

Discrete Exterior Calculus

Overview and Perspective

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Welcome!

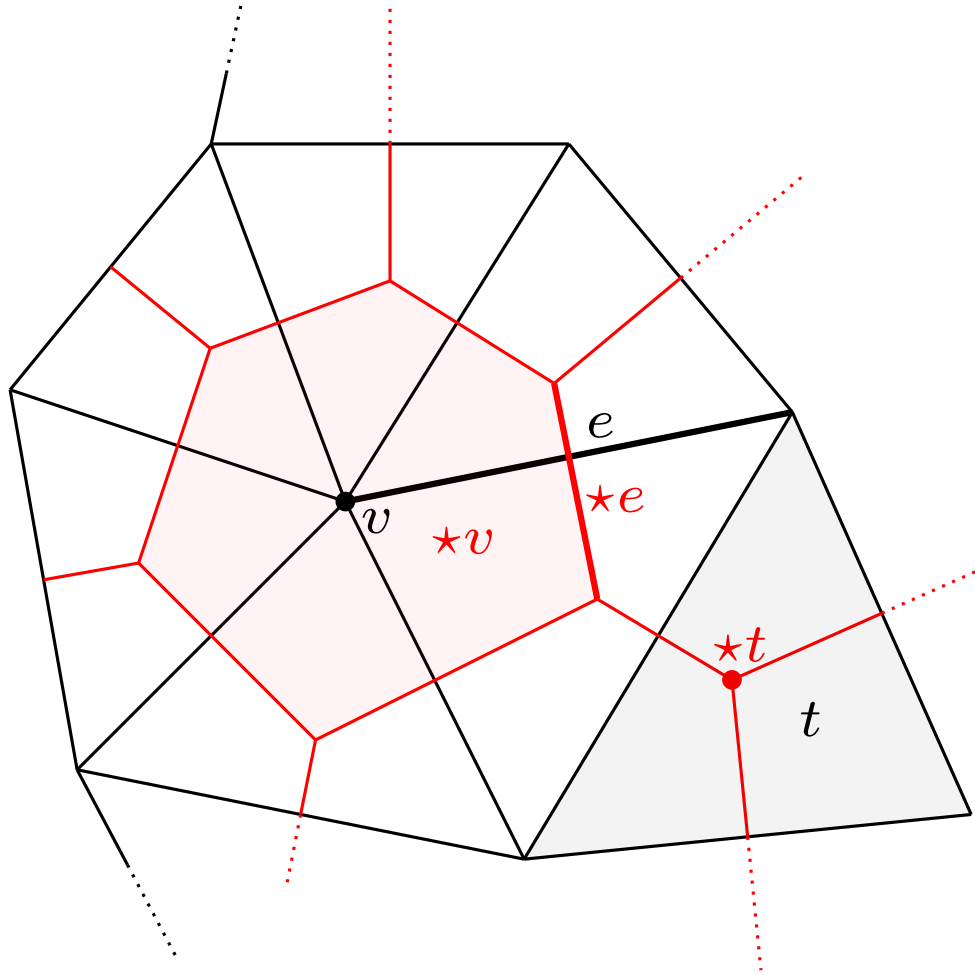
Setting the stage for next 3 days

- Early history of DEC
- Recent years
- New developments
- Near term future
- Longer term future



