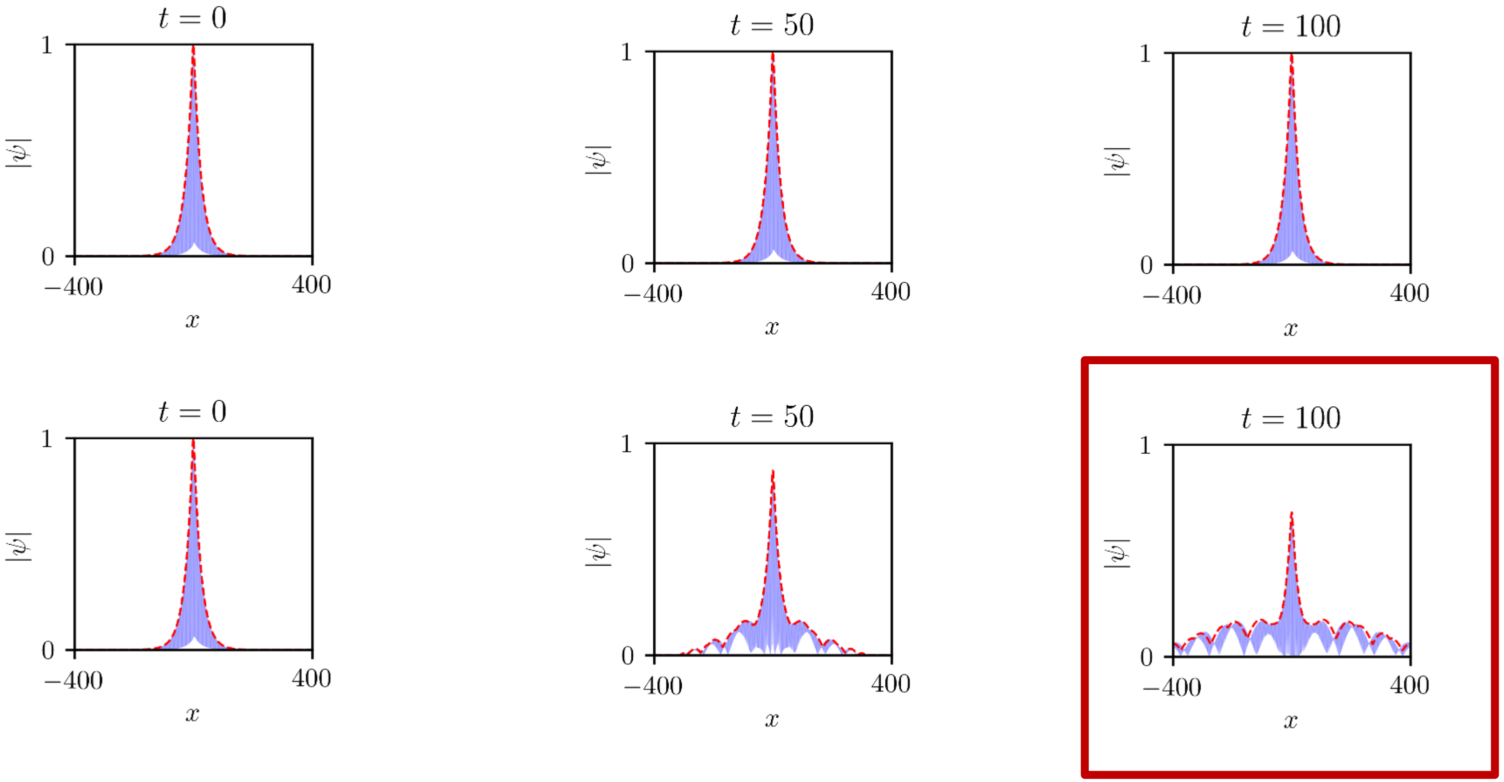
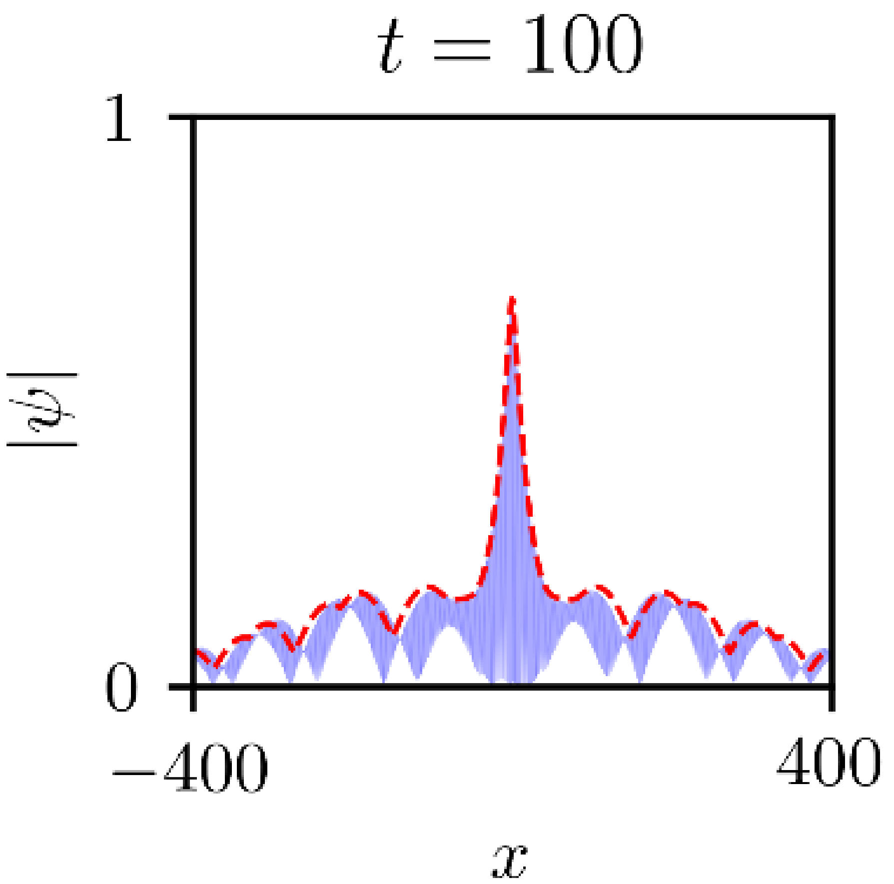



