

Linearly-implicit energy-momentum preserving scheme for geometrically nonlinear mechanics

A mixed finite element approach

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Summary

An introductory example

The Poisson structure of geometrically nonlinear continuum mechanics

- Abstract framework

- Von Kármán nonlinearity in thin structures

- Finite strain mechanics

Discretization

- Finite element discretization

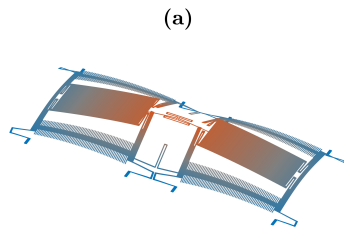
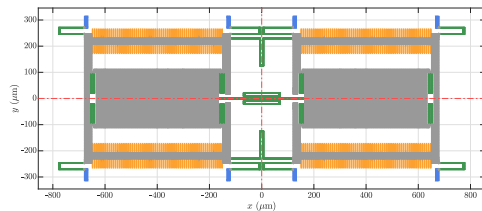
- Time integration

Numerical experiments

Numerical modelling of geometrical nonlinearities

Geometrical nonlinearities: arise from large displacements; occur in aeronautics, wind energy, musical acoustics, MEMS.

Simulation is crucial: animation, sound synthesis, control and optimization.



MEMS device (left) and First mode (right) Schiwietz et al., “Shape optimization of geometrically nonlinear modal coupling coefficients: an application to MEMS gyroscopes”

Summary

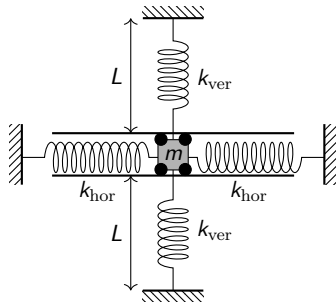
An introductory example

The Poisson structure of geometrically nonlinear continuum mechanics

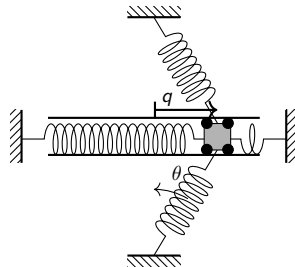
Discretization

Numerical experiments

Duffing oscillator



Undeformed configuration



Deformed configuration

The material behavior of the springs is linear elastic (only geometrical nonlinearities)

$$m\ddot{q} = -2k_{\text{hor}}q - 2k_{\text{ver}}\delta \sin(\theta),$$

- ▶ $\delta = \sqrt{L^2 + q^2} - L$ is the elongation of the vertical springs;
- ▶ $\sin \theta = q / \sqrt{L^2 + q^2}$

Cubic approximation and canonical Hamiltonian form

Consider a Taylor expansion

$$(L^2 + q^2)^{-1/2} = \frac{1}{L} - \frac{q^2}{2L^3} + \mathcal{O}(q^4).$$

Then the Duffing oscillator is obtained

$$\ddot{q} = -\alpha q - \beta q^3, \quad \alpha := 2k_{\text{hor}}/m, \quad \beta := k_{\text{ver}}/(mL^3).$$

Canonical Hamiltonian equations ($p := \dot{q}$):

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix}.$$

The Hamiltonian is given by

$$H(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\alpha q^2 + \frac{1}{4}\beta q^4.$$

Poisson formulation of a Hamiltonian system

A general Hamiltonian system can be written as

$$\dot{x} = J(x)\nabla H(x), \quad x \in \mathbb{R}^n, \quad J(x) = -J(x)^\top.$$

Define the **Poisson bracket** for smooth functions $F, G : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$\{F, G\} := (\nabla F)^\top J(x) \nabla G.$$

Then the system can equivalently be written in Poisson form:

$$\dot{x}_i = \{x_i, H\}, \quad i = 1, \dots, n, \quad \frac{dF}{dt} = \{F, H\}.$$

Properties:

- ▶ Bilinearity: $\{aF + bG, H\} = a\{F, H\} + b\{G, H\},$
- ▶ Skew-symmetry: $\{F, G\} = -\{G, F\},$
- ▶ Jacobi identity: $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0,$
- ▶ Energy conservation: $\frac{dH}{dt} = \{H, H\} = 0.$

Energy preservation via Discrete Gradient method

Energy preservation can be achieved using a discrete gradient.

A discrete gradient $\bar{\nabla}H(x, y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a mapping that satisfies:

1. **Discrete chain rule:** $H(y) - H(x) = \bar{\nabla}H(x, y)^\top (y - x)$,
2. **Consistency:** $\bar{\nabla}H(x, x) = \nabla H(x)$.

Examples:

- Mean Value Discrete Gradient

$$\bar{\nabla}_{\text{MV}}H(x, y) = \int_0^1 \nabla H((1 - \theta)x + \theta y) d\theta.$$

- Gonzales Midpoint Discrete Gradient

$$\bar{\nabla}_{\text{G}}H(x, y) = \nabla H\left(\frac{x + y}{2}\right) + \frac{H(y) - H(x) - \nabla H\left(\frac{x + y}{2}\right)^\top (y - x)}{\|y - x\|^2} (y - x), \quad y \neq x.$$

Time stepping

Given a discrete gradient $\overline{\nabla}H$, the corresponding time-stepping method is

$$\frac{x^{n+1} - x^n}{\Delta t} = J\left(\frac{x^n + x^{n+1}}{2}\right) \overline{\nabla}H(x^n, x^{n+1}).$$

This one-step method preserves H exactly.

Disadvantages

- ▶ **Computational cost:** Implicit methods, Newton solver required. Mean value gradients require numerical quadrature; midpoint gradients require multiple function evaluations.
- ▶ **Order of accuracy:** Discrete gradient methods are second-order accurate. Constructing higher-order energy-preserving schemes is challenging¹.

