Quantum spin systems: old and new

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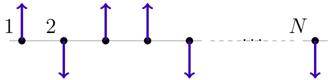
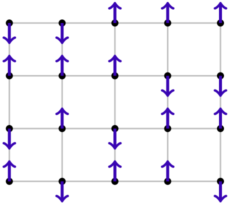
What is a spin system?

A **spin system** is a collection of N fixed sites, each carrying an arrow $+$ (up) or $-$ (down). They model e.g. **magnets** and **quantum computer hardware**.



(spin chain)



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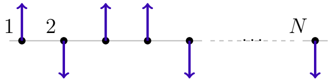

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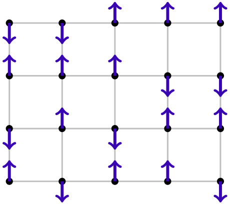

(spin grid)

The **state** of the system is the element $s \in \{+,-\}^N$ collecting the arrows at sites $1,\ldots,N$ (2^N possibilities). Its **energy** H is measured by the **Hamiltonian** of the system: a function

$$H : \{+,-\}^N \to \mathbb{R}$$

In statistical mechanics: **the lower the energy, the more likely the state is.**

Example: the Ising model

For the Ising model on $\{1, 2, \ldots, N\}$:

$$H = -\sum_x s_x s_{x+1}$$

"You vote like your neighbors."

The energy is lowest when all adjacent spins are aligned; each misalignment adds $+2$ units of energy. **Two ground states:** $\{+, +, \ldots, +\}$ and $\{-, -, \ldots, -\}$.

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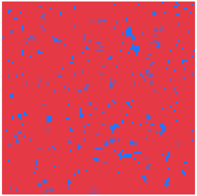
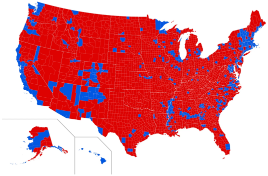
More generally, one can consider the Ising model on a **graph** (e.g. a grid):

$$H = -\sum_{x \sim y} s_x s_y.$$



Low energy state for Ising grid



US 2024 presidential election

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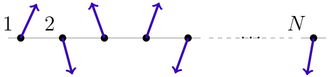
This motivates our structural assumptions on Hamiltonians.

Locality: $H = \sum_x H_x$, where H_x affects only spins adjacent to x (i.e. $O(1)$-away from x).

One analyses these systems in the **large N limit**.

Quantum spin systems

A **quantum spin system** is the quantum analogue: for a graph Λ, each vertex carries a spin – an element of $\mathbb{C}^2$.



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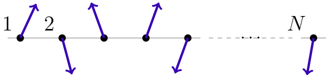

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The **state** of the system is a vector $|\psi\rangle \in (\mathbb{C}^2)^{\otimes\Lambda}$ (dimension $2^{|\Lambda|}$). The **Hamiltonian** is a Hermitian operator

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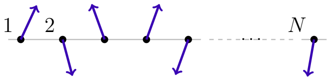

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Example: the quantum Ising model.

$$H = -\sum_{x\sim y} \sigma_x^3 \sigma_y^3,$$

where σ_x^3 measures the spin at site x:

$$\sigma_x^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \mathbf{1}_{\Lambda\setminus\{x\}}.$$

Propagation in quantum spin systems

The dynamics follows the **Schrödinger equation**:

$$i\frac{d\psi(t)}{dt} = H\psi(t), \qquad \psi(0) = \psi_0 \in (\mathbb{C}^2)^{\otimes\Lambda}.$$

There are **no moving particles**. But measurements of an observable A change over time. In the **Heisenberg picture**:

$$\langle \psi(t), A\psi(t)\rangle = \langle \psi_0, A(t)\psi_0\rangle, \qquad A(t) = e^{iHt}Ae^{-iHt}.$$

For instance, $\langle \psi_0, \sigma_x^3(t)\psi_0\rangle$ measures the x-th spin of ψ at time t.

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Which spins of ψ at time 0 influence the x-th spin of ψ at time t?