Separation of
derivative
and metric

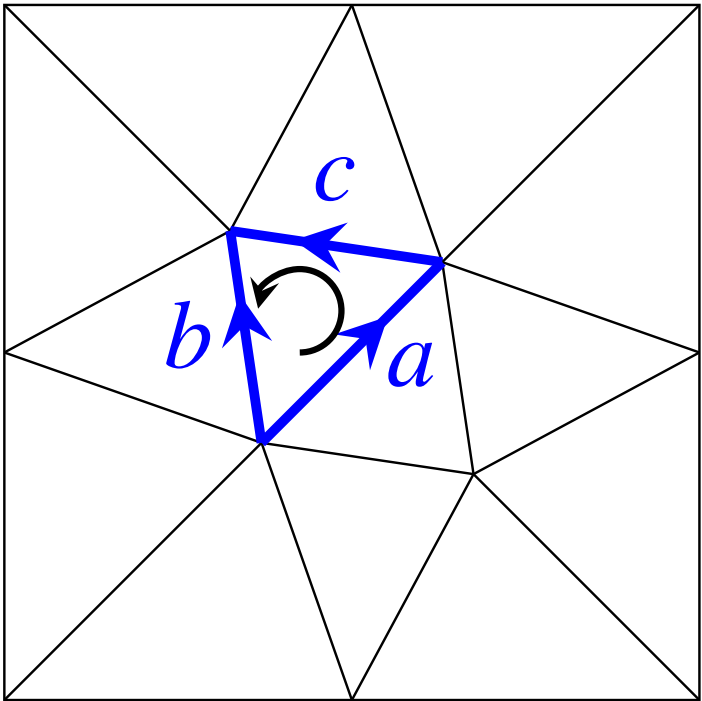
$$\begin{array}{ccccc}
 C^0(X) & \xrightarrow{d_0} & C^1(X) & \xrightarrow{d_1} & C^2(X) \\
 \uparrow \scriptstyle *^{-1}_0 \downarrow \scriptstyle *_0 & & \uparrow \scriptstyle *^{-1}_1 \downarrow \scriptstyle *_1 & & \uparrow \scriptstyle *^{-1}_2 \downarrow \scriptstyle *_2 \\
 D^2(\star X) & \xleftarrow{d^{\text{dual}}_1} & D^1(\star X) & \xleftarrow{d^{\text{dual}}_0} & D^0(\star X)
 \end{array}$$



Primal and
dual mesh

DEC

Exterior derivative
as coboundary



a-b+c

Discrete Hodge star via
circumcentric duality

$$\frac{1}{|\star\sigma|} \int_{\star\sigma} \star\alpha = \frac{1}{|\star\sigma|} \langle \star\alpha, \star\sigma \rangle = \frac{1}{|\sigma|} \langle \alpha, \sigma \rangle = \frac{1}{|\sigma|} \int_{\sigma} \alpha$$

Wedge product as
anti-symmetrized
cup product



Early history of DEC

A personal view

2002 Computation

$$\Delta u = \star d \star u$$

cotan formula
computation with
Mathieu Desbrun

2003 Thesis

Discrete
Exterior
Calculus



Advisor: Jerry Marsden

2003 Workshop

Discrete Geometry
and Mechanics

Bossavit
Forman
Harrison
Nicolaidis

...
Caltech groups



Meanwhile ...

Finite element exterior calculus (FEEC)