¹Eidnes, “Order theory for discrete gradient methods”

Towards a Poisson formulation: energy quadratisation

The Hamiltonian

$$H = \frac{1}{2}v^2 + \frac{1}{2}\alpha q^2 + \frac{1}{4}\beta q^4, \quad v := \dot{q}$$

can be rewritten using the stresses. The strains are given by

$$\varepsilon_1 = q, \quad \varepsilon_2 = q^2$$

The stresses are given by (Legendre transform conjugate variables)

$$\sigma_1 = \alpha \varepsilon_1, \quad \sigma_2 = \frac{\beta}{2} \varepsilon_2$$

The energy is now rewritten as a quadratic form

$$H = \frac{1}{2} \begin{pmatrix} v \\ \sigma_1 \\ \sigma_2 \end{pmatrix}^\top \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\alpha & 0 \\ 0 & 0 & 2/\beta \end{bmatrix} \begin{pmatrix} v \\ \sigma_1 \\ \sigma_2 \end{pmatrix} = \frac{1}{2} \mathbf{x}^\top \mathbf{H} \mathbf{x}.$$

Poisson structure

Idea: introduce the dynamics of the stress variables. The augmented system reads

$$\begin{aligned}\dot{q} &= \mathbf{e}_1^\top \mathbf{x}, \\ \mathbf{H}\dot{\mathbf{x}} &= \mathbf{J}(q)\mathbf{x},\end{aligned}$$

where $\mathbf{e}_1 = [1 \ 0 \ 0]^\top$ is the first element of the Euclidean canonical basis, and

$$\mathbf{H} = \text{Diag} \begin{bmatrix} 1 \\ 1/\alpha \\ 2/\beta \end{bmatrix}, \quad \mathbf{J}(q) = \begin{bmatrix} 0 & -1 & -2q \\ 1 & 0 & 0 \\ 2q & 0 & 0 \end{bmatrix}.$$

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A general abstract framework

Geometrically nonlinear mechanical models admit an analogous non-canonical Hamiltonian form:

$$\begin{aligned}\partial_t \mathbf{q} &= \mathbf{v}, \\ \mathcal{H} \partial_t \mathbf{x} &= \mathcal{J}(\mathbf{q}) \mathbf{x},\end{aligned}$$

with state $\mathbf{x} = (\mathbf{v}, \mathbf{S})$, $\mathcal{H} = \text{Diag}[\rho, \mathbb{C}]$

- ▶ $\mathbf{q}$: displacement, $\mathbf{v}$: velocity, $\mathbf{S}$: stress-like variable,
- ▶ ρ : density, $\mathbb{C}$: compliance tensor.

Poisson operator:

$$\mathcal{J}(\mathbf{q}) = \begin{bmatrix} 0 & -\mathcal{L}^*(\mathcal{D}\mathbf{q}) \\ \mathcal{L}(\mathcal{D}\mathbf{q}) & 0 \end{bmatrix}.$$

Here $\mathcal{D}\mathbf{q}$ is a parameter and $\mathcal{L}$ designates the formal adjoint of $\mathcal{L}$

Energy and Variational Basis

Skew-adjointness of $\mathcal{J}(\mathbf{q})$ ensures conservation of

$$H(\mathbf{v}, \mathbf{S}) = \frac{1}{2} \int_{\Omega} \rho \|\mathbf{v}\|^2 + \mathbf{S} \cdot \mathbb{C} \mathbf{S} \, d\Omega, \quad \dot{H} = 0.$$

This system is derived from the least action principle $\delta S = 0$ using the Reissner expression of the deformation energy V_R :

$$\begin{aligned} S(\mathbf{q}, \mathbf{v}, \mathbf{S}) &:= \int_0^{t_f} \int_{\Omega} \dot{\mathbf{q}} \cdot \rho \mathbf{v} \, d\Omega \, dt - \int_0^{t_f} T(\mathbf{v}) + V_R(\mathbf{q}, \mathbf{S}) \, dt, \\ T(\mathbf{v}) &:= \frac{1}{2} \int_{\Omega} \rho \|\mathbf{v}\|^2 \, d\Omega, \\ V_R(\mathbf{q}, \mathbf{S}) &:= \int_{\Omega} \mathbf{S} \cdot \mathbf{E}(\mathbf{q}) - \frac{1}{2} \mathbf{S} \cdot \mathbb{C} \mathbf{S} \, d\Omega, \end{aligned}$$

where $\mathbf{S} \cdot \mathbf{E}$ denotes the tensor contraction and $\mathbf{E}$ is a geometrically non linear strain. The formulation is local and requires boundary conditions.

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Abstract framework

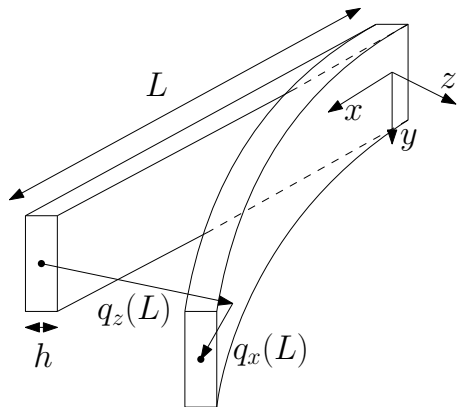
Von Kármán nonlinearity in thin structures

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Von Kármán beams



Assumptions:

- ▶ Out-of-plane deflection comparable to thickness
- ▶ Axial stretching terms negligible vs. rotation squares

⇒ Quadratic bending–membrane coupling term retained

Strain and energies

The axial strain and the (linearized) curvature are given by

$$\varepsilon_a := \frac{\partial q_x}{\partial x} + \frac{1}{2} \left(\frac{\partial q_z}{\partial x} \right)^2, \quad \kappa := \frac{\partial^2 q_z}{\partial x^2}.$$

Total strain $\varepsilon = \varepsilon_a + \kappa$. Consider the kinetic and potential energy

$$T = \frac{1}{2} \int_0^L \rho A (v_x^2 + v_z^2) dx,$$
$$V_R = \int_0^L \left\{ N \varepsilon_a(q_x, q_z) + M \kappa(q_z) - \frac{1}{2EA} N^2 - \frac{1}{2EI} M^2 \right\} dx.$$