Natural but non-trivial question:

- e^{itH} is not a local operator (not a sum of terms acting on nearby spins), even for small t.
- In fact even H^2 is not a local operator:

$$H^2 = \sum_{x,y} H_x H_y, \qquad H_x H_y \text{ supported near } \{x, y\}.$$

Detour: single particle systems

Dynamics for quantum spins contrasts with that of **single particles**. Single particles are described by a short-range Hamiltonian $H : \ell^2(\Lambda) \to \ell^2(\Lambda)$:

$$|H(x,y)| \leq Ce^{-\nu|x-y|}, \qquad H(x,y) := \langle \delta_x, H\delta_y \rangle$$

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Theorem [Combes–Thomas '73]

For some constants $C', \alpha > 0$,

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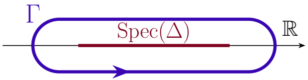
Proof. Write



$$e^{itH} = \frac{1}{2\pi i} \oint_\Gamma e^{itz} \, (z - H)^{-1} \, dz.$$

For $z \in \Gamma$, $d(z, \mathrm{Spec}(H)) \geq 1$ and by the Combes–Thomas inequality:

$$\left| (z-H)^{-1}(x,y) \right| \leq Ce^{-\alpha|x-y|}$$

for some $C, \alpha > 0$ independent of $|\Gamma|$. Then, using $e^{itz} \leq e^{|t||\operatorname{Im} z|} \leq e^{|t|}$:

$$\left| e^{itH}(x,y) \right| \leq C|\Gamma| \, e^{|t|} \, e^{-\alpha|x-y|}. \qquad \square$$

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Interpretation:

- If ψ is localized near y then $e^{itH}\psi_0$ is exponentially small away from $|t| - \alpha|x-y| > 0$ i.e. away from $[y - \alpha^{-1}|t|, y + \alpha^{-1}|t|]$.
- This means that information propagates at **finite speed**.
- It turns out this remains true in **quantum spin systems**, despite lack of locality of e^{-itH}!

Return to quantum spin systems

Definition

We say that an observable A *on* $(\mathbb{C}^2)^{\otimes\Lambda}$ *has* **support in** X *if* $A = a \otimes \mathbf{1}_{\Lambda\setminus X}$ *for some observable* a *on* $(\mathbb{C}^2)^{\otimes X}$.

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Lemma

A has support in X iff for every B supported in $\Lambda \setminus X$, we have $[A, B] = 0$.

Interpretation: A measurement depends only on spins in X iff it corresponds to an observable that commutes with all operators supported in $\Lambda \setminus X$.

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*We say that an observable A on $(\mathbb{C}^2)^{\otimes\Lambda}$ has ε-***support in** *X if $\|A - a \otimes \mathbf{1}_{\Lambda\setminus X}\| \leq \varepsilon$ for some observable a on $(\mathbb{C}^2)^{\otimes X}$.*

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Lemma

A has ε-support in X iff for every B supported in $\Lambda \setminus X$, we have $\|[A, B]\| \leq 2\varepsilon\|A\|\|B\|$.

Interpretation: A measurement depends (nearly) only on spins in X iff it corresponds to an observable that (nearly) commutes with all operators supported in $\Lambda \setminus X$.

Lieb–Robinson bound

Recall: $H = \sum_x H_x$ where each H_x has support in $B_r(x)$, with r independent of x and $|\Lambda|$.

Theorem [Lieb–Robinson '72]

There exist constants $C, v, \alpha > 0$ (depending only on r and $\sup_x \|H_x\|$) such that, for any observables A with support in X and B with support in Y,

$$\left\|[e^{itH} A e^{-itH}, B]\right\| \leq C\|A\|\|B\| \, e^{-\alpha(d(X,Y)-v|t|)}.$$

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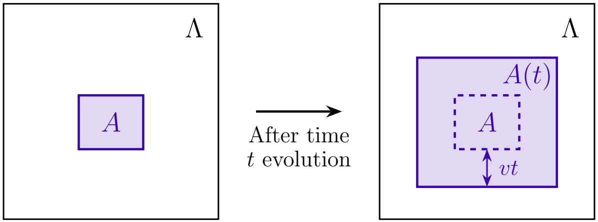

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Interpretation:

- $A(t) = e^{itH} A e^{-itH}$ is exponentially small outside $B_{vt}(X)$: it essentially has support within vt of X.
- The spin at site x at time t depends (essentially) only on spins in $B_{vt}(x)$.