Acta Numerica (2006), pp. 1–155
doi: 10.1017/S0962492906210018

© Cambridge University Press, 2006
Printed in the United Kingdom

Finite element exterior calculus,
homological techniques, and applications



Doug Arnold



Rick Falk



Ragnar Winther

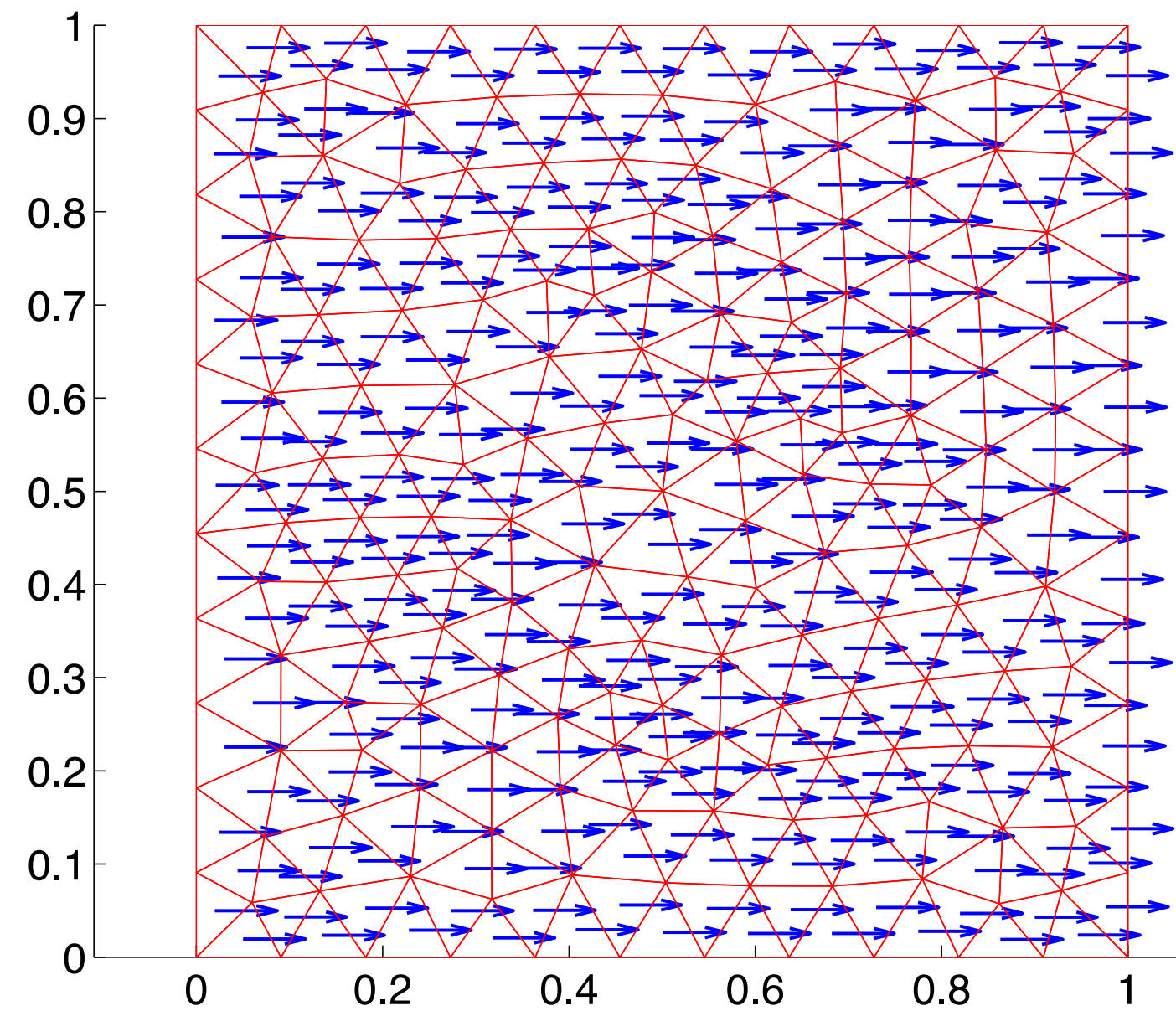