Euler Lagrange equations

Kinematics

$$\partial_t q_x = v_x,$$

$$\partial_t q_z = v_z,$$

Dynamics

$$\rho A \partial_t v_x = \partial_x N,$$

$$\rho A \partial_t v_z = -\partial_{xx}^2 M + \partial_x (N \partial_x q_z),$$

Constitutive laws

$$(EA)^{-1} N = (\partial_x q_x + 1/2(\partial_x q_z)^2),$$

$$(EI)^{-1} M = \partial_{xx}^2 q_z.$$

Poisson system

The time derivative of the stress variables gives

$$C_a \partial_t N = \partial_x v_x + (\partial_x q_z) \partial_x v_z,$$

$$C_a := (EA)^{-1},$$

$$C_b \partial_t M = \partial_{xx} v_z,$$

$$C_b := (EI)^{-1}.$$

The Poisson system reads

$$\partial_t q_x = v_x,$$

$$\partial_t q_z = v_z,$$

$$\text{Diag} \begin{bmatrix} \rho A \\ \rho A \\ C_a \\ C_b \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} v_x \\ v_z \\ N \\ M \end{pmatrix} = \begin{bmatrix} 0 & 0 & \partial_x & 0 \\ 0 & 0 & \partial_x (\circ \partial_x q_z) & -\partial_{xx}^2 \\ \partial_x & (\partial_x q_z) \partial_x \circ & 0 & 0 \\ 0 & \partial_{xx}^2 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_x \\ v_z \\ N \\ M \end{pmatrix}.$$

For this example, the operator $\mathcal{L}(\partial_x q_z)$ and its adjoint are given by

$$\mathcal{L}(\partial_x q_z) = \begin{bmatrix} \partial_x & (\partial_x q_z) \partial_x \circ \\ 0 & \partial_{xx}^2 \end{bmatrix}, \quad \mathcal{L}^*(\partial_x q_z) = - \begin{bmatrix} \partial_x & 0 \\ \partial_x (\circ \partial_x q_z) & -\partial_{xx}^2 \end{bmatrix}.$$

Von Kármán plate model: Strain and energies

The axial strain and curvature generalize the previous model

$$\boldsymbol{\varepsilon}_m = \text{def}(\mathbf{q}_m) + \frac{1}{2} \nabla q_z \otimes \nabla q_z, \quad \boldsymbol{\kappa} = \text{Hess } q_z.$$

- ▶ The displacement vector $\mathbf{q}$ has been split into the membrane displacement $\mathbf{q}_m = (q_x \quad q_y)^\top$ and the out-of-plane component q_z ;
- ▶ $\text{def} := \frac{1}{2}(\nabla + \nabla^\top)$ is the symmetric gradient;
- ▶ $\otimes$ is the dyadic product of the two vectors, i.e. $\mathbf{a} \otimes \mathbf{b} = \mathbf{ab}^\top$;

Consider the kinetic and Reissner deformation energy

$$T = \frac{1}{2} \int_0^L \rho h (\|\mathbf{v}_m\|^2 + \|v_z\|^2) dx,$$
$$V_R = \int_0^L \left\{ \mathbf{N} \cdot \boldsymbol{\varepsilon}_m(\mathbf{q}_m, q_z) + \mathbf{M} \cdot \boldsymbol{\kappa}(q_z) - \frac{1}{2} \mathbf{N} \cdot \mathbb{C}_m \mathbf{N} - \frac{1}{2} \mathbf{M} \cdot \mathbb{C}_m \mathbf{M} \right\} dx.$$

Poisson Structure

Applying the generalized Hamilton principle and taking the derivative of the membrane and bending stress tensors $\mathbf{N}$, $\mathbf{M}$

$$\begin{aligned}\partial_t \mathbf{q}_m &= \mathbf{v}_m, \\ \partial_t q_z &= v_z, \\ \text{Diag} \begin{bmatrix} \rho h \\ \rho h \\ \mathbf{C}_m \\ \mathbf{C}_b \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{v}_m \\ v_z \\ \mathbf{N} \\ \mathbf{M} \end{pmatrix} &= \begin{bmatrix} 0 & 0 & \text{div} & 0 \\ 0 & 0 & \mathcal{C}(q_z) & -\text{div div} \\ \text{def} & -\mathcal{C}^*(q_z) & 0 & 0 \\ 0 & \text{Hess} & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{v}_m \\ v_z \\ \mathbf{N} \\ \mathbf{M} \end{pmatrix}.\end{aligned}$$

The operator $\mathcal{C}(q_z)(\cdot)$ acting on symmetric tensors is defined by

$$\mathcal{C}(q_z)(\mathbf{N}) := \text{div}(\mathbf{N} \nabla q_z).$$

Its adjoint is given by

$$\mathcal{C}(q_z)^*(\cdot) = -\text{sym} [\nabla(\cdot) \otimes \nabla(q_z)].$$

Von Kármán in Airy form: the 2D elasticity complex

The model can be simplified by neglecting the membrane inertia $\rho h \partial_t \mathbf{v}_m = 0$, leading to

$$\operatorname{div} \mathbf{N} = 0.$$

For a simply connected domain, the 2D elasticity complex is exact

$$\mathbb{P}_1 \xrightarrow{\subset} C^\infty \xrightarrow{\text{Air}} C^\infty \otimes \mathbb{S} \xrightarrow{\operatorname{div}} C^\infty \otimes \mathbb{V}$$

where $\mathbb{P}_1 := \mathbb{R} + \mathbf{x}^\perp \cdot \mathbb{R}^2$ is the space of first order polynomial. So the membrane stress field can be represented by an Airy potential

$$\mathbf{N} = \text{Air } \varphi.$$

Plugging this expression in the dynamics

$$\operatorname{div}(\mathbf{N} \nabla q_z) = \text{Air } \varphi \cdot \operatorname{Hess} q_z, \quad \text{since } \operatorname{div} \text{Air} \equiv 0.$$

The adjoint complex

$$\mathcal{B}(f, g) := \text{Air } f \cdot \text{Hess } g.$$

The following properties of the bilinear form have been proven²

- ▶ Symmetry: $\mathcal{B}(f, g) = \mathcal{B}(g, f)$;
- ▶ Self-adjointness: $(\mathcal{B}(f, g), h)_\Omega = (g, \mathcal{B}(f, h))_\Omega$ (function f is a parameter).

