

Numerical Convergence of DEC Approximations to Hodge-Laplace Problems on Nice and Less Nice Meshes

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Joint work with
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The Hodge-Laplace Problem(s)

For a contractible domain Ω , given a k -form f , find a k -form u such that

$$\begin{cases} (d\delta + \delta d)u = f \\ \text{tr}_{\partial\Omega} \star u = 0 \\ \text{tr}_{\partial\Omega} \star du = 0 \end{cases}$$

DEC discretization:

$$(d\delta^{\text{DEC}} + \delta^{\text{DEC}}d)u = \mathcal{R}f$$

where $\delta^{\text{DEC}} = \star_h^{-1}d\star_h$.

As a matrix equation,

$$\left(\begin{bmatrix} d \end{bmatrix} \begin{bmatrix} \star_h^{-1} \end{bmatrix} \begin{bmatrix} d \end{bmatrix} \begin{bmatrix} \star_h \end{bmatrix} + \begin{bmatrix} \star_h^{-1} \end{bmatrix} \begin{bmatrix} d \end{bmatrix} \begin{bmatrix} \star_h \end{bmatrix} \begin{bmatrix} d \end{bmatrix} \right) \vec{u} = \vec{f}$$

The discrete Hodge star

- Using a circumcentric dual mesh, we get that $\star_h$ is diagonal and

$$(\star_h)_{ii} = \frac{|\star \sigma_i|}{|\sigma_i|}.$$

for a k -simplex σ_i in the mesh $\mathcal{T}_h$.

- If the mesh $\mathcal{T}_h$ is well centered, then $\star_h$ and $\star_h^{-1}$ are positive matrices.
- But even if $\mathcal{T}_h$ is well-centered, it can have simplices with degenerating duals!

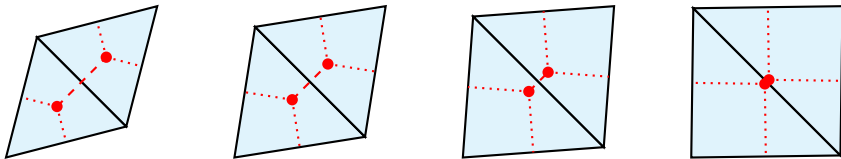


Figure: Example of degenerating dual edge

What About Less Nice Meshes?

Can we modify the DEC scheme to allow for convergence on meshes with nearly degenerate or completely degenerate duals?

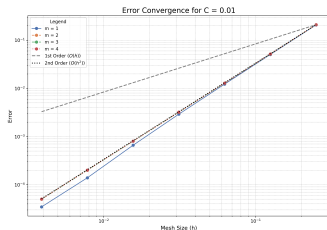
Possible solution: Introduce a modified discrete Hodge star $\star_{h,\gamma}$:

$$(\star_{h,\gamma})_{ii} = \frac{\max(|\star \sigma_i|, \gamma_h)}{|\sigma_i|}.$$

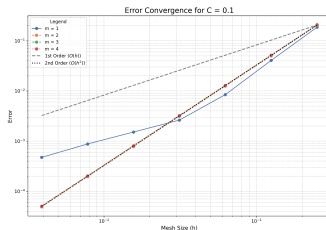
for an appropriately chosen parameter γ_h .

Example 1: Unit Square Mesh (Numerical Experiments)

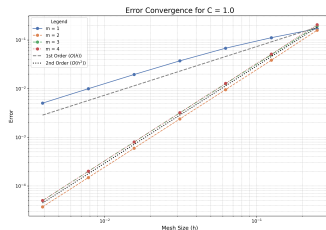
$k = 1$ **problem:** ($\gamma_h = Ch^m$). Manufactured solution: $\mathbf{u} = [\sin(2\pi x), \sin(2\pi y)]$.



(a) $C = 0.01$



(b) $C = 0.1$

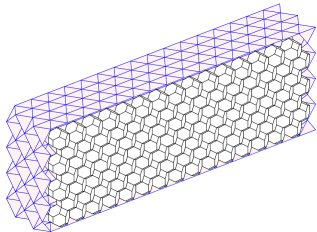


(c) $C = 1.0$

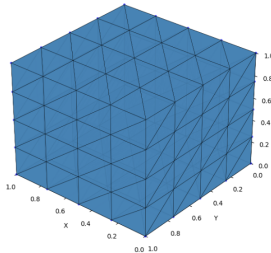
Figure: u error

Example 2: 3D meshes

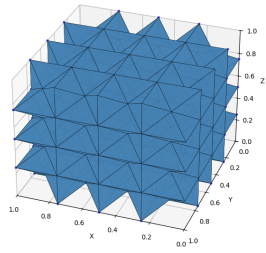
- It is difficult to construct 3D well-centered meshes in a structured way for convergence analysis.
- One method: Use the Delaunay tetrahedrization of the vertices of the body centered cubic (BCC) lattice.