No! The decay occurs for the Dirac equation as well

# Zoom in

**Forced**
$\beta = 0.01$

$A(T) = \beta cos(\omega T)$




Recall, $T = \varepsilon t$
Here $\varepsilon = 0.5$

Takeaway –

The approximation constants are "not so bad"

Numerics vs. "*there exists* $\varepsilon > 0$ ..."

# Analysis

$$i\partial_T \alpha(T,X) = \big(D_0 + \boldsymbol{\beta} D_1(\omega t)\big)\alpha$$

Perturbation in the small forcing parameter $\boldsymbol{\beta}$

Autonomous part:

$D_0 = i\partial_X \sigma_3 + \kappa(X)\sigma_1$

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

# Analysis

$$i\partial_T \alpha(T,X) = \big(D_0 + \boldsymbol{\beta} D_1(\omega t)\big)\alpha$$

$$g(T) \equiv \langle \alpha_\star, \alpha(T,\cdot)\rangle_{L^2(\mathbb{R};\mathbb{C}^2)}$$

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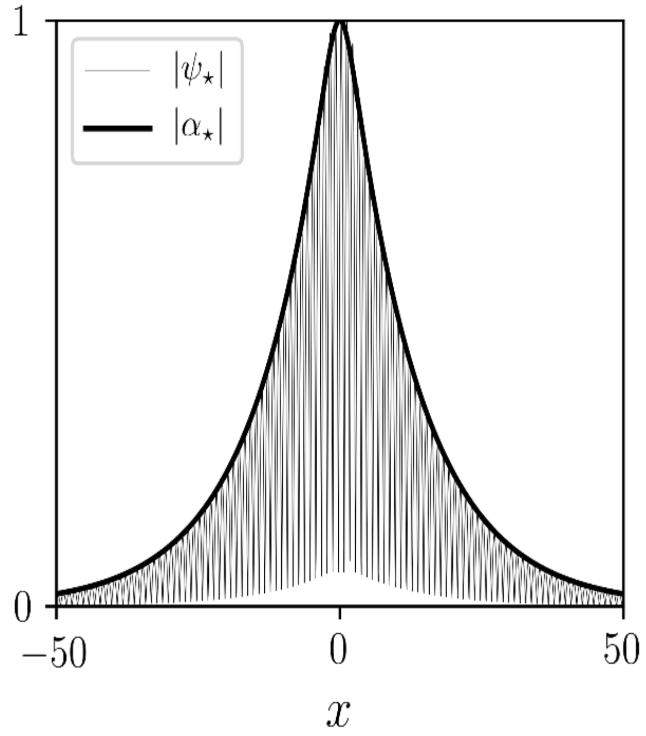
Quantity of interest-
projection onto the edge mode



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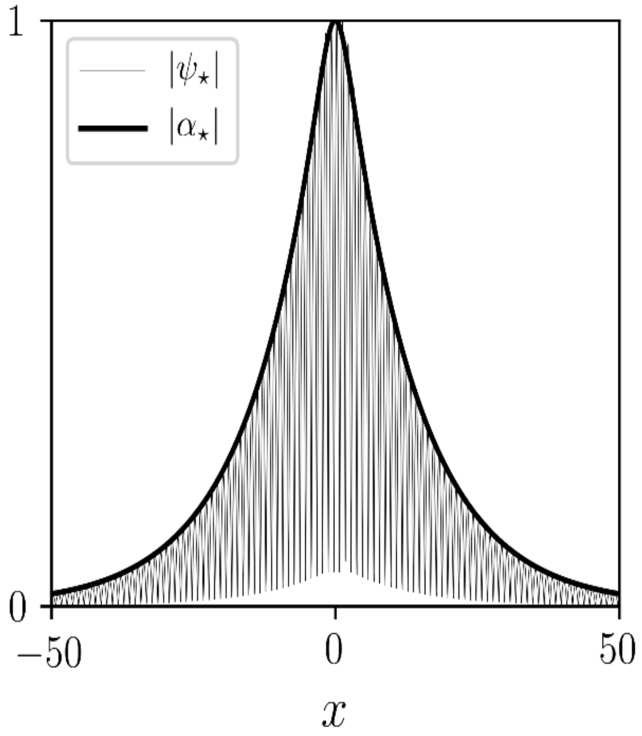


Methodology:
multi-scale modeling in time

For a full proof, as $t \to \infty$

we are missing the Floquet dispersive estimates

# Radiative Decay Formula

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For $\beta > 0$ sufficiently small and $\boldsymbol{T < T_0 \beta^{-2}}$,

$$\boldsymbol{g(T)} = \exp[-\beta^2(i\Lambda_0 + \Gamma_0)T](1+o(1))$$

where

$$\Gamma_0(\omega) = \frac{\pi}{4}\langle P_d\sigma_3\alpha_\star, [\delta(D_0+\omega) + \delta(D_0-\omega)]P_d\sigma_3\alpha_\star\rangle$$