Recent years

Applications - Theory feedback loop

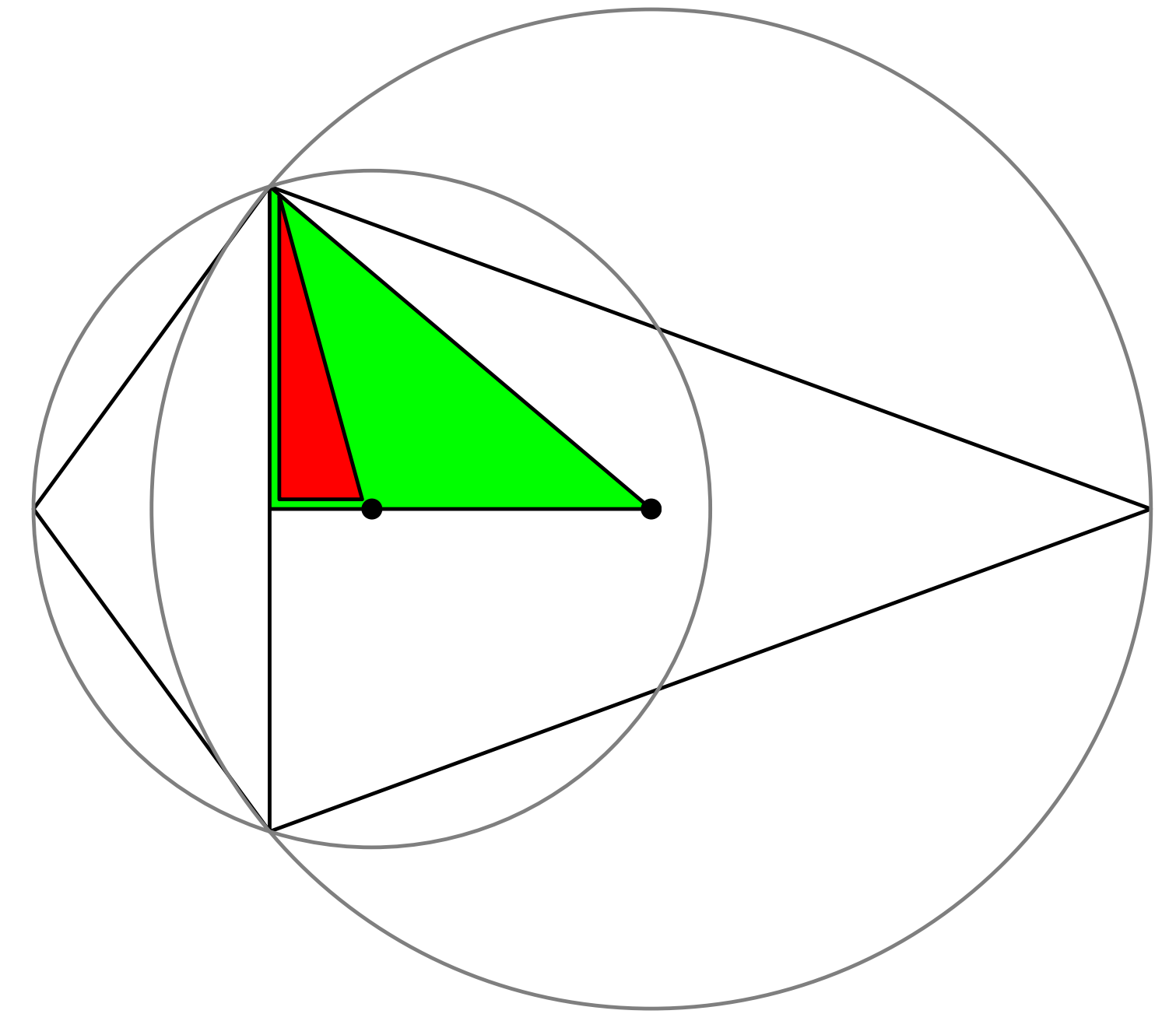
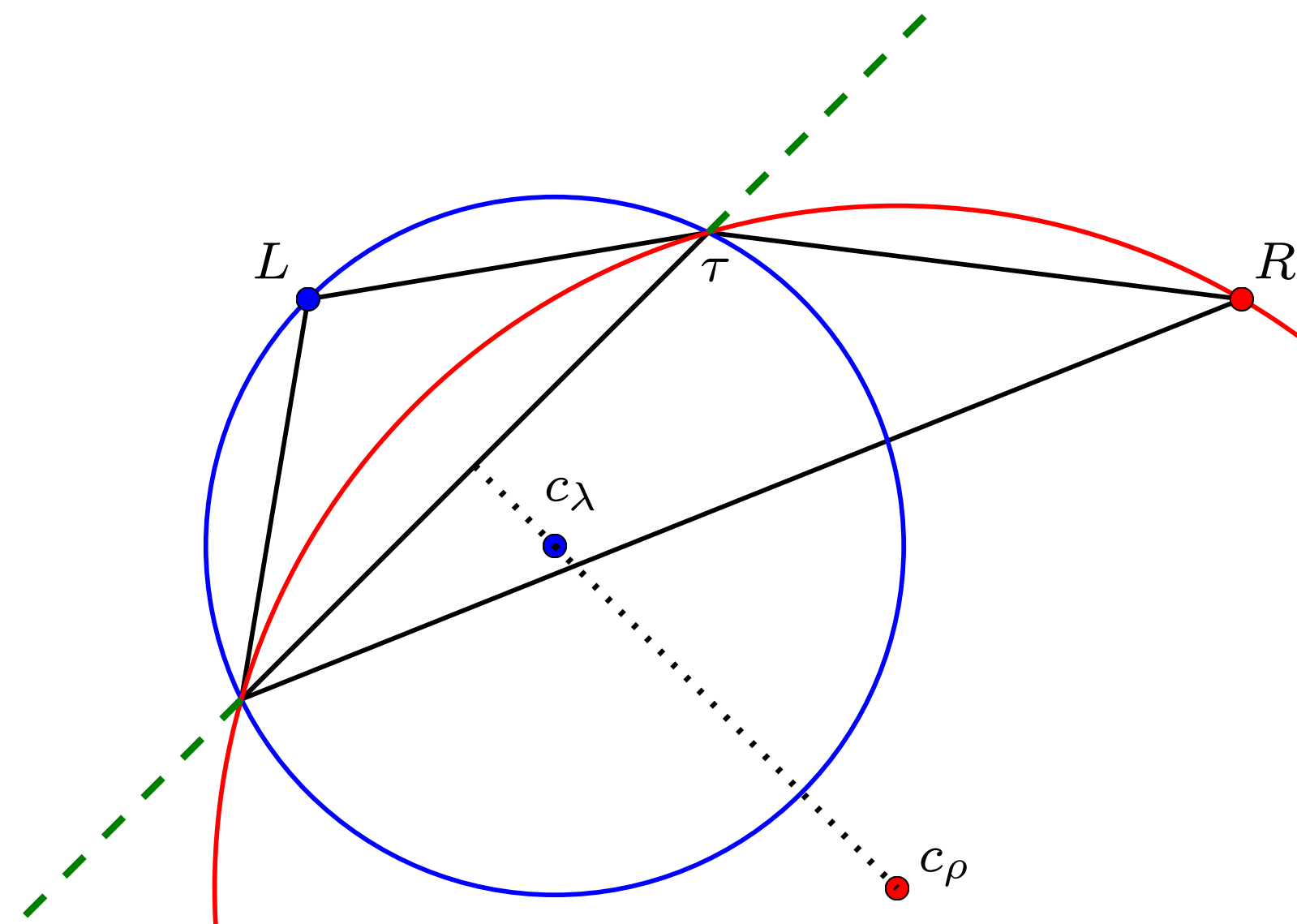
- Darcy flow + Delaunay Hodge star
- Navier-Stokes + Discrete wedge product
- ...
- Spacetime formulation + Minkowski dual (Jyväskylä)