(a) BCC Tiling



(b) Truncation to unit cube



(c) Well-centered interior of truncated mesh

Poincaré Operators in Discrete Exterior Calculus

A Flash Talk

Anil N. Hirani Bingyan Liu

University of Illinois at Urbana-Champaign

IMSI Workshop *Discrete Exterior Calculus*, Chicago, 2025

From Smooth to Discrete

Smooth Setting

- For contractible manifold M with homotopy $\Phi : M \times I \rightarrow M$
- $\Phi(x, 1) = x$ and $\Phi(x, 0) = c$ (fixed point)
- Family $\mathfrak{p}_k : \Lambda^k(M) \rightarrow \Lambda^{k-1}(M)$ with:
 - $\mathfrak{p}_{k-1} \circ \mathfrak{p}_k = 0$ (chain property)
 - $d\mathfrak{p}_k + \mathfrak{p}_{k+1}d = \text{id}$ (homotopy property)
- **Explicit formula:**
$$\mathfrak{p}_k \omega = \int_0^1 i_{\frac{\partial}{\partial t}} \Phi^* \omega$$

Example: $\mathbb{R}^3$ with $\omega = zdx \wedge dy$

$$\mathfrak{p}_2 \omega = \frac{1}{3} z(xdy - ydx)$$

Discrete Setting

- Need discrete analogues of:
 - Homotopy $\Phi : M \times I \rightarrow M$
 - Pullback Φ^*
 - Contraction $i_{\frac{\partial}{\partial t}}$
 - Integration $\int_0^1$

Goal: Explicit discrete Poincaré operators for collapsible domains

Triangulating $X \times I$

- Given simplicial complex X with vertex ordering $v_0 < v_1 < \dots$
- Product order on $X^{(0)} \times I^{(0)}$: $(v_i, s) \leq (v_j, t)$ iff $v_i \leq v_j$ and $s \leq t$
- Create simplices from totally ordered subsets

Triangle Example: $X = [v_0 v_1 v_2] \times I$ gives 3 tetrahedra:

$$[(v_0, 0)(v_1, 0)(v_2, 0)(v_2, 1)] \quad (1)$$

$$[(v_0, 0)(v_1, 0)(v_1, 1)(v_2, 1)] \quad (2)$$

$$[(v_0, 0)(v_0, 1)(v_1, 1)(v_2, 1)] \quad (3)$$

Discrete Homotopy via Abstract Simplicial Maps

Abstract Simplicial Maps

- Induced by vertex maps $f^{(0)} : X^{(0)} \rightarrow Y^{(0)}$
- Preserve simplicial structure
- Enable discrete pullback: $(f^*\alpha)(c) := \alpha(f_\#(c))$

From Simple to Discrete Homotopy

- Start with elementary simplicial collapses
- Apply subdivisions to ensure abstract simplicial maps
- Build discrete homotopy sequence

Result: Systematic construction extending smooth homotopy theory to simplicial complexes

Discrete Poincaré Construction

Contraction via Extrusion

- **Smooth principle:** $\int_S i_X \alpha = \left. \frac{d}{dt} \right|_{t=0} \int_{E_X(S,t)} \alpha$
- Here S is submanifold extruded by vector field X into higher-dimensional manifold $E_X(S, t)$
- **Discrete version:** For d -simplex σ and $(d+1)$ -form ω :

$$\int_{\sigma} i_{\frac{\partial}{\partial t}} \omega = \sum_{\tau \in (\sigma \times I)_{d+1}} \text{sign}(\tau) \cdot \langle \omega, \tau \rangle$$

- Sum over all $(d+1)$ -simplices τ in simplicial complex $\sigma \times I$ with appropriate orientation signs

Algorithm

- 1 Start with collapsible complex X and collapse sequence
- 2 Apply subdivisions to ensure abstract simplicial maps
- 3 Define discrete homotopy $\Phi : X \times I_n \rightarrow X$
- 4 Compute: discrete pullback + contraction + summation

A_∞ -Algebras

- DEC wedge product fails to be associative
- A_∞ -algebras capture non-associativity via higher operations
- Key observation: $\mathfrak{p}$ operators satisfy $d\mathfrak{p} + \mathfrak{p}d = \text{id}$
- Recursive construction of higher operations

Electromagnetic Potentials

- Given closed 2-cochain ω
- Find potential: $\alpha = \mathfrak{p}_2\omega$
- Direct combinatorial computation
- Explicit combinatorial formula

Contribution: First rigorous discrete Poincaré operators for general collapsible domains

Thank You!

Questions?

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IMSI Workshop *Discrete Exterior Calculus*
Chicago, 2025

Submanifold Optimization: A Complete Classification

Lek-Heng Lim, Rongbiao Wang, Ke Ye

University of Chicago, Chinese Academy of Sciences

September 4, 2025

Manifold optimization

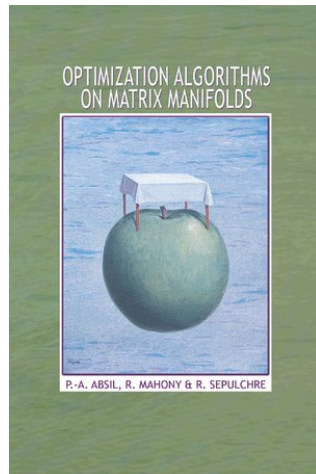
- Optimization over Riemannian manifolds is ubiquitous and efficient:

$$\min_{x \in \mathcal{M}} f(x)$$

- The manifolds of interest are often equipped with symmetries, i.e., of the form G/H
- To implement the optimization algorithms, we need a concrete way to represent them, i.e., an embedding $\varepsilon : G/H \rightarrow \mathbb{R}^{m \times n}$

Question 1

Can we classify all homogeneous spaces G/H that could be equivariantly embedded into matrices of dimension at most n^2 ?