The expression of the membrane stress still contains the in-plane displacement

$$\mathbf{C}_m \text{Air } \varphi = \text{def } \mathbf{q}_m + \frac{1}{2} \nabla q_z \otimes \nabla q_z.$$

The idea is to exploit the relation

$$\text{rot rot def} = 0$$

from the adjoint complex

$$\mathbf{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{rot rot}} C^\infty$$

²Bilbao, “A family of conservative finite difference schemes for the dynamical von Kármán plate equations”

Elimination of $\mathbf{q}_m$, $\mathbf{v}_m$

Applying rot rot to the inplane

$$\text{rot rot } \mathbf{C}_m \text{ Air } \varphi = \frac{1}{2} \text{rot rot}(\nabla q_z \otimes \nabla q_z).$$

The operator $\text{rot rot } \mathbf{C}_m \text{ Air} = \text{Air}^* \mathbf{C}_m \text{ Air}$ is self-adjoint. A little algebra provides

$$(\text{Air}^* \mathbf{C}_m \text{ Air}) \varphi = \frac{1}{2} \text{rot rot}(\nabla q_z \otimes \nabla q_z) = -\frac{1}{2} \mathcal{B}(q_z, q_z).$$

Taking the time derivative provides the dynamics of the Airy potential φ

$$(\text{Air}^* \mathbf{C}_m \text{ Air}) \frac{\partial \varphi}{\partial t} = -\mathcal{B}(q_z, v_z).$$

Poisson formulation of the Airy form

$$\partial_t q_z = v_z,$$
$$\text{Diag} \begin{bmatrix} \rho h \\ \text{Air}^* \mathbf{C}_m \text{Air} \\ \mathbf{C}_b \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} v_z \\ \varphi \\ \mathbf{M} \end{pmatrix} = \begin{bmatrix} 0 & \mathcal{B}(q_z, \circ) & -\text{div div} \\ -\mathcal{B}(q_z, \circ) & 0 & 0 \\ \text{Hess} & 0 & 0 \end{bmatrix} \begin{pmatrix} v_z \\ \varphi \\ \mathbf{M} \end{pmatrix}.$$

A self-adjoint differential operator $\mathcal{H}$ takes the place of an algebraic energy matrix (implicit Hamiltonian system³).

The differential operator and its adjoint read

$$\mathcal{L}(\text{Air } q_z) = \begin{bmatrix} -\mathcal{B}(q_z, \circ) \\ \text{Hess} \end{bmatrix}, \quad \mathcal{L}^*(\text{Air } q_z) = - \begin{bmatrix} \mathcal{B}(q_z, \circ) & -\text{div div} \end{bmatrix}.$$

³Bendimerad-Hohl et al., “Stokes-Lagrange and Stokes-Dirac representations of N -dimensional port-Hamiltonian systems for modeling and control”

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Canonical Hamiltonian structure for continuum mechanics

Given a solid body $\mathcal{B} \subset \mathbb{R}^3$ and an embedding

$$\mathbf{x} = \varphi(\mathbf{X}, t) = \mathbf{X} + \mathbf{q}(\mathbf{X}, t) \in \mathcal{S} \subset \mathcal{A} = \mathbb{R}^3,$$

the dynamics of a deformable body can be written in canonical Hamiltonian form as

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\tilde{\pi}} \end{pmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{pmatrix} \delta_{\varphi} H \\ \delta_{\tilde{\pi}} H \end{pmatrix},$$

where the Hamiltonian is given by

$$H(\tilde{\pi}, \mathbf{F}) = \int_{\mathcal{B}} \frac{\|\tilde{\pi}\|^2}{2\rho} + \tilde{\rho} W(\mathbf{F}) d\mathbf{X}, \quad W \text{ hyperelastic potential}$$

The quantity $\tilde{\pi} := \tilde{\rho} \tilde{\mathbf{v}}$ is the material representation linear momentum (and velocity)

$$\tilde{\mathbf{v}}(\mathbf{X}, t) : \mathbf{X} \in \mathcal{B} \rightarrow T_{\varphi(\mathbf{X}, t)} \mathcal{S}$$

Rate of change strain tensor (cartesian coordinates)

To ensure frame indifference the deformation energy must depend on the right Cauchy strain tensor or equivalently the Green Lagrange strain

$$\mathbf{E}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \rightarrow T_{\mathbf{X}}\mathcal{B}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I}), \quad \mathbf{F} = \mathbf{I} + \nabla \mathbf{q}.$$

The kinetic and potential energies are given by

$$T = \frac{1}{2} \int_{\mathcal{B}} \rho \|\tilde{\mathbf{v}}\|^2 d\mathbf{X},$$
$$V_R = \int_{\Omega} \mathbf{S} \cdot \mathbf{E}(\mathbf{q}) - \frac{1}{2} \mathbf{S} \cdot \mathbb{C} \mathbf{S} \, d\Omega.$$

Euler Lagrange equations

$$\partial_t \mathbf{q} = \tilde{\mathbf{v}}, \quad \rho \partial_t \tilde{\mathbf{v}} = \operatorname{div}(\mathbf{F} \mathbf{S}), \quad \mathbb{C} \mathbf{S} = \frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I}).$$

Poisson structure

The rate of change of the Green Lagrange strain is given by⁴

$$\mathbb{C} \partial_t \mathbf{S} = \text{sym}(\mathbf{F}^\top \nabla \tilde{\mathbf{v}}).$$

Dynamical system⁵

$$\begin{aligned} \partial_t \mathbf{q} &= \tilde{\mathbf{v}}, \\ \begin{bmatrix} \rho & 0 \\ 0 & \mathbf{C} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \tilde{\mathbf{v}} \\ \mathbf{S} \end{pmatrix} &= \begin{bmatrix} 0 & \text{div}(\mathbf{F} \circ) \\ \text{sym}(\mathbf{F}^\top \nabla \circ) & 0 \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{v}} \\ \mathbf{S} \end{pmatrix}, \end{aligned}$$

The differential operator and its adjoint read

$$\mathcal{L}(\nabla \mathbf{q}) = \text{sym}(\mathbf{F}^\top \nabla \circ), \quad \mathcal{L}^*(\nabla \mathbf{q}) = -\text{div}(\mathbf{F} \circ).$$

⁴Marsden and Hughes, Mathematical foundations of elasticity

⁵Thoma, Kotyczka, and Egger, “On the velocity-stress formulation for geometrically nonlinear elastodynamics and its structure-preserving discretization”

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Von Kármán beams: Finite element spaces

Denote by E a generic element in an interval mesh $\mathcal{I}_h$.