Darcy flow + Delaunay Hodge star



H., Nakshatrala, Chaudhry, *Numerical method for Darcy flow derived using Discrete Exterior Calculus*.
Int. J. Comp. Meth. Eng. Sci. Mech. (2015)

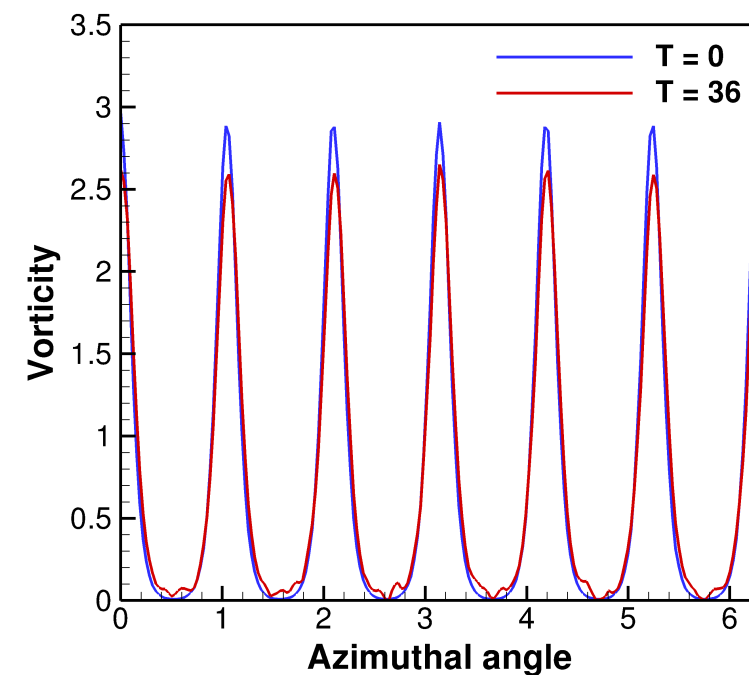
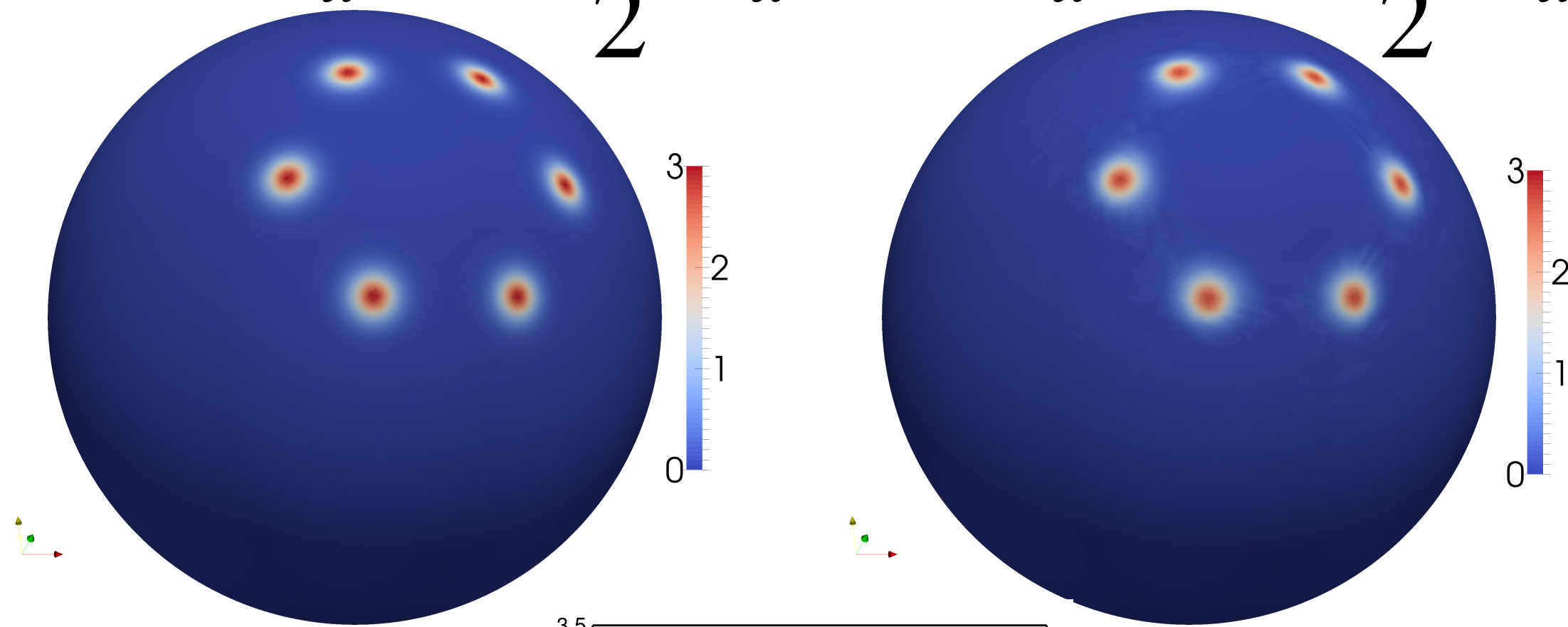