Classification

Goal

To answer the question for

$$G \in \{\mathrm{SL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{R}), \mathrm{SU}(n), \mathrm{SO}_n(\mathbb{C}), \mathrm{SO}_n(\mathbb{R}), \mathrm{SO}(p, q), \mathrm{Sp}_n(\mathbb{C}), \mathrm{Sp}_n(\mathbb{R}), \mathrm{Sp}(n)\}$$

- **Representations:**

- ▶ Ambient space has to be a representation of G
- ▶ Weyl's dimension formula tells us exactly the representations of dimension at most n^2

$$\dim \mathbb{V}_\lambda = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}$$

- **Orbit-stabilizer:** for each G -representation $\mathbb{V}$, the space G/H is embedded as $G \cdot v = G/H$ for some $v \in \mathbb{V}$.
- **Matrix factorization:** Gives us concrete ways to compute stabilizer of $v \in \mathbb{V}$ under different actions.

Classification

- Complete classification of ambient spaces and stabilizers
- Proof explicitly constructs the embeddings as orbits
- Include various familiar spaces from practice

Family	Manifold	As homogeneous space
Grassmannian	real	$SO_n(\mathbb{R})/S(O_k(\mathbb{R}) \times O_{n-k}(\mathbb{R}))$
	complex	$SU(n)/S(U(k) \times U(n-k))$
	quaternionic	$Sp(n)/(Sp(k) \times Sp(n-k))$
	complex locus of real	$SO_n(\mathbb{C})/(SO_k(\mathbb{C}) \times SO_{n-k}(\mathbb{C}))$
	Lagrangian	$SU(n)/SO_n(\mathbb{R})$
	isotropic	$SO_{n+2}(\mathbb{R})/(U(k+1) \times SO_{n-2k}(\mathbb{R}))$
Flag	real	$SO_n(\mathbb{R})/S(O_{n_1, \dots, n_{p+1}}(\mathbb{R}))$
	complex	$SU(n)/S(U(n_1, \dots, n_{p+1}))$
	quaternionic	$Sp(n)/Sp(n_1, \dots, n_{p+1})$
Stiefel	real	$SO_n(\mathbb{R})/SO_{n-r}(\mathbb{R})$
	complex	$SU(n)/SU(n-r)$
	quaternionic	$Sp(n)/Sp(n-r)$

Mimimal dimension model

Question 2

For all these homogeneous spaces, what are the smallest ambient spaces to embed them in?

- Follows from the proof
- The proof gives the embeddings as orbits, but we can also find implicit characterizations via invariants

Examples:

- 1 $SL_n(\mathbb{F})/SL_{n-r}(\mathbb{F}) = \{Y \in \mathbb{F}^{n \times r} : \text{rank}(Y) = r\}$
- 2 $SL_n(\mathbb{R})/S(O_{n_1, \dots, n_s}(\mathbb{R}) \times GL_{n_{s+1}}(\mathbb{R})) = \{Y \in S^2(\mathbb{R}^n) : \text{intertia}(Y) = (p, q), \det(Y) = c\}$
- 3 $SO_n(\mathbb{F})/SO_{n-r}(\mathbb{F}) = \{Y \in \mathbb{F}^{n \times r} : Y^T Y = A\}$

$$\textcircled{4} \quad SU(n)/S(U(n_1) \times \cdots \times U(n_{s+1})) = \left\{ Y \in \mathfrak{su}(n) : \begin{aligned} &\prod_{i=1}^{s+1} (Y - a_i I) = 0, \\ &\text{tr}(Y^m) = \sum_{i=1}^{s+1} n_i a_i^m, \quad m = 1, \dots, s \end{aligned} \right\}$$

Convergence and Stability of Discrete Exterior Calculus for the Hodge-Laplace Problem in 2D

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¹University of Illinois at Urbana-Champaign
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²University of Oslo

³University

September 4, 2025

The Problem

The Challenge

- **Discrete Exterior Calculus (DEC)** is a popular framework for discretizing exterior calculus on simplicial complexes.
- While there is strong numerical evidence for its convergence, a general proof of convergence and stability for the Hodge-Laplace problem has been an open question.

- We prove convergence and stability of DEC for the 2D Hodge-Laplace problem on meshes that are **non-degenerate Delaunay and shape regular**.
- We establish a **Poincaré Inequality** specifically for the DEC framework.
- We prove that DEC harmonic forms approximate FEEC harmonic forms with $O(h)$ accuracy.

Johnny and **Pratyush** also prove the convergence of Hodge-Laplace problem (assuming harmonic part is trivial)

Our Approach: Connecting DEC to FEEC

Our core strategy is to relate DEC solutions to the well-understood **Finite Element Exterior Calculus (FEEC)** solutions.

How are they related?

- DEC and lowest-order FEEC use the same degrees of freedom (cochains) and differential operators.
- The main difference lies in the mass matrices, which define the inner product:
 - **FEEC** uses a standard, non-diagonal mass matrix $(*^F)$ from integrating basis functions.
 - **DEC** uses a modified, **diagonal** mass matrix $(*^D)$, or "Hodge star," computed from the geometry of the primal and dual meshes.