Velocity (and displacement) space $V_h = \text{CG}_1 \times \text{Her}$:

$$\text{CG}_1 = \{v_h \in C^0([0, L]), v_h|_E \in \mathbb{P}_1\},$$

$$\text{Her} = \{v_h \in C^1([0, L]), v_h|_E \in \mathbb{P}_3\}.$$

Stress space $\Sigma_h = \text{DG}_4 \times \text{DG}_1$

$$\text{DG}_k = \{v_h|_E \in \mathbb{P}_k, \forall E \in \mathcal{I}_h\}$$

The reason for DG_4 comes from the axial resultant dynamics

$$C_a \partial_t N = \partial_x v_x + \partial_x q_z \partial_x v_z$$

$$q_z, v_z \in \text{Her} \Rightarrow \partial_x q_z \partial_x v_z \in \text{CG}_4,$$

$$v_x \in \text{CG}_1 \Rightarrow \partial_x v_x \in \text{DG}_0$$

Since $\text{CG}_4 + \text{DG}_0 \subset \text{DG}_4$, choosing a quartic DG space avoids locking.

Weak formulation

Integration by parts of the lines of the linear momentum balance.

$$\begin{aligned}
 \partial_t q_{z,h} &= v_{z,h}, \\
 (\psi_x, \rho A \partial_t v_{x,h}) &= -(\partial_x \psi_x, N_h), \\
 (\psi_z, \rho A \partial_t v_{z,h}) &= -(\partial_x \psi_z \partial_x q_{z,h}, N_h) - (\partial_{xx} \psi_z, M_h), \\
 (\psi_N, C_a \partial_t N_h) &= (\psi_N, \partial_x q_z^h \partial_x v_z^h + \partial_x v_{x,h}), \\
 (\psi_M, C_b \partial_t M_h) &= (\psi_M, \partial_{xx} v_z^h).
 \end{aligned}$$

Integration by parts of the lines of the linear momentum balance.

$$\begin{aligned}
 \dot{\mathbf{q}}_z &= \mathbf{v}_z, \\
 \text{Diag} \begin{bmatrix} \mathbf{M}_{\rho A} \\ \mathbf{M}_{\rho A} \\ \mathbf{M}_{C_a} \\ \mathbf{M}_{C_b} \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \mathbf{v}_x \\ \mathbf{v}_z \\ \mathbf{n} \\ \mathbf{m} \end{pmatrix} &= \begin{bmatrix} 0 & 0 & -\mathbf{D}_{\partial_x}^\top & 0 \\ 0 & 0 & -\mathbf{L}^\top(\mathbf{q}_z) & -\mathbf{D}_{\partial_{xx}}^\top \\ \mathbf{D}_{\partial_x} & \mathbf{L}(\mathbf{q}_z) & 0 & 0 \\ 0 & \mathbf{D}_{\partial_{xx}} & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{v}_x \\ \mathbf{v}_z \\ \mathbf{n} \\ \mathbf{m} \end{pmatrix}
 \end{aligned}$$

Nonlinear elasticity: Finite element spaces

Denote with T a generic cell of the computational mesh $\mathcal{T}_h$.

Velocity (and displacement) space $V_h = \text{CG}_1 \otimes \mathbb{R}^d$:

$$\text{CG}_1 \otimes \mathbb{R}^d = \{v_h \in C^0(\Omega), v_h|_T \in \mathbb{P}_1 \otimes \mathbb{R}^d\},$$

Stress space $\Sigma_h = \text{DG}_0 \otimes \mathbb{R}_{\text{sym}}^{d \times d}$. The second Piola stress dynamics is

$$\mathbb{C} \partial_t \mathbf{S} = \text{sym}(\mathbf{F}^\top \nabla \tilde{\mathbf{v}}).$$

Hence $\mathbf{S} \propto \nabla \tilde{\mathbf{v}} \nabla \mathbf{q}$.

Since $\nabla \tilde{\mathbf{v}} \in \text{DG}_0$, $\nabla \mathbf{q} \in \text{DG}_0$, it holds $\mathbf{S} \in \text{DG}_0$.

Weak formulation

$$\begin{aligned}\partial_t \mathbf{q}_h &= \tilde{\mathbf{v}}_h, \\ (\psi, \rho \partial_t \tilde{\mathbf{v}}_h)_\Omega &= -(\mathbf{F}_h^\top \nabla \psi, \mathbf{S}_h)_\Omega, \\ (\Psi, \mathbf{C} \partial_t \mathbf{S}_h)_\Omega &= +(\Psi, \mathbf{F}_h^\top \nabla \tilde{\mathbf{v}}_h)_\Omega,\end{aligned}$$

Algebraic system

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{v}, \\ \begin{bmatrix} \mathbf{M}_\rho & 0 \\ 0 & \mathbf{M}_C \end{bmatrix} \frac{d}{dt} \begin{pmatrix} \tilde{\mathbf{v}} \\ \mathbf{s} \end{pmatrix} &= \begin{bmatrix} 0 & -\mathbf{L}^\top(\mathbf{q}) \\ \mathbf{L}(\mathbf{q}) & 0 \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{v}} \\ \mathbf{s} \end{pmatrix}.\end{aligned}$$

The algebraic system can be written more compactly in the following form

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{v}, \\ \mathbf{H} \dot{\mathbf{x}} &= \mathbf{J}(\mathbf{q}) \mathbf{x}.\end{aligned}$$

Summary

An introductory example

The Poisson structure of geometrically nonlinear continuum mechanics

Discretization

Finite element discretization

Time integration

Numerical experiments

Recall on symplectic integrators

Consider the ODE

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{v}, \\ \mathbf{M}\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{q}).\end{aligned}$$

Störmer Verlet preserves symmetry by staggering

$$\begin{aligned}\frac{\mathbf{q}_{n+\frac{1}{2}} - \mathbf{q}_{n-\frac{1}{2}}}{\Delta t} &= \mathbf{v}_n, \\ \mathbf{M}\frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{\Delta t} &= \mathbf{f}(\mathbf{q}_{n+\frac{1}{2}})\end{aligned}$$

This leads to angular momentum preservation.

Consider the generic ODE $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$. Implicit midpoint update:

$$\frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\Delta t} = \mathbf{g}\left(\frac{\mathbf{x}_{n+1} + \mathbf{x}_n}{2}\right)$$

If system is linear and Hamiltonian then energy is preserved.