# Exponential Decay

$$i\partial_T \alpha(T,X) = \left(D_0 + \beta D_1(\omega t)\right)\alpha$$

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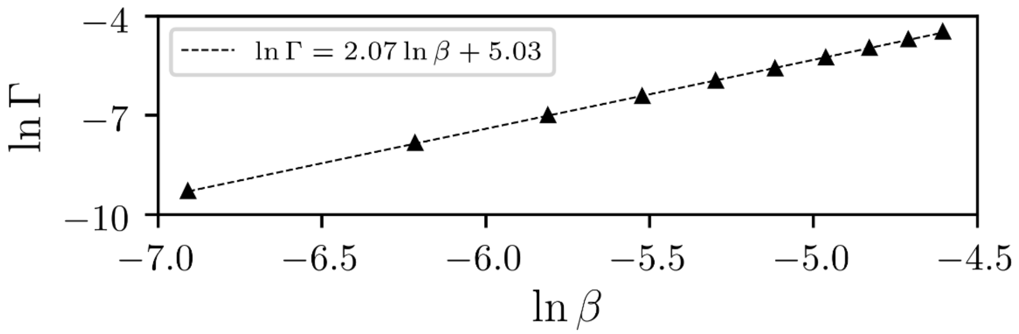
**1) Decay rate ~ $\boldsymbol{\beta}^2$**



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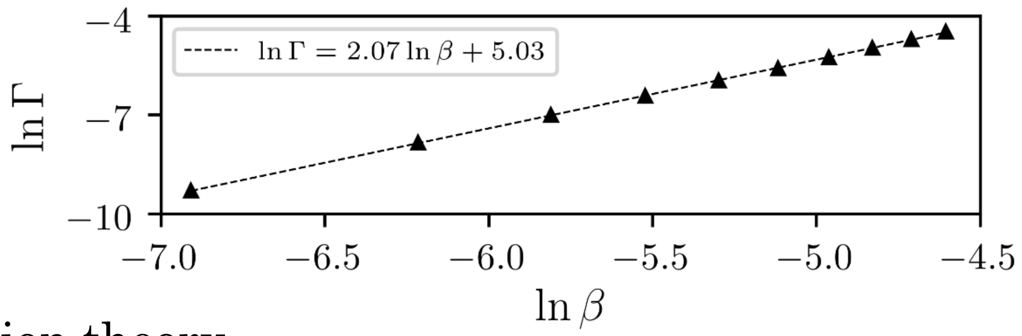


Technically – second order in perturbation theory

Quantum Mechanically – ionization of a "particle" by a field.
Known as ***Fermi Golden Rule***

# Resonant Decay

$$i\partial_T \alpha(T,X) = \big(D_0 + \boldsymbol{\beta} D_1(\omega t)\big)\alpha$$

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$$g(T) = \exp[\,-\beta^2(i\Lambda_0 + \Gamma_0)T](1+o(1)\,)$$

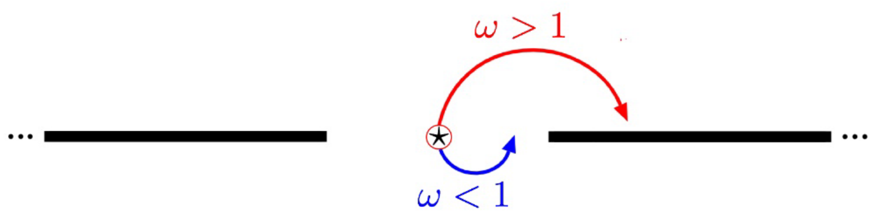
where

$$\Gamma_0(\omega) = \frac{\pi}{4}\langle P_d\sigma_3\alpha_\star, [\boldsymbol{\delta}(\boldsymbol{D_0}+\boldsymbol{\omega}) + \boldsymbol{\delta}(\boldsymbol{D_0}-\boldsymbol{\omega})]P_d\sigma_3\alpha_\star\rangle$$

**2) Decay as resonance**

Applies only if $\omega$ "bridges" the spectral gap

$\sigma(D_0)$



# Resonant Decay

$$i\partial_T \alpha(T,X) = \big(D_0 + \boldsymbol{\beta} D_1(\omega t)\big)\alpha$$

$$g(T) \equiv \langle \alpha_\star, \alpha(T,\cdot)\rangle_{L^2(\mathbb{R};\mathbb{C}^2)}$$

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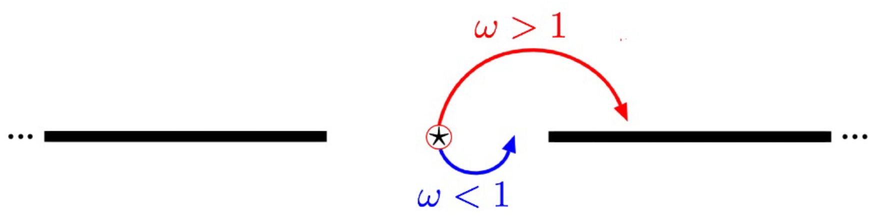


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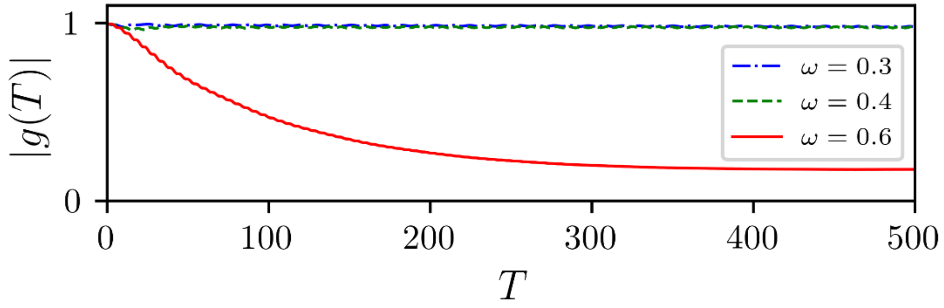