This difference can be analyzed as a "variational crime". The key is to control the error this introduces.

Key Ingredient: Norm Equivalence

To control the error, it is important to show that the norm relationship between DEC and FEEC.

Theorem (Norm Equivalence, Thm. 4.9)

For a family of triangulations that is **shape regular** and **uniformly Delaunay**, there exist constants c_1, c_2 independent of mesh size h such that for any differential form α :

$$c_1 \|\alpha\|_F^2 \leq \|\alpha\|_D^2 \leq c_2 \|\alpha\|_F^2$$

where $\|\cdot\|_F$ is the FEEC norm and $\|\cdot\|_D$ is the DEC norm.

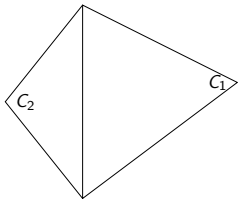


Figure 1: This requires the **uniform Delaunay condition**
($C_1 + C_2 \leq \pi - \delta_\pi$).

The Payoff: Stability and Convergence

Stability and Convergence only requires $\|\alpha\|_D^2 \leq c_2 \|\alpha\|_F^2$. This is ensured by shape regularity. In addition, nondegenerate Delaunay is needed to ensure DEC norm is positive definite.

1 Consistency of Inner Products (Thm. 5.4)

The DEC and FEEC inner products are close, with an error of order h :

$$|\langle \alpha_h, \beta_h \rangle_D - \langle \alpha_h, \beta_h \rangle_F| \leq Ch \|\alpha_h\|_V \|\beta_h\|_V$$

2 Stability (Thm. 6.2)

This consistency allows us to prove a **discrete inf-sup condition** for the mixed formulation of the Hodge-Laplace problem, which guarantees the discrete problem is well-posed.

3 Convergence (Thm. 6.3 & 6.6)

Finally, stability leads to our main convergence theorem, which bounds the error between the DEC solution ($\hat{u}_h$) and the FEEC solution (u_h):

$$\|\hat{u}_h - u_h\|_V + \dots \leq Ch (\|u_h\|_V + \dots)$$

Summary

We proved the convergence and stability of the DEC method for the 2D Hodge-Laplace problem and a DEC Poincaré inequality.

- **How We Did It:**

- By formally connecting DEC to lowest-order FEEC.
- By proving norm equivalence under specific geometric conditions on the mesh (uniformly Delaunay and shape regular).
- By using one side of the equivalence to prove a discrete inf-sup condition and derive error estimates.

- **Impact:**

- This work provides a rigorous theoretical foundation for DEC, which was previously supported primarily by numerical evidence.

Thank you.

Boundary Dual Complex and Combinatorial Orientation in DEC

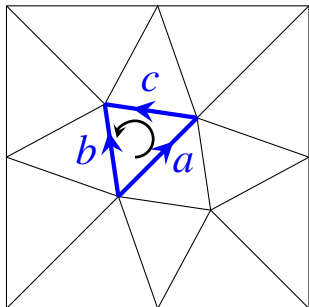
Siqi Jiao, Anil N. Hirani, Mark D. Schubel

Department of Mathematics, University of Illinois at Urbana-Champaign

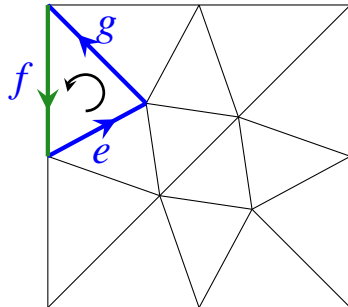
IMSI, Chicago, September 4th, 2025

Primal exterior derivative $d_k : C^k(X) \rightarrow C^{k+1}(X)$

Does not need special treatment near boundary

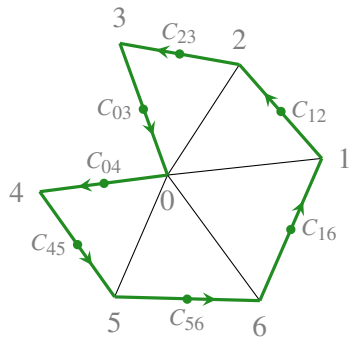
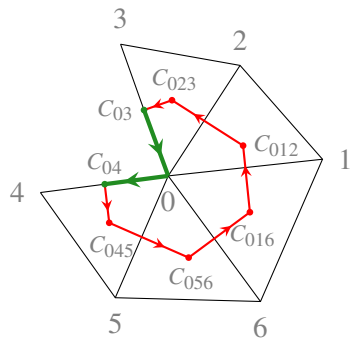


$$a - b + c$$



$$e + f + g$$

Extended dual exterior derivative



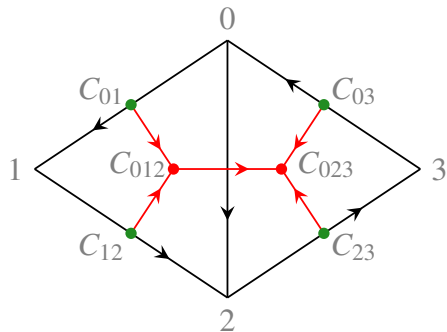
$$D^1(\star X) \oplus D^1(\star \partial X) \rightarrow D^2(\star X)$$

Near boundary, with boundary terms incorporated.