H., Kalyanaraman, VandeZee, *Delaunay Hodge Star*,
Computer-Aided Design (2013)



Navier-Stokes + Discrete wedge product

$$\mathcal{L}_u u^b - \frac{1}{2} di_u u^b = i_u du^b + \frac{1}{2} di_u u^b$$



$$i_X \alpha = \pm \star (\star \alpha \wedge X^b)$$

H., *Discrete Exterior calculus*, (2003)

- Averaging property
- Wilson cochain product

Schubel, H., Berwick-Evans, *Averaging property of wedge product and naturality in discrete exterior calculus*, Adv. Comp. Math. (2024)

Mohamed, H., Samtaney, *Discrete exterior calculus discretization of incompressible Navier-Stokes equations on simplicial meshes*, J. Comp. Phys. (2016)



Recent developments

Software, Convergence, ML, Double forms, Vector bundles

1. Software

- Decapodes (Fairbanks)
- Sandia software (Eldred)
- Jyväskylä (Mönkölä, Kettunen, ...)

2. Hodge-Laplace convergence

- Guzmán and Potu
- Zhu, Christiansen, Hu, H.

3. ML and exterior calculus

- Trask

4. Double forms

- Arnold, Hu, Čap, Kupferman, Leder, ...
- Gawlik, Berchenko-Kogan

7. Bundle valued DEC

- Eldred et al.
- Berwick-Evans, Schubel, H.
- Braune, Tong, Gay-Balmaz, Desbrun
- Christiansen, Hu

8. Bundles in continuum mechanics

- Stramigioli, Brugnoli, ...



Near term future

- Discretization of bundle-valued forms
- Relationship with geometric mechanics
- Double forms and general tensors in the cochain world



Longer term future ...

Quantum systems

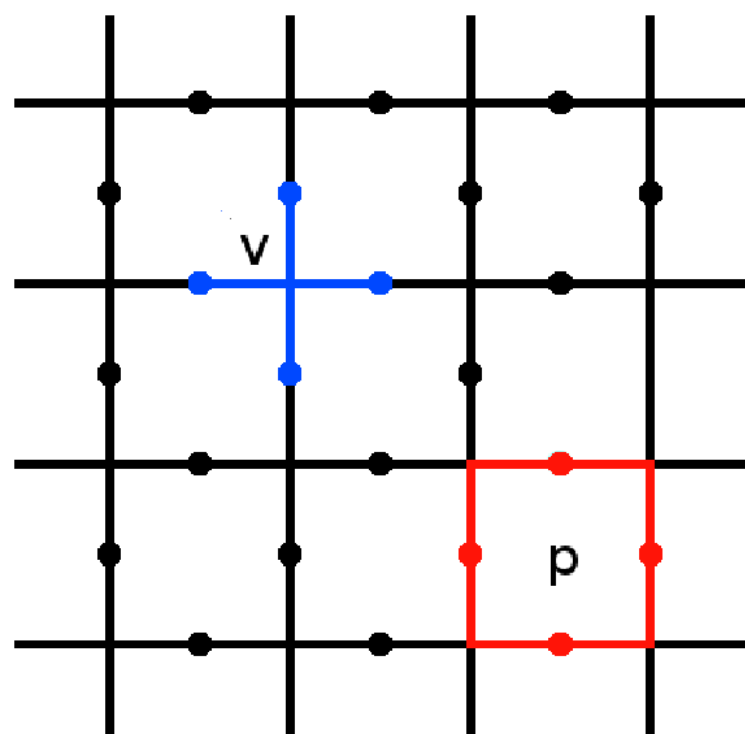
- Many quantum systems are approachable from discrete vector bundle view
- Some are inherently discrete
- Some are discretizations of continuum