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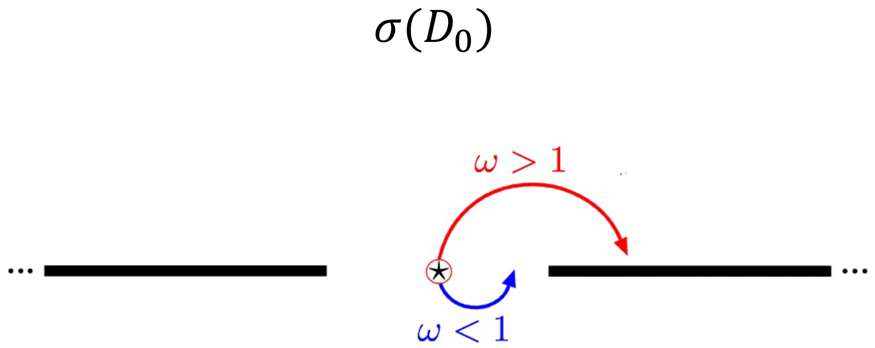
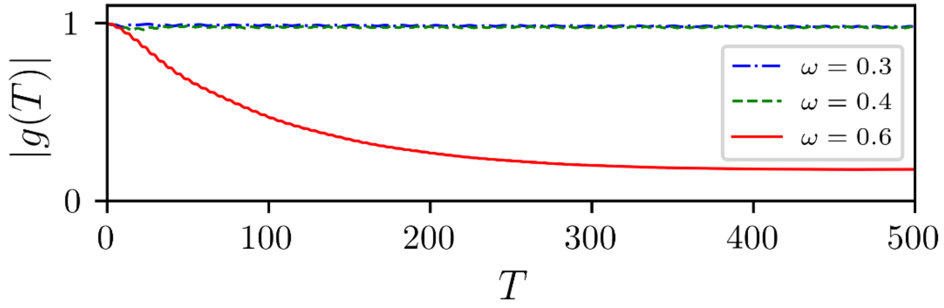
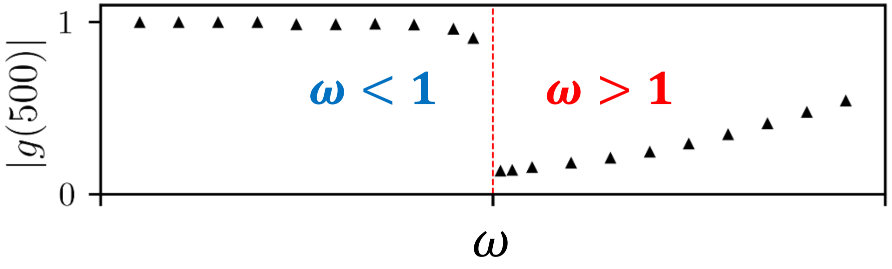


# Agenda

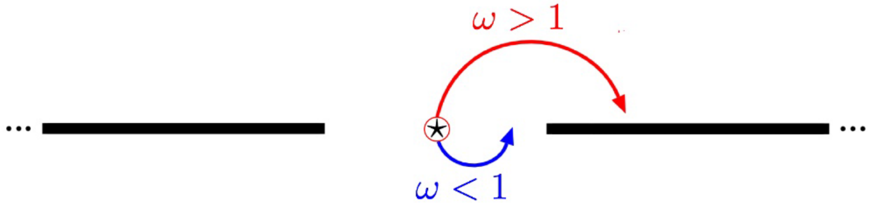
- The time-periodic Dirac equation
- How to get very slow dispersion
- Application to Floquet edge modes
- A discrete Analog

With:

Michael Weinstein (Columbia)
Remy Kassem (Columbia -> GA Tech)