Boundary dual mesh in dimension 2.

Dual exterior derivative d_0^{dual}

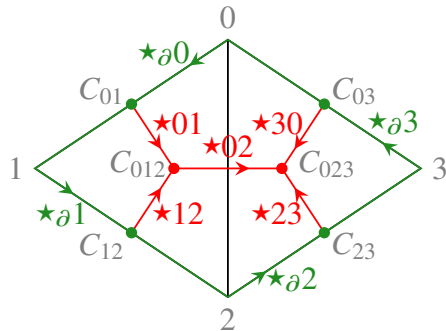
Boundary terms incorporated



	$\star 012$	$\star 023$	$\star_{\partial 01}$	$\star_{\partial 30}$	$\star_{\partial 12}$	$\star_{\partial 23}$
$\star 01$	1	0	-1	0	0	0
$\star 02$	-1	1	0	0	0	0
$\star 30$	0	1	0	-1	0	0
$\star 12$	1	0	0	0	-1	0
$\star 23$	0	1	0	0	0	-1

Dual exterior derivative d_1^{dual}

Boundary terms incorporated

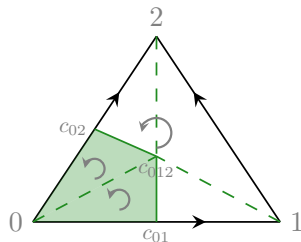


	$\star 01$	$\star 02$	$\star 30$	$\star 12$	$\star 23$	$\star \partial 0$	$\star \partial 1$	$\star \partial 2$	$\star \partial 3$
$\star 0$	1	1	-1	0	0	1	0	0	0
$\star 1$	-1	0	0	1	0	0	1	0	0
$\star 2$	0	-1	0	-1	1	0	0	1	0
$\star 3$	0	0	1	0	-1	0	0	0	1

General formula: $d_{n-k}^{\text{dual}} := [(-1)^k d_{k-1}^T, (-1)^{k-1} i_{k-1}^\partial]$

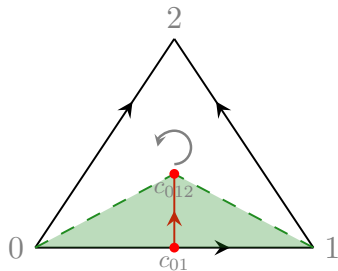
Combinatorial Orientation in DEC

Flip sign rule and algorithm to orient elementary duals

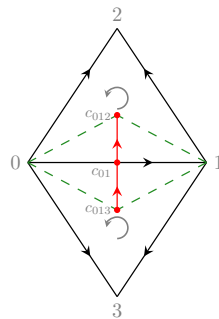


- ▶ $[0, c_{01}, c_{012}] \Leftrightarrow [\{0\}, \{0, 1\}, \{0, 1, 2\}]$
- ▶ $sgn([0, c_{02}, c_{012}]) = sgn([0, 2, 1]) = -sgn([0, 1, 2])$
- ▶ Algorithm to orient elementary duals:
$$sgn([c(\sigma^p), \dots, c(\sigma^n)]) =$$
$$sgn([c(\sigma^0), \dots, c(\sigma^p)], \sigma^p) \times sgn([c(\sigma^0), \dots, c(\sigma^n)], \sigma^n)$$

Consistency within a Single Simplex and across Adjacent Simplices



Our combinatorial method ensures that within a single n -simplex, the orientation of any elementary dual simplex is uniquely determined.



The method's power extends to an entire mesh, ensuring a consistent orientation for duals across different primal simplices.

Discrete exterior calculus on general polygonal meshes

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September 2025

Work initiated under the supervision of Luiz Velho, IMPA, Rio de Janeiro

Overview

We shall expose a framework presented in

L. Ptackova and L. Velho. A simple and complete discrete exterior calculus on general polygonal meshes. Computer Aided Geometric Design, 88, 2021.

L. Ptackova. A discrete wedge product on general polygonal meshes. ArXiv, 2025.

Its implementation can be found at

<https://github.com/lenka-ptackova/poly-DEC>

Motivation

Why discrete exterior calculus on general polygonal meshes?

- We are interested in geometry processing and simulations on **curved surface meshes**.
- **Exterior calculus** is a coordinate-free calculus that greatly simplifies analysis and calculations on curved spaces of differential manifolds.
- In geometric design and engineering there is a **prevalence of non-triangle surface meshes**.

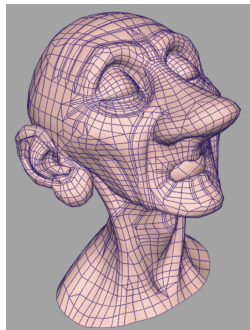


Figure: The control mesh for Geri's head used for recursive Catmull-Clark subdivision. Catmull-Clark subdivision produces meshes consisting only of quadrilaterals. Image taken from [DKT98].