Best of both worlds

Idea Combine Störmer–Verlet (leapfrog) and implicit midpoint.

$$\frac{\mathbf{q}_{n+\frac{1}{2}} - \mathbf{q}_{n-\frac{1}{2}}}{\Delta t} = \mathbf{v}_n,$$
$$\mathbf{H} \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\Delta t} = \mathbf{J}(\mathbf{q}_{n+\frac{1}{2}}) \frac{\mathbf{x}_{n+1} + \mathbf{x}_n}{2}$$

Features:

- ▶ **Energy preservation**, since $H = \frac{1}{2} \mathbf{x}^\top \mathbf{H} \mathbf{x}$ and the implicit midpoint preserves quadratic invariants;
- ▶ **Angular momentum preservation**. Since the integration of displacement and velocity is performed on a staggered grid;
- ▶ **Static condensation of the stress**, since it is discretized by a discontinuous space;
- ▶ **Linearly implicit scheme**

Connection with the Scalar Auxiliary Variable (SAV) approach

- ▶ Introduced for gradient flows⁶, extended to Hamiltonian dynamics⁷;
- ▶ Efficient and energy-stable scheme for systems with diagonal $\mathbf{M}$.

Separable Hamiltonian system (velocity formulation):

$$\dot{\mathbf{q}} = \mathbf{v}, \quad \mathbf{M}\dot{\mathbf{v}} = -\nabla_{\mathbf{q}} V, \quad V \geq 0$$

Introduce scalar variable ξ :

$$V = \frac{1}{2}\xi^2, \quad \nabla_{\mathbf{q}} V = \xi \nabla_{\mathbf{q}} \xi, \quad \dot{\xi} = (\nabla_{\mathbf{q}} \xi)^{\top} \mathbf{v}$$

Time integration of Bilbao, Ducceschi, and Zama equivalent to the proposed one:

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{v}, \\ \mathbf{H}\dot{\mathbf{x}} &= \mathbf{J}(\mathbf{q})\mathbf{x}, \quad \mathbf{x} = [\mathbf{v}^{\top} \ \xi]^{\top}, \quad H = \frac{1}{2}\mathbf{x}^{\top} \mathbf{H} \mathbf{x} \end{aligned}$$

⁶Shen, Xu, and Yang, "The scalar auxiliary variable (SAV) approach for gradient flows"

⁷Bilbao, Ducceschi, and Zama, "Explicit exactly energy-conserving methods for Hamiltonian systems"

Comparison with the energy-momentum preserving scheme of Simo⁸

The idea applies to generic constitutive laws.

Mechanical system:

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{v}, \\ \mathbf{M}\dot{\mathbf{v}} &= -\mathbf{L}^\top(\mathbf{q})\boldsymbol{\sigma}(\mathbf{q})\end{aligned}$$

Strain energy (geometric nonlinearity only):

$$V_{\text{def}} = \frac{1}{2}\boldsymbol{\varepsilon}^\top \mathbf{W}\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \mathbf{G}(\mathbf{q}),$$

Stress is obtained by derivation $\boldsymbol{\sigma} = \mathbf{W}\mathbf{G}(\mathbf{q})$.

Modified midpoint: use average stress

$$\hat{\boldsymbol{\sigma}}_{n+1/2} = \mathbf{W} \frac{\mathbf{G}(\mathbf{q}_{n+1}) + \mathbf{G}(\mathbf{q}_n)}{2}$$

⁸Simo and Tarnow, “The discrete energy-momentum method. Conserving algorithms for nonlinear elastodynamics”

Summary

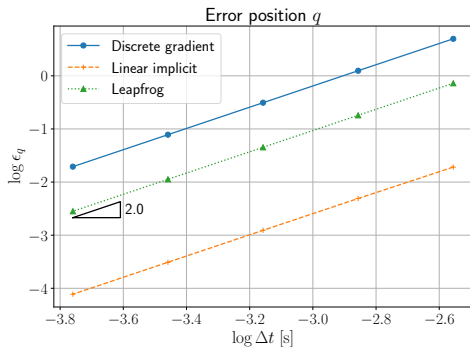
An introductory example

The Poisson structure of geometrically nonlinear continuum mechanics

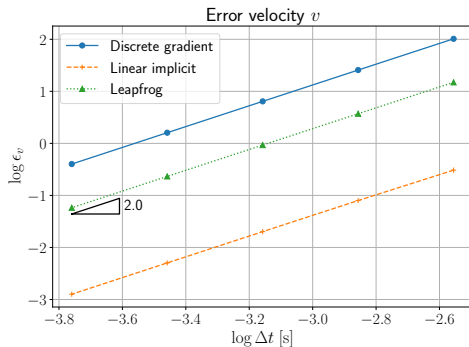
Discretization

Numerical experiments

Duffing oscillator: convergence rates



(a) Displacement q



(b) Velocity v

Figure: Convergence rate for the Duffing oscillator

Duffing Oscillator: energy preservation and efficiency

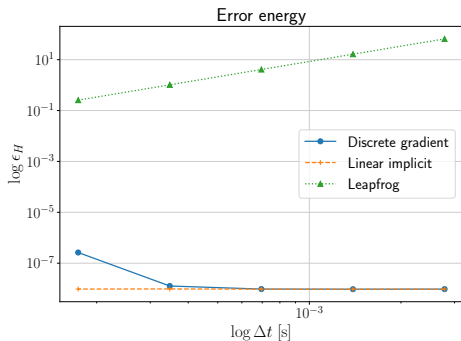


Figure: Energy error for Duffing

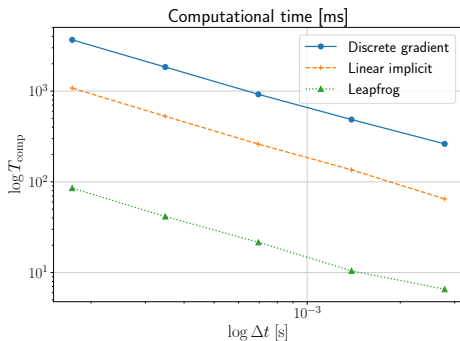
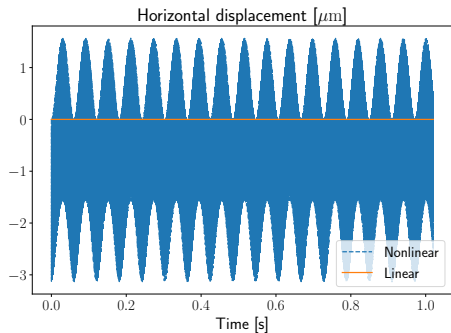
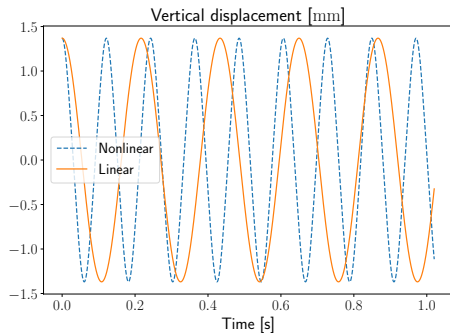