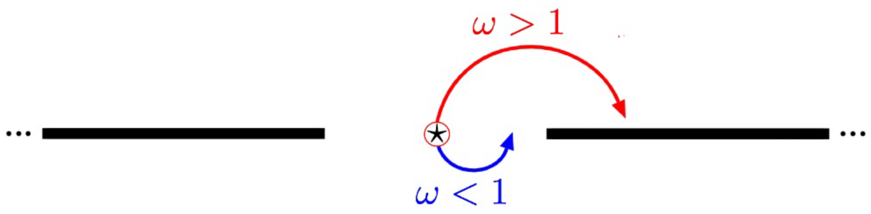
# Back to Discrete Models

Dirac spectrum

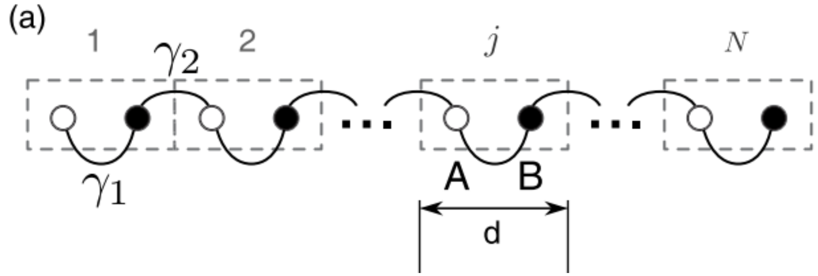


# Back to Discrete Models

Dirac spectrum



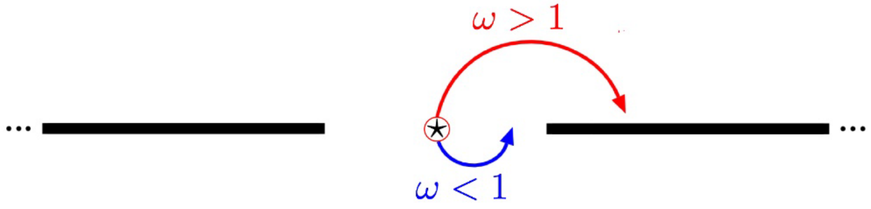

SSH – standard discrete model




Su, Schrieffer, Heeger, PRL 1979

# Back to Discrete Models

Dirac spectrum

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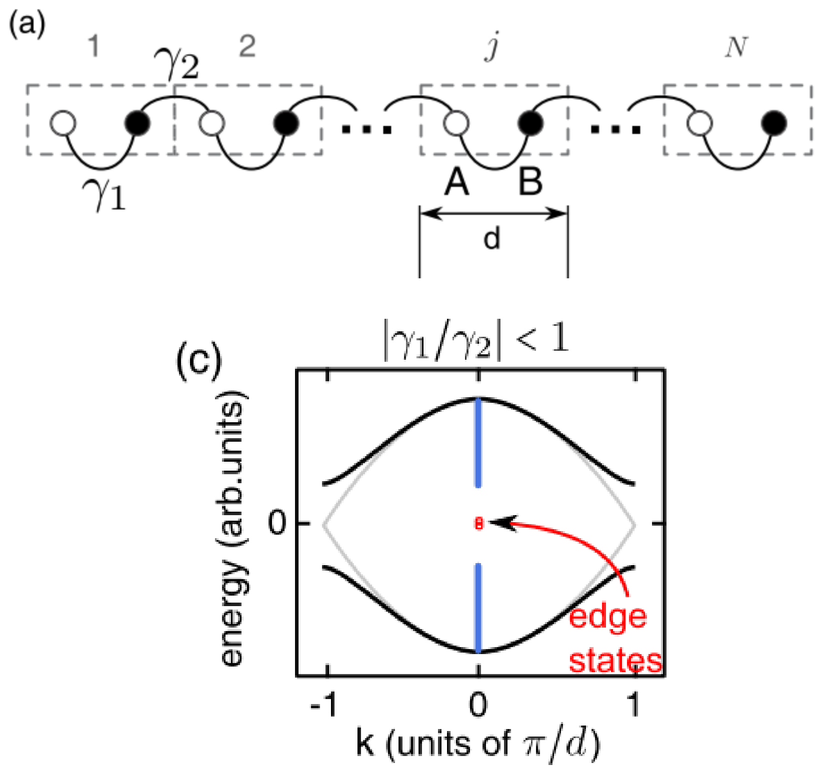


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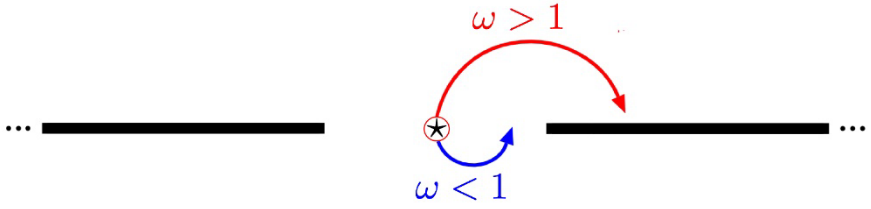
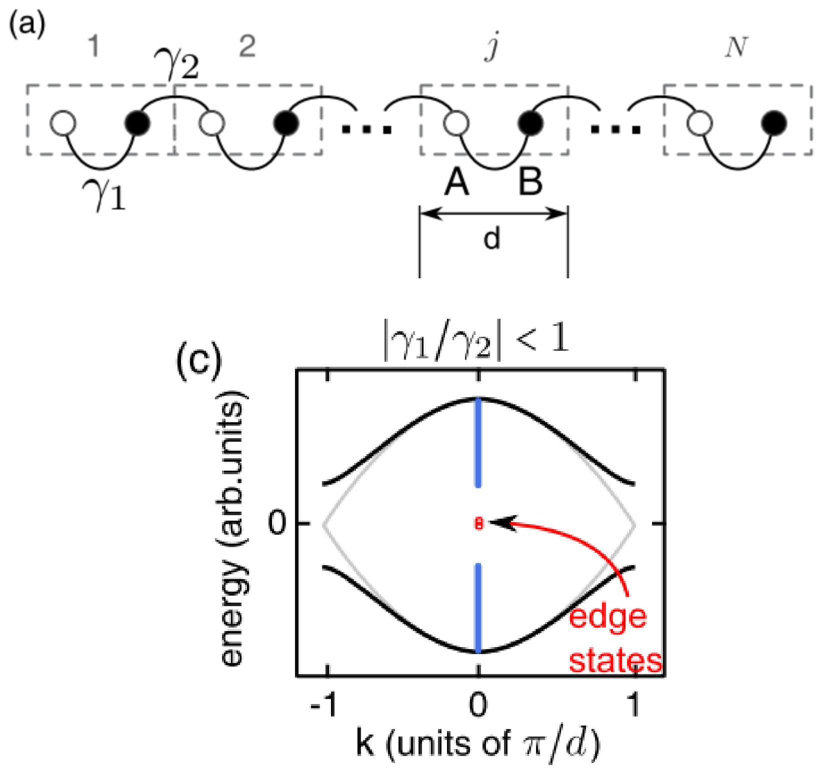
Dirac spectrum




In discrete models -

- spectrum is bounded

SSH – standard discrete model

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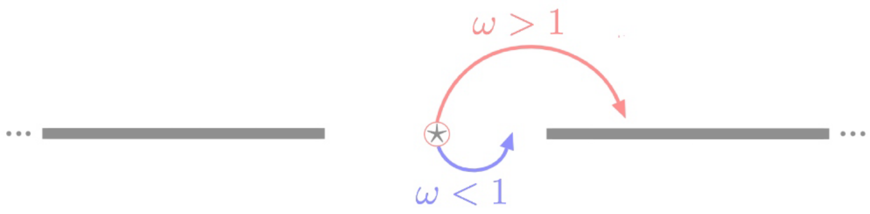
# Back to Discrete Models