A mesh and the discrete differential forms

2-dimensional orientable
pseudomanifold = **mesh**

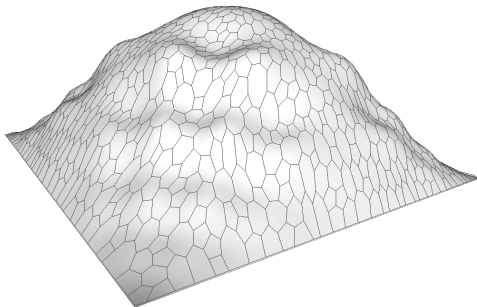


chains



cochains =
discrete differential forms

discretization ↑



discretization ↑

2-dimensional orientable
differential manifold



tangent spaces



differential forms

Our Approach

- We work directly with polygonal meshes, no combinatorial subdivision involved.
- Our construction operates solely on primal meshes, **no dual mesh** is involved.
- Our discretization is intrinsic — we only need to know the lengths of edges and the magnitudes of vector areas of faces.
- Our method is based on three basic operators: exterior derivative (coboundary operator) $\mathbf{d}$, wedge product (cup product) $\wedge$, and Hodge star operator $\star$.
- Using these three operators ($\mathbf{d}$, $\wedge$, $\star$) we derive further discrete differential operators, such as codifferential δ , Laplacian Δ , or Lie derivative.
- We evaluate the adequacy of our approach by numerical tests and by applying our operators on task such as Lie advection or Helmholtz–Hodge decomposition of vector fields.

The polygonal wedge product

- Just like the wedge product of differential forms Ω , our discrete wedge product is a metric-independent **bilinear operation** such that

$$\wedge : \Omega^k \times \Omega^l \rightarrow \Omega^{k+l}.$$

- Our polygonal wedge product satisfies the **Leibniz product rule** and is **skew-commutative**, just like its differential analog. It is actually a **proper cup product** on 2-dimensional pseudomanifolds. For a proof, see [Pta25].

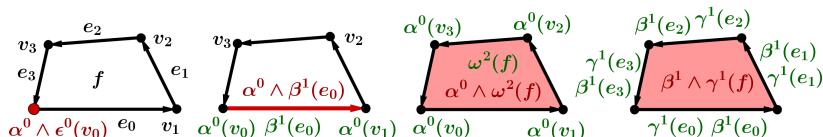


Figure: The wedge product on a quadrilateral: the product of two 0-forms is a 0-form located on vertices (far left). The product of a 0-form with a 1-form is a 1-form located on edges (center left). The product of a 0-form with a 2-form is a 2-form located on faces (center right), and the product of two 1-forms is a 2-form located on faces (far right).

The Hodge star operator

- Discrete Hodge star operator is defined as a linear operator such that

$$\star : \Omega^k \rightarrow \Omega^{2-k}.$$

- Since dual forms are attributed to primal elements, we can compute discrete wedge products of primal and dual forms and define a contraction operator later on.
- On the other hand, there is no isomorphism between the groups of k - and $(2 - k)$ -dimensional cells, in general. Hence our Hodge star **is not an isomorphism**, unlike the Hodge star on Riemannian manifolds.

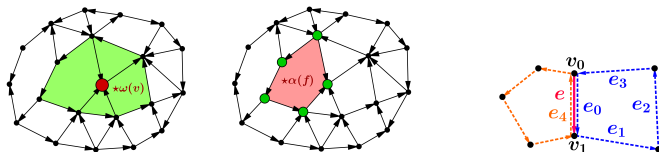


Figure: Left: Hodge dual of a 2-form ω is a 0-form $\star\omega$, whose value on vertex v (colored red) is a linear combination of values of ω on adjacent faces (colored green). Center: Hodge dual of a 0-form α is a 2-form $\star\alpha$, the value of $\star\alpha$ on face f (colored red) is a linear combination of values of α on vertices (green) of that face. Right: the Hodge dual of an 1-form on the edge e (red) is a linear combination of the given 1-form on adjacent edges (orange and blue).

The contraction operator and the Lie derivative

We define our **discrete contraction operator** $\mathbf{i}_X$ on a polygonal mesh M by the following property that holds on Riemannian 2-manifolds ([Hir03, Lemma 8.2.1]):

$$\mathbf{i}_X \alpha = (-1)^{k(2-k)} \star (\star \alpha \wedge X^\flat), \quad \alpha \in \Omega^k, \quad k = 1, 2, \quad (1)$$

where X is a tangent vector field.

We then define **discrete Lie derivative** $L_X : \Omega^k \rightarrow \Omega^k$ using the Cartan's homotopy formula and our discrete contraction operator:

$$L_X \alpha = \mathbf{i}_X d\alpha + d\mathbf{i}_X \alpha, \quad \alpha \in \Omega^k, \quad k = 0, 1, 2, \quad X \in TM. \quad (2)$$

The Leibniz product rule of the above operators with $\mathbf{d}$ is satisfied only if α or β is a closed 0-form, unfortunately.