Longer term future ...

Quantum error correction

Kitaev toric code



Woottonjames (Wikipedia)

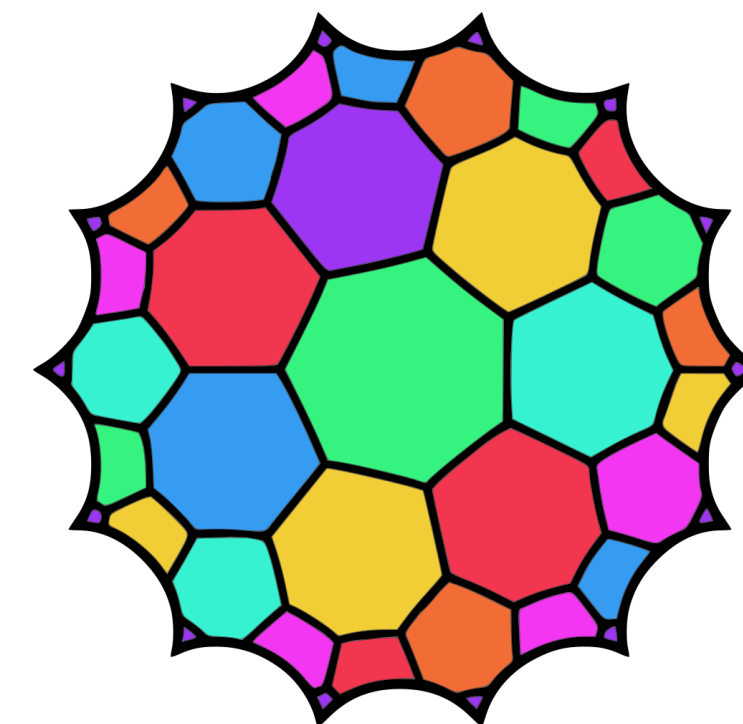
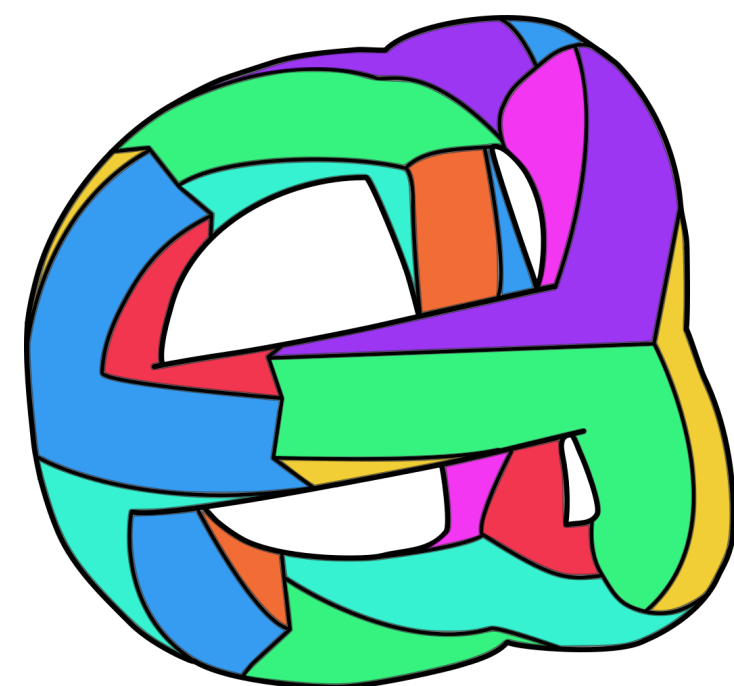
$\mathbb{Z}_2$ valued bundle