Dirac spectrum

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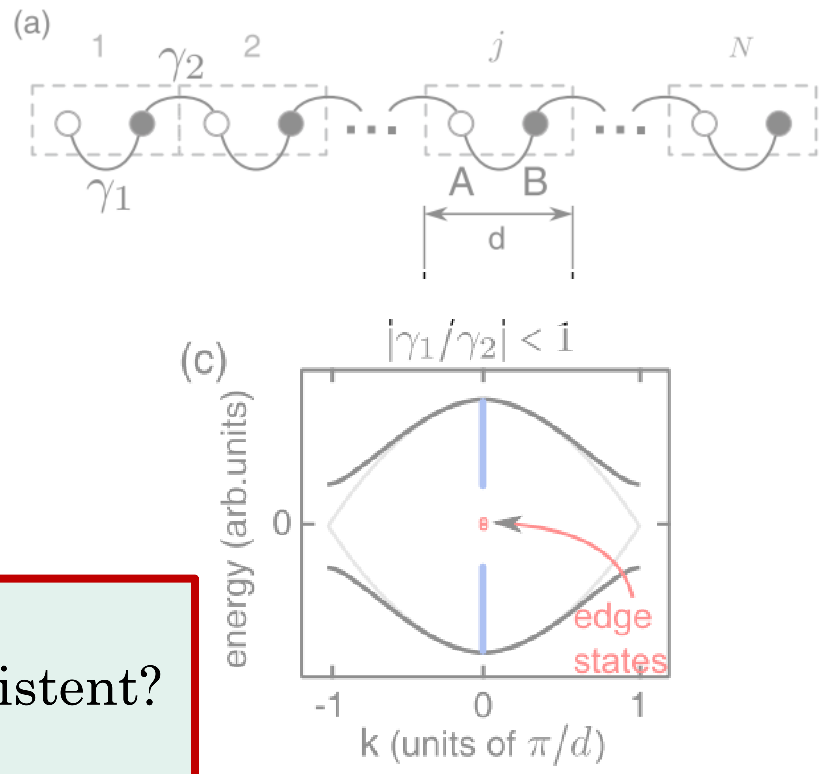



In discrete models -

- spectrum is bounded

Question:

Are the discrete and continuum models consistent?

or

Does Floquet SSH undergoes resonant decay?

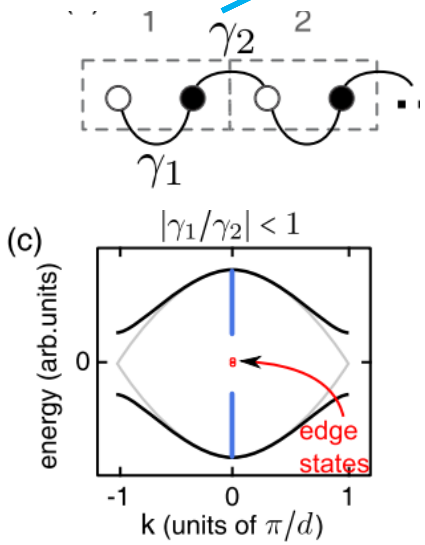
Su, Schrieffer, Heeger, PRL 1979

# Preliminary Results

Driven SSH

$$i\partial_T \alpha(T,X) = \big(D_0 + \boldsymbol{\beta} D_1(\omega t)\big)\alpha$$

time-periodic*




Discrete: underlying space is $\ell^2(\mathbb{N};\mathbb{C}^2)$

Question:

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Ongoing work with R Kassem & MI Weinstein

*Asboth, Tarasinski, Delpalce, Phys. Rev. B 2014
Dal Lago, Atalla, Foa Torres, Phys. Rev. A 2015

# Preliminary Results

Projection on edge state at $T \gg 1$

Driven SSH

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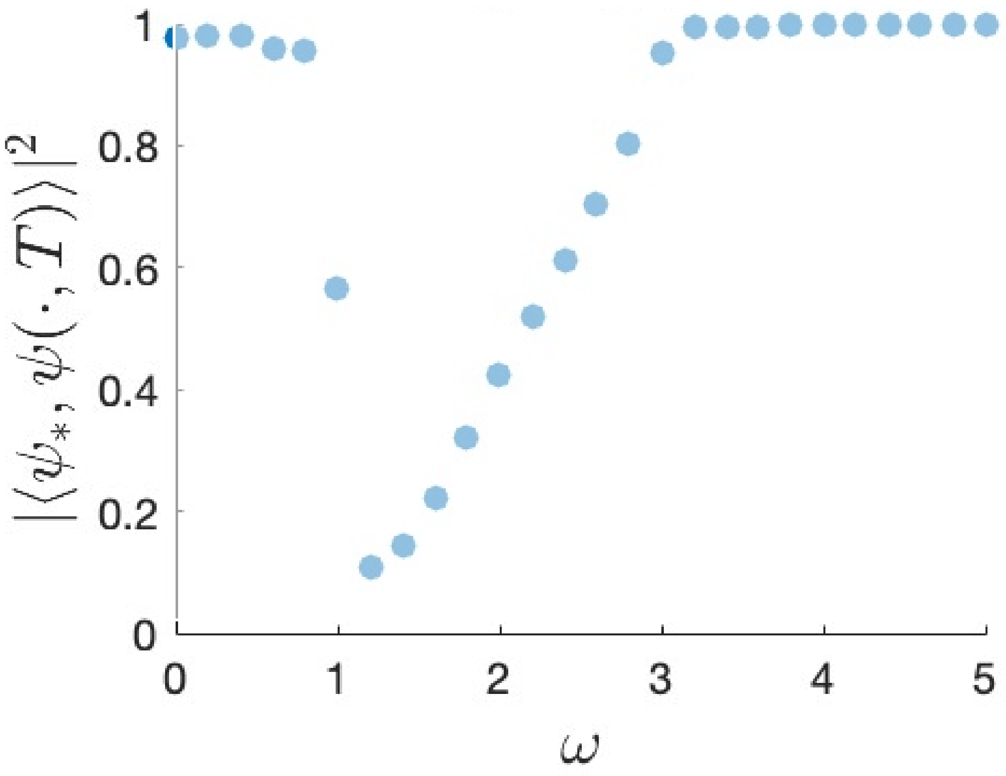
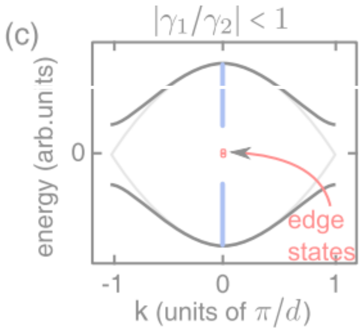