The Codifferential and Laplacian

Just like on Riemannian n -manifolds M , the Hodge star operator is employed to define **codifferential operator** $\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ by

$$\delta(\alpha^k) = (-1)^{n(k-1)+1} \star d \star \alpha.$$

Then using the codifferential operator, the **Laplace-de Rham operator** is given as

$$\Delta := \delta d + d\delta.$$

Applications: Implicit mean curvature flow

If f is a vector-valued 0-form representing the coordinates of points on a smooth surface M in $\mathbb{R}^3$ and $\vec{H}$ is the mean curvature vector, then

$$\Delta f(x) = \vec{H}(f(x)), \quad x \in M \setminus \partial M.$$

Let V_0 be vertex positions of an initial mesh, we employ backward Euler method, as in [DMSB99], and solve the following linear system to find the new vertex positions with decreased mean curvature:

$$(I + dt\Delta)V_{k+1} = V_k, \quad k \geq 0, \quad dt > 0.$$

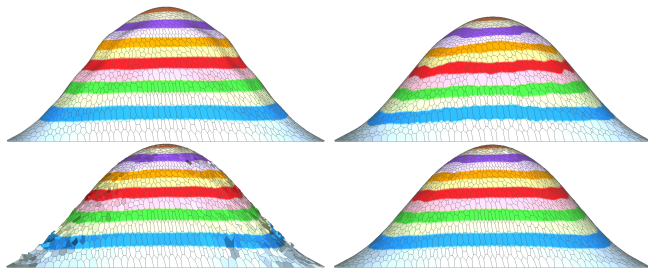


Figure: An original polygonal mesh (top left) after one iteration of curvature flow with $dt = 0.1$ using the Laplacians of [Fuj95] (top right), [AW11] (bottom left), and [PV21] (bottom right).

Applications: Implicit mean curvature flow

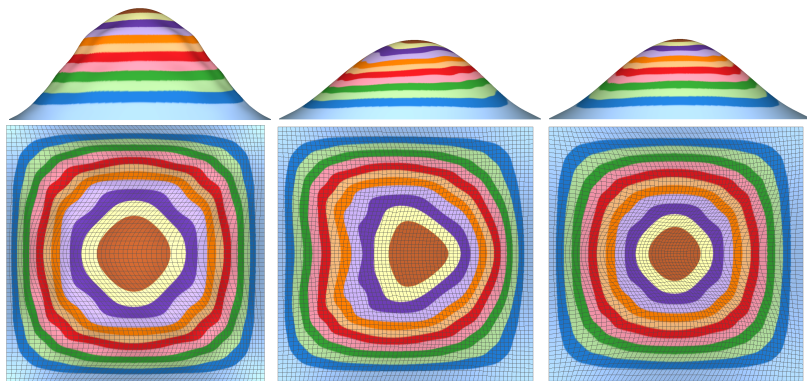


Figure: Mesh smoothing through mean curvature flow of a mesh with a texture reflecting its heightmap. In the left column is the original mesh. In the center column mesh smoothed by employing the Laplacian of [Fuj95], corresponding to the standard second order five point finite difference discretization, and in the right column we employed our discrete Laplacian after 1 iteration with time step $dt = 0.05$. We can observe a significant lateral shifting of the texture when we employ the Laplacian of [Fuj95]; the shifting is caused by the non-zero tangential component of the mean curvature vector.

Applications: Discrete Helmholtz–Hodge Decomposition

The Hodge theorem states that a differential k -form ω on an oriented compact Riemannian manifold without boundary can be uniquely decomposed into three parts

$$\omega = d\alpha + \delta\beta + \gamma \quad (3)$$

for some $(k-1)$ -form α , $(k+1)$ -form β , and a harmonic k -form γ ($\Delta\gamma = 0$).

If we decompose a vector field as a differential 1-form ω^1 , then

1. $d\alpha$ corresponds to a curl-free component of the vector field,
2. $\delta\beta$ corresponds to a divergence-free (incompressible) component,
3. γ corresponds to a harmonic component.

This is the **three component form** of the HHD. As mentioned in [Bhatia et al. 2013], some applications employ the **two component form** of the HHD, where the harmonic component is “included” either into the curl-free or the divergence-free part.

Applications: Discrete Helmholtz–Hodge Decomposition

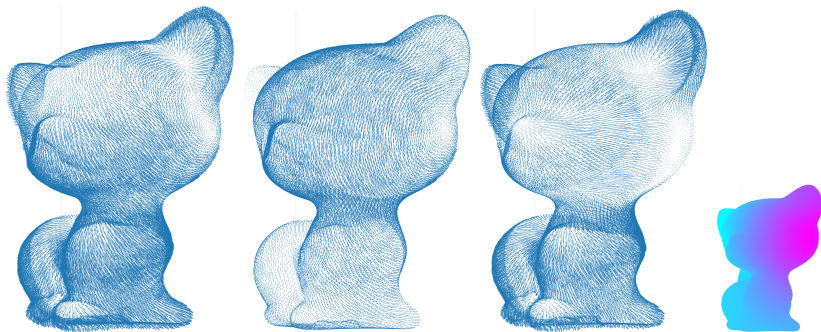


Figure: Helmholtz–Hodge decomposition on a hexagonal mesh. From left to right: original vector field X , its incompressible vortical part, its curl-free part, and corresponding vector potential β in pseudo-colors. The color magenta correctly reflects the distance to the center of CCW rotation of the vector field.