Gauge invariance

$$H = -J_e \sum_{\text{vertices}} A_s - J_m \sum_{\text{plaquettes}} B_p,$$

$$A_s = \prod_{\text{star}(s)} \sigma_j^x, \quad B_p = \prod_{\text{boundary}(p)} \sigma_j^z.$$

Extendability

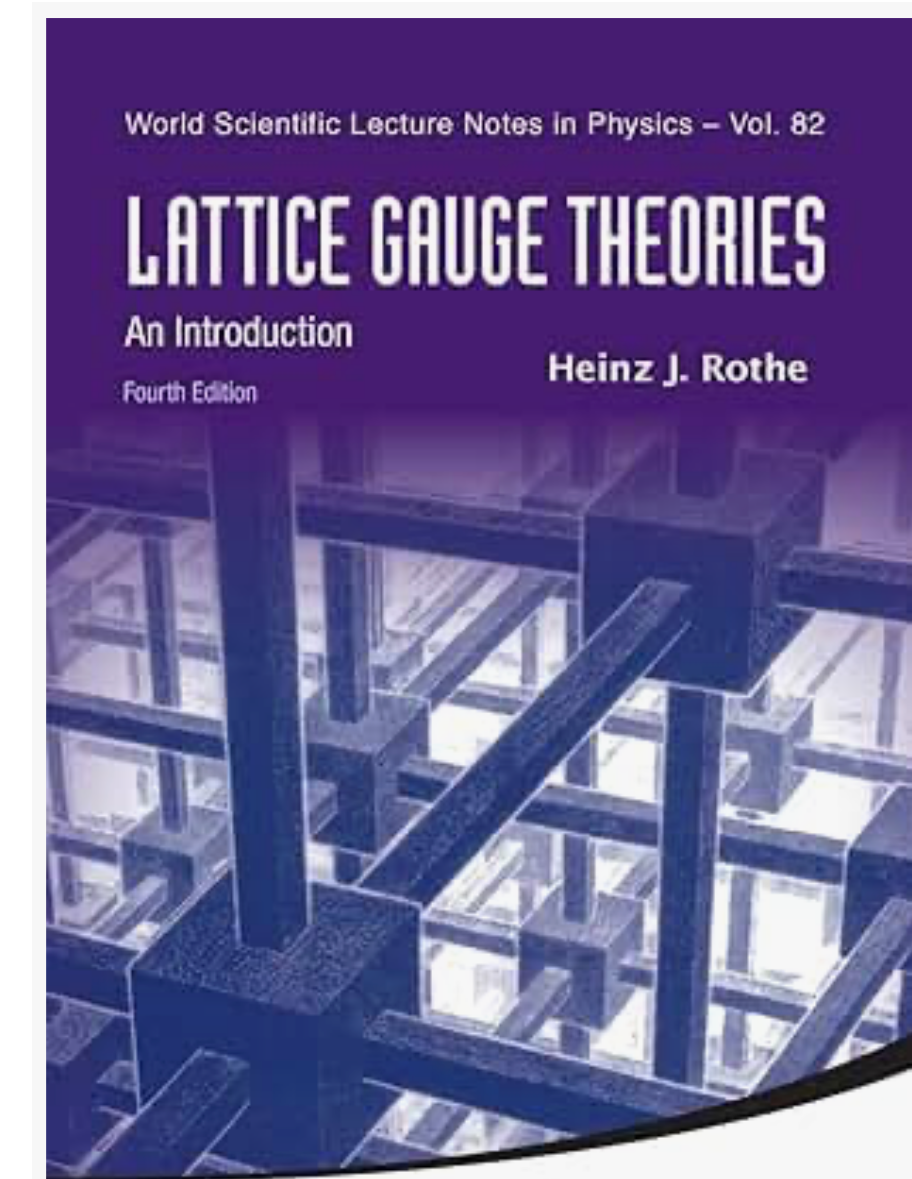


Breuckmann, Eberhardt, *Quantum LDPC codes* (2021)

Longer term future ...

Quantum field theory

- Yang-Mills action $\int F \wedge \star F$. Find connection ∇ whose curvature F makes action stationary
- Quantum observable:
 - Fix a loop
 - Integral over all connections mod gauge
 - Integrand holonomy with $\exp\left(-\int F \wedge \star F\right)$ weighting
 - Classical solutions have large weight
- Lattice gauge theory is well established
- How about simplicial gauge theory?
- Discrete Hodge star for endomorphism and vector bundle valued forms



Thank you!

Let's get started with the workshop