Ongoing work with R Kassem & MI Weinstein

# Preliminary Results

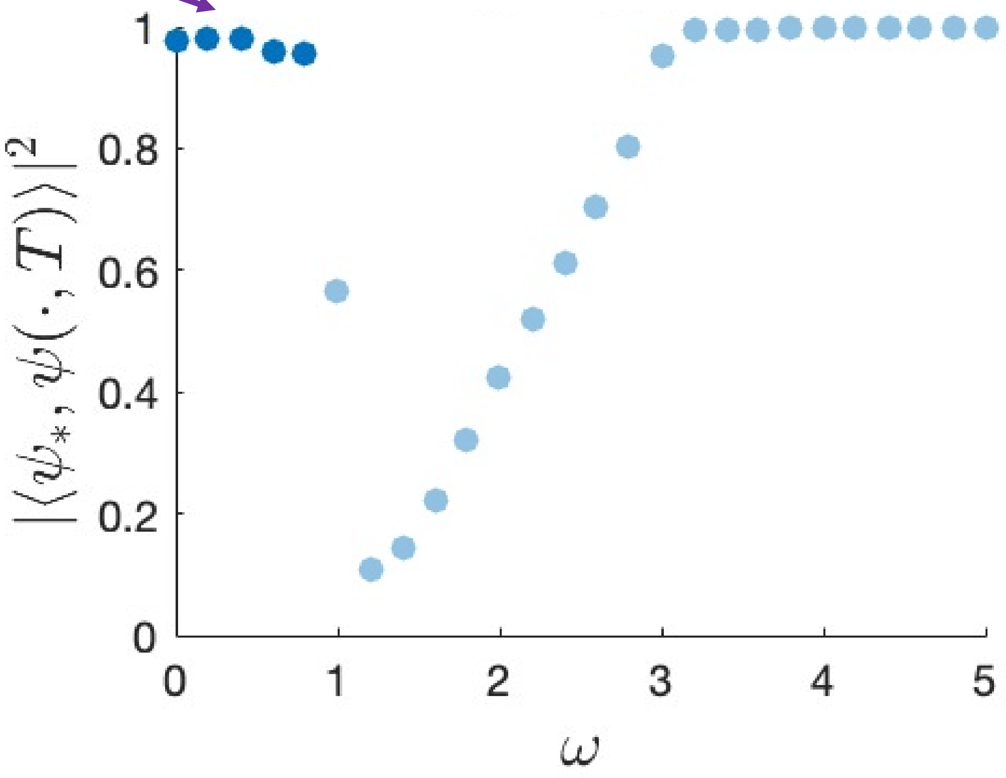
Projection on edge state at $T \gg 1$

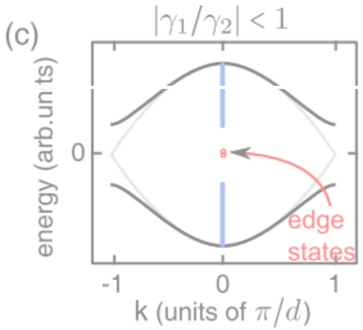
Three phases of $\omega$

Driven SSH

$$i\partial_T \alpha(T,X) = \big(D_0 + \boldsymbol{\beta} D_1(\omega t)\big)\alpha$$