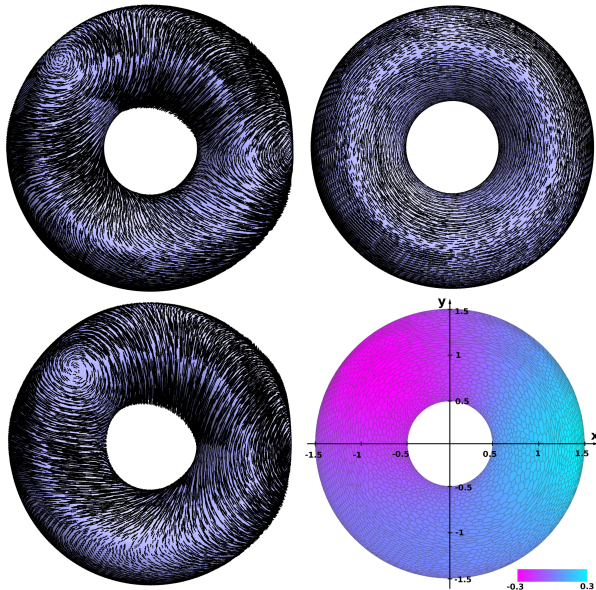


Figure: HHD of an incompressible vector field X (top left) on a torus. The original vector field is $X = X_H + X_R$, where $X_H = (-y, x, 0)$ is a harmonic field, and X_R is a rotational vector field. Our discrete decomposition gives approximate expected results. The calculated decomposition consists of harmonic part $\gamma^\sharp$ (top right) and rotational part $(\delta\beta)^\sharp$ (bottom left).

Applications: Lie Advection

We can model advection of a conserved q -form β in a flow generated by vector field X by solving the Lie advection equation:

$$\frac{\partial \beta}{\partial t} + L_X \beta = 0.$$

Thus to advect a discrete q -form β by the flow of a vector field X , we can iterate over discrete solutions using a simple forward Euler method:

$$\beta_{k+1} = \beta_k - dt L_X \beta_k, \quad k = 0, \dots, \quad (4)$$

where dt is the time step, k is the number of iterations.

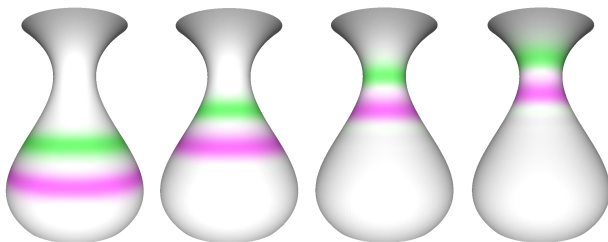


Figure: Lie advection of a color function (a (R,G,B)-valued 0-form) on a mesh of a vase.

Applications: Lie Advection

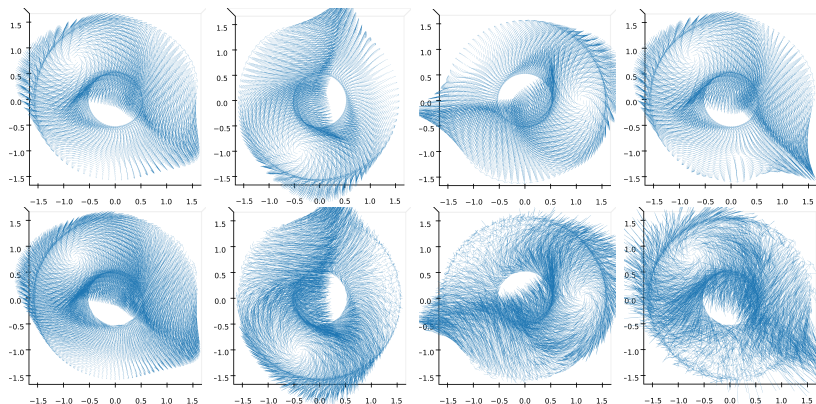


Figure: Lie advection of a vortical vector field Y along the flow of $X = (-y, y, 0)^T$, which is a Killing vector field with periodic orbit. We show the advection with $dt = 0.01$ after 0, 200, 400, 635 iterations on a regular quad mesh with 5k vertices (top row) and jittered quad mesh with 8k vertices (bottom row). We observe that on regular meshes the advection is almost flawless, whereas on irregular meshes some artifacts develop and not even mesh refinement nor decreasing the time steps solves the issue completely.

Numerical behavior

- Experiments show at least linear convergence of our wedge product and Hodge star operator both in L^2 and L^∞ norms, but they are exact on constant differential forms on planar surfaces.
- Experimental convergence of Lie derivative was observed on regular meshes for 0-, 1-, and 2-forms (area potentials). On irregular meshes the errors stay rather constant for 1- and 2-forms, and decrease at least linearly on 0-forms.
- Our codifferential and Laplace-de Rham operator exhibit convergent behavior on regular grids. On irregular meshes the error of approximation stays rather constant after an initial decrease. However, our Laplacian is **linearly precise**, i.e., it is zero on linear forms in the plane (unlike the standard second order five point finite difference discretization).

Ongoing and Future Work

- **Various boundary conditions** — apply domain decomposition methods for mean curvature flow of surface polygonal meshes and perform other geometry processing tasks on meshes with boundary.
- **2D fluid flow simulation** — apply our framework for fluid flow simulation, see also [WIL11, MHS16].
- **DEC on 3-dimensional pseudomanifolds** — examine the possibility of extending our framework to 3-dimensional to volumetric meshes made of tetrahedrons or 3-dimensional (topological) cubes. See [Arn12] for treatment of cubical complexes.

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