Proof of Lieb–Robinson

Setup. Let A have support $\{0\}$, and B supported on an edge $e \subset \Lambda$ with $\|B\| \leq 1$. Write $H = \sum_{e'} H_{e'}$ with $\|H_{e'}\| \leq 1$.

1. Because of unitarity:

$$\big\|[A(t+\varepsilon), B]\big\| = \big\|[A(t), B(-\varepsilon)]\big\|.$$

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2. Up to $O(\varepsilon^2)$ terms, using the **Heisenberg equation** and the **Jacobi identity**:

$$[A(t), B(-\varepsilon)] = [A(t),\, B - i\varepsilon[H, B]] = [A(t),\, B] - i\varepsilon \sum_{e' : e' \cap e \neq \emptyset} \big[A(t), [H_{e'}, B]\big]$$

$$= [A(t),\, B] - i\varepsilon \sum_{e' : e' \cap e \neq \emptyset} \big[H_{e'}, [A(t), B]\big] + i\varepsilon \sum_{e' : e' \cap e \neq \emptyset} \big[B, [A(t), H_{e'}]\big]$$

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3. Taking norms, using unitarity again:

$$\big\|[A(t+\varepsilon),\, B]\big\| \leq \big\|[A(t),\, B]\big\| + 2\varepsilon \sum_{e' \,:\, e' \cap e \neq \emptyset} \big\|[A(t),\, H_{e'}]\big\|.$$

$$\frac{d}{dt}\big\|[A(t),\, B]\big\| \leq 2 \sum_{e' \,:\, e' \cap e \neq \emptyset} \big\|[A(t),\, H_{e'}]\big\|.$$

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4. We rewrite this as:

$$\frac{du(t,e)}{dt} \le 2 \sum_{e' \,:\, e' \cap e \neq \emptyset} u(t,e') = 2\Delta u(t,e)$$

$$u(t,e) = \sup_{\mathrm{supp} X \subset e, \|X\| \le 1} \big\|[A(t),\, X]\big\|.$$

Hence $u(t,e)$ is a **subsolution** of the heat equation (for the *line graph Laplacian* Δ *on edges:* two edges are connected if they share a vertex).

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5. Subsolutions of heat equations are **below actual solutions**. Because A is supported at 0, $u(0,e)$ is 0 unless $0\in e$ and

$$\left\|[A(t),\,B]\right\| \leq u(t,e) \leq e^{2t\Delta}u(0,e) = \sum_{0\in e'}\left|e^{2t\Delta}(e,e')\right|.$$

We have bounded $\left\|[A(t),\,B]\right\|$ by the kernel of $e^{2t\Delta}$!

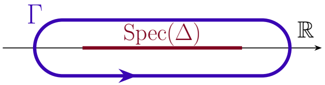
Proof of Lieb–Robinson

6. It remains to bound the kernel of $e^{2t\Delta}$. Δ is an honest graph Laplacian, hence short-range. Apply the Combes–Thomas inequality to

$$e^{2t\Delta}(e,e') = \frac{1}{2\pi i} \oint_\Gamma e^{2tz} \, (z-\Delta)^{-1}(e,e') \, dz.$$

With $R = \operatorname{diam}\Gamma$, it gives:

$$\left| e^{2t\Delta}(e,e') \right| \leq e^{2Rt} \cdot C \, e^{-\alpha \, d(e,e')},$$



and hence with $v = 2R/\alpha$,

$$\left\| [A(t),\, B] \right\| = u(t,e) \leq C \, e^{-\mu(d(0,e) - v|t|)}. \qquad \square$$

Applications

Hastings popularized (forgotten) Lieb–Robinson bounds in 2000s. Applications:

- **Decay of correlations in gapped ground states.** If H has a unique, gapped ground state $|\psi\rangle$, then for A, B with supports X, Y,

$$\left|\langle\psi| AB |\psi\rangle - \langle\psi| A |\psi\rangle\langle\psi| B |\psi\rangle\right| \lesssim e^{-\mu\, d(X,Y)}.$$

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  Consequences:

  - Good **matrix product state** representations, efficient simulation [Verstraete–Cirac '05, Hastings '07],
  - 1D gapped GS are **short-range entangled** (i.e. obtained as a locally generated unitary acting on a product state) [Fraas–de Roeck '25].