Figure: Computational time for Duffing

Von Kármán beam: vibration starting from first bending mode



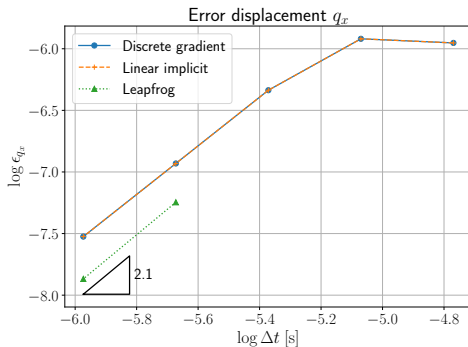
(a) Horizontal displacement $q_x(L/4, t)$



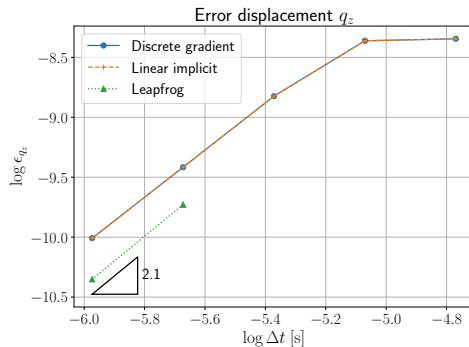
(b) Linear Vertical displacement $q_z(L/4, t)$

Figure: Comparison between linear and nonlinear case at $x = L/4$ for the von-Kármán beam

Von Kármán beam: Convergence rates



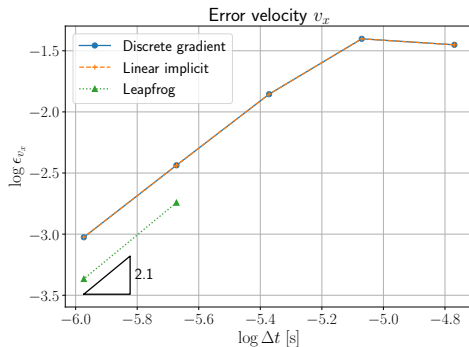
(a) Horizontal displacement q_x



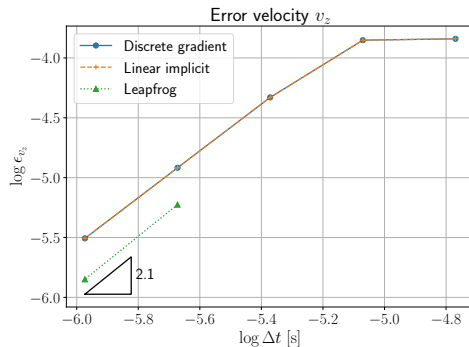
(b) Vertical displacement q_z

Figure: Convergence of q_x , q_z , v_x , v_z over $[0, T_{1,\text{bend}}/10]$

Von Kármán beam: Convergence rates



(a) Horizontal velocity v_x



(b) Vertical velocity v_z

Figure: Convergence of q_x , q_z , v_x , v_z over $[0, T_{1,\text{bend}}/10]$

Von Kármán beam: energy preservation and computational time

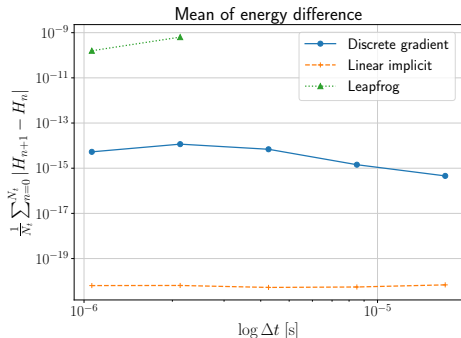


Figure: Mean of $|H_{n+1} - H_n|$ for the von-Kármán beam

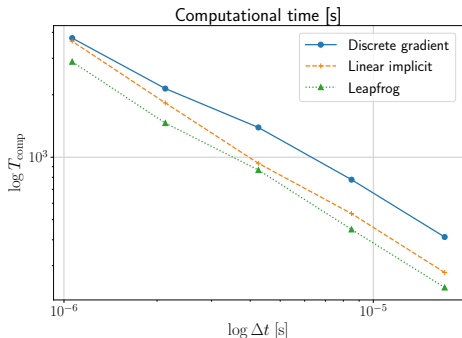
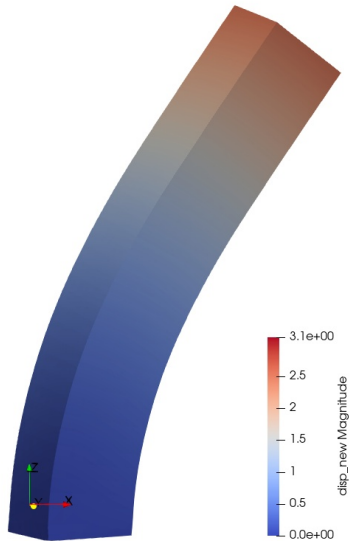
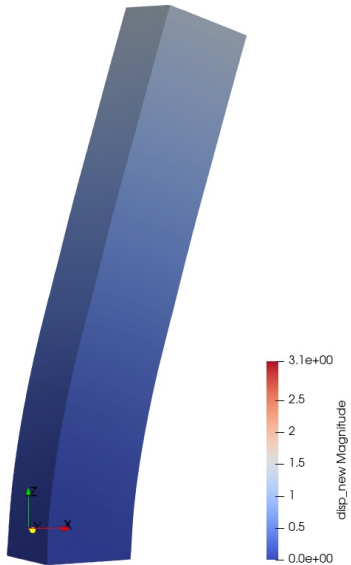
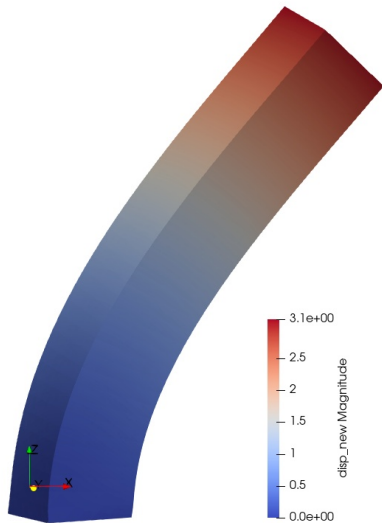
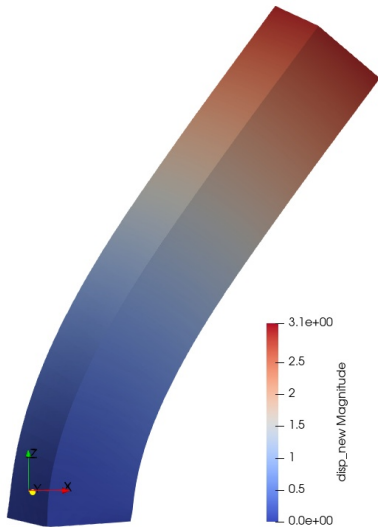