I) Below resonance, negligible decay




Ongoing work with R Kassem & MI Weinstein

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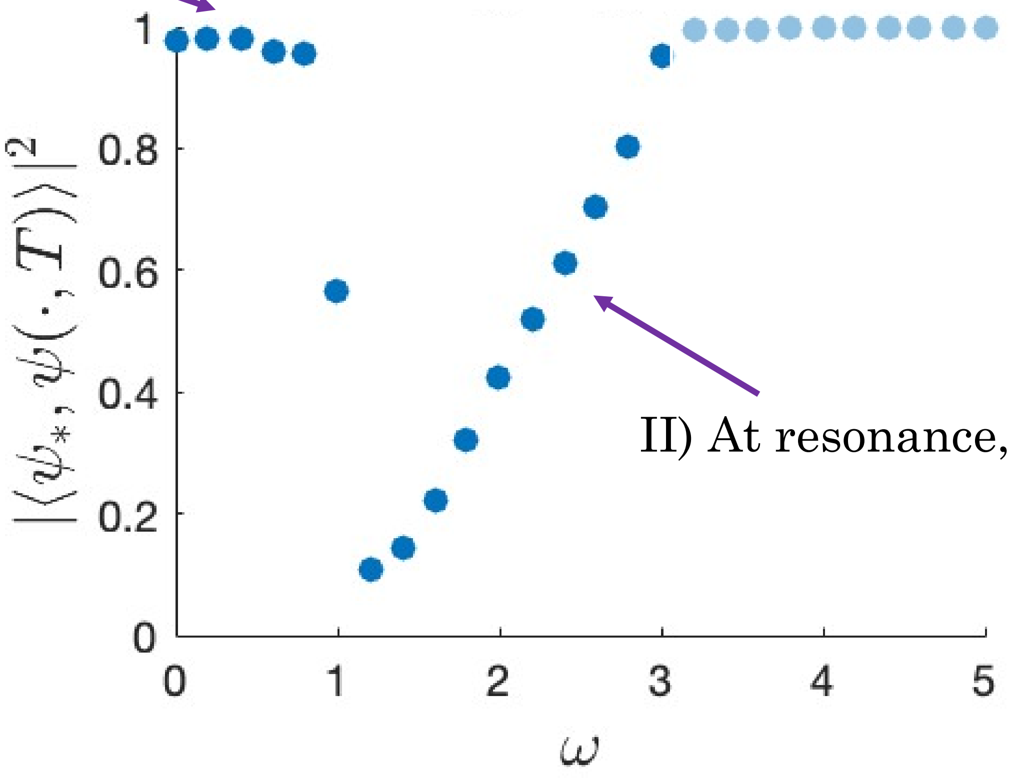
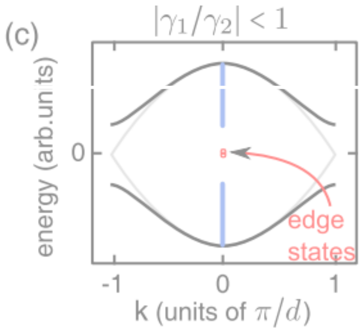



II) At resonance, decay




Ongoing work with R Kassem & MI Weinstein

# Preliminary Results

Projection on edge state at $T \gg 1$
Three phases of $\omega$

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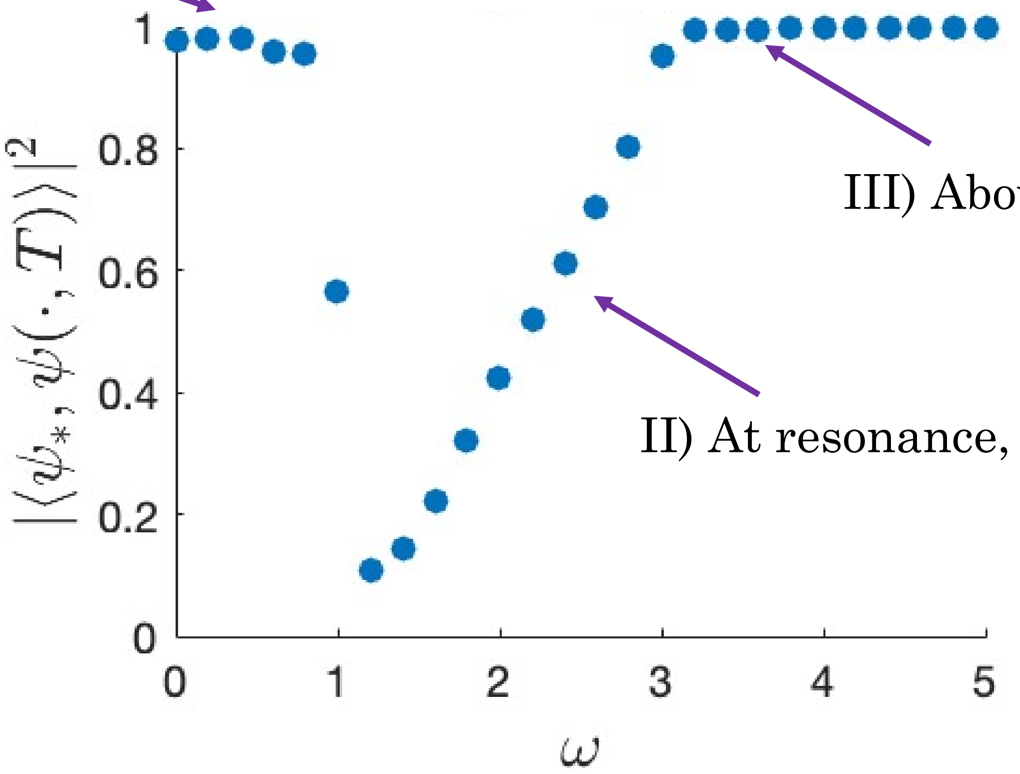
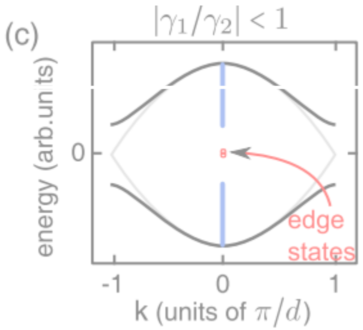



III) Above resonance,
*no decay?*

II) At resonance, decay




Ongoing work with R Kassem & MI Weinstein

# Thank you!

**References**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- *On the Time Decay of Solutions Arising from Periodically Forced Dirac Hamiltonians*

  *Krasiler, Sagivm & Weinstein, J Differential Equations, 2025*

- *Effective Gaps in Floquet Hamiltonians,*

  *Sagiv & Weinstein, SIAM J Mathematical Analysis 2022*

- *Radiative Decay of Edge States in Floquet Media,*

  *Hameedi, Sagiv, & Weinstein, SIAM MMS 2023*