Conjecture. The entropy of gapped ground states in a region X is bounded by $C\,|\partial X|$: **area law in higher dimension!**

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- **Thermalization** and Gibbs state sampling [Davies '70s, Chen–Kastoryano –Brandao–Gilyen '25, Becker–Rouzé–Salzmann '26]
- Under certain conditions, **local perturbations** of ground states affect the ground state **only locally (LPPL)** [Hastings 04', Bachmann–Michalakis –Nachtergaele–Sims '11] $\Rightarrow$ **What we study next!**

Motivation: study of interface effects between topological phases of matter. For one-particle systems, **very solid mathematical theories but they rely on LPPL...**

Do local perturbations affect ground state far away?

- In **one-particle systems**, local perturbations do not affect far-away ground state measurements. This is the **LPPL principle**
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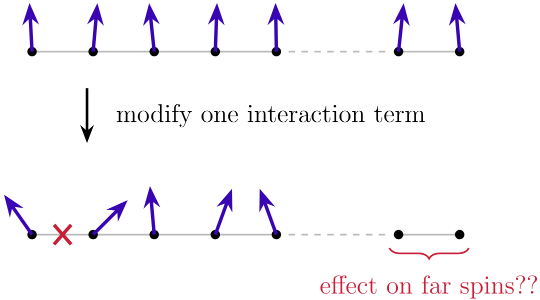

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LPPL does not hold without additional assumptions!

Goal. Find good assumptions on gapped Hamiltonians H_+, H_- for LPPL to hold.

Hastings' quasi-adiabatic continuation

Assumption

H_+ deforms to H_- **without closing the gap** *through local perturbations: there exists a C^1 family $H(s)$, $s \in [0,1]$, gapped for all $s \in [0,1]$, with*

$$H(0) = H_-, \qquad H(1) = H_+, \qquad \mathrm{supp}\ \partial_s H(s) \text{ controlled.}$$

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The ground state $\psi(s)$ exists and is continuous in s. Write

$$\psi(s) = U(s)\,\psi(0),$$

for some unitary family $U(s)$. **Lots of flexibility in $U(s)$!**

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[Hastings '04] One can pick $U(s)$ **locally generated**:

$$i\,\partial_s U(s) = G(s)\,U(s), \qquad G(s) = \int_{\mathbb{R}} W(t)\, e^{itH(s)}\, \partial_s H(s)\, e^{-itH(s)}\, dt,$$

$$\widehat{W}(\omega) = \frac{1}{i\omega} \text{ for } |\omega| \geq \delta \;\; \textit{(uniform gap)}, \qquad \underline{\widehat{W}\ \textbf{smooth}}$$

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$$\boxed{\text{Fast decay of } W + \text{Lieb–Robinson} \Rightarrow \operatorname{supp} G(s) \text{ is near } \operatorname{supp} \partial_s H(s).}$$

Ground state: $\psi(s) = U(s)\,\psi(0)$, so $\partial_s\psi(s) = -i\,G(s)\,\psi(s)$.

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So $\langle \psi_+, A\psi_+\rangle \simeq \langle \psi_-, A\psi_-\rangle.$ **(LPPL)**

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$$\boxed{\text{So} \quad \langle \psi_+, A\psi_+ \rangle \simeq \langle \psi_-, A\psi_- \rangle. \quad \textbf{(LPPL)}}$$

So LPPL holds for ground states in the same topological phase [Hastings 04', Bachmann–Michalakis–Nachtergaele–Sims '11]

Unsatisfying condition to study interface effects in topological insulators.

What if H_+, H_- are in disjoint phases?

H_+, H_- gapped with ground states ψ_+, ψ_- and $\mathrm{supp}(H_+ - H_-)$ bounded.

Assumption

The ground states ψ_+, ψ_- **have non-small overlap**:

$$|\langle \psi_+, \psi_- \rangle| \ \geq \ e^{-\nu N}, \qquad \nu = 10^{-5}\,\delta \quad (\delta : \textit{ joint gap of } H_\pm).$$

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Theorem [Drouot '26]

When dist $\left(\mathrm{supp} A,\ \mathrm{supp}(H_+ - H_-)\right) \geq N/2,$

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- If H_+, H_- **are** in the same phase: $\psi_+ = \psi(0)$ and $\psi(s_1)$ have overlap $\geq 1/2$ for s_1 small; same for $\psi(s_1)$ and $\psi(s_2)$; and so on. Applying the theorem numerous times:
$$\langle \psi_+, A\psi_+ \rangle \simeq \langle \psi(s_1), A\psi(s_1) \rangle \simeq \cdots \simeq \langle \psi_-, A\psi_- \rangle.$$

So we **recover the previous LPPL result!**

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- **Hope:** this LPPL principle would be a more versatile tool to study conduction between distinct topological phases.

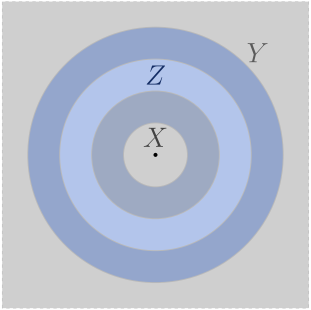
Idea of proof

Lemma [Hastings '07]

Let $X = B_0(N/2)$, $Y = \Lambda \setminus B_0(2N/3)$, $Z = B_0(3N/4) \setminus B_0(N/4)$. For any gapped ground state ψ:

$$|\psi\rangle\langle\psi| = B_Z\, B_X\, B_Y + O(e^{-3\nu N}),$$

B_X near X, B_Y away from S, B_Z near Z.



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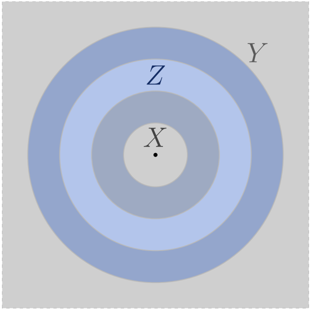

Apply to H_+, H_-: since $H_+ = H_-$ away from $\mathrm{supp}(H_+ - H_-) \subset X$, the far pieces coincide: $B_Y^+ = B_Y^-$, $B_Z^+ = B_Z^-$; only $B_X^\pm$ differ. As $\mathrm{supp}\, A$ is far from X, **decay of correlations** between A and B_X gives (after some algebra)

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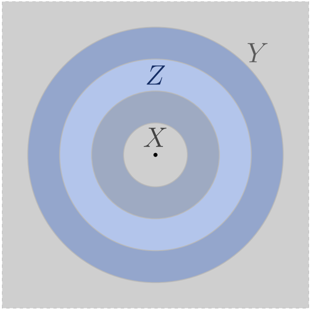

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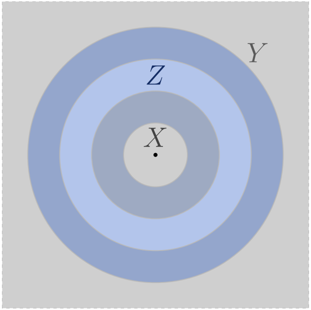

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$$|\langle\psi_+, \psi_-\rangle|^2 \left(\langle\psi_+, A\psi_+\rangle - \langle\psi_-, A\psi_-\rangle\right) = O\left(e^{-3\nu N}\right),$$

and dividing by $|\langle\psi_+, \psi_-\rangle|^2 \geq e^{-2\nu N}$:

$$\langle\psi_+, A\psi_+\rangle - \langle\psi_-, A\psi_-\rangle = O\left(e^{-\nu N}\right). \qquad \square$$

Conclusion

- We studied **dynamics** in quantum spin systems.
- We connected it to properties of **ground states**.

Thank you!