Figure: Computational time for the von-Kármán beam

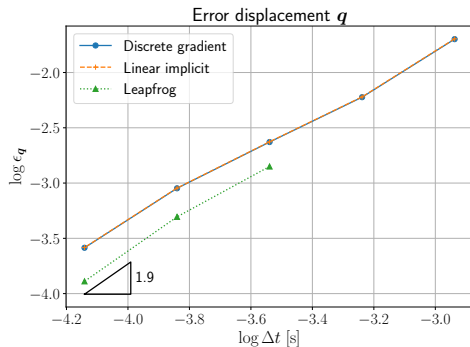
Nonlinear elasticity: screenshots



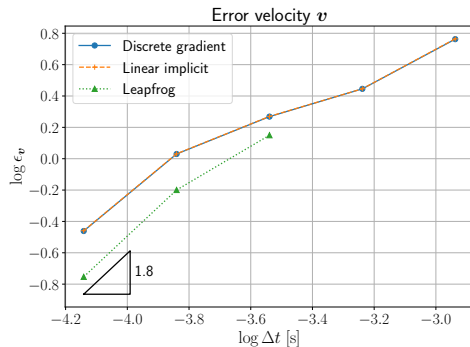
Nonlinear elasticity: screenshots



Nonlinear elasticity: convergence rates



(a) Displacement q



(b) Velocity v

Convergence rate for q , v in geometrically nonlinear elasticity

Nonlinear elasticity: energy preservation and computational time

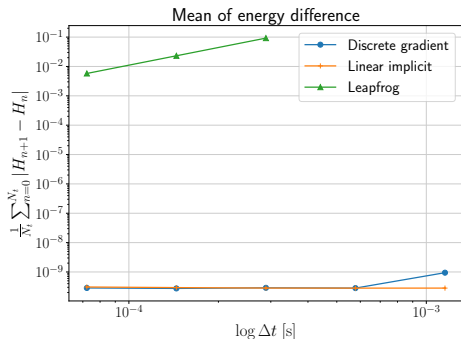


Figure: Mean of $|H_{n+1} - H_n|$ for the geometrically nonlinear elasticity problem

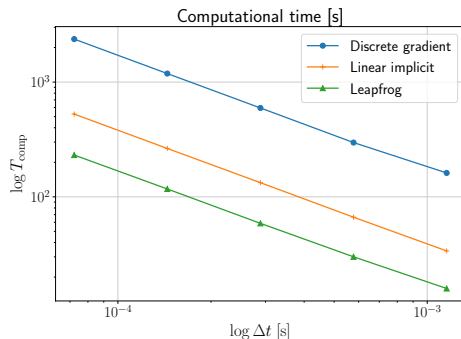


Figure: Computational time for geometrically nonlinear elasticity problem

Conclusions and outlook

Method	Accuracy	Stability	Efficiency
Discrete Gradient	✓	✓	✗
Linear Implicit	✓	✓	✓
Leapfrog	✓	✗	✓

Comparison of time integration methods for geometrically nonlinear mechanics

- ▶ General framework: recast geometrically nonlinear problems into a **non-canonical Hamiltonian formulation**
- ▶ Extendable to rods, shells, solids
- ▶ Mixed FEM with discontinuous stress $\Rightarrow$ static condensation
- ▶ Time integration: Störmer–Verlet + implicit midpoint $\Rightarrow$ exact energy-momentum conservation without nonlinear solves
- ▶ Outlook: improved linear algebra solvers, FEM–FDM coupling for efficiency

Andrea Brugnoli, Denis Matignon, and Joseph Morlier. “A linearly-implicit energy-momentum preserving scheme for geometrically nonlinear mechanics based on non-canonical Hamiltonian formulations”. In: [Nonlinear Dynamics](#) (July 2025). ISSN: 1573-269X. DOI: 10.1007/s11071-025-11601-6. URL: <https://doi.org/10.1007/s11071-025-11601-6>

Different representations leads to qualitative different systems

Geometrically exact beams⁹, all variables are represented in the body attached frame (convective representation):

$$\text{Diag} \begin{bmatrix} \rho A \\ \rho J \\ \mathbf{C}_t \\ \mathbf{C}_r \end{bmatrix} \partial_t \begin{pmatrix} \mathbf{V} \\ \mathbf{W} \\ \mathbf{N} \\ \mathbf{M} \end{pmatrix} = + \begin{bmatrix} 0 & 0 & \partial_s & 0 \\ 0 & 0 & 0 & \partial_s \\ \partial_s & 0 & 0 & 0 \\ 0 & \partial_s & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{W} \\ \mathbf{N} \\ \mathbf{M} \end{pmatrix} + \begin{bmatrix} 0 & [\boldsymbol{\Pi}_V]_\times & [\mathbf{K}]_\times & 0 \\ [\boldsymbol{\Pi}_V]_\times & [\boldsymbol{\Pi}_W]_\times & [\boldsymbol{\Gamma} + \mathbf{e}_1]_\times & [\mathbf{K}]_\times \\ [\mathbf{K}]_\times & [\boldsymbol{\Gamma} + \mathbf{e}_1]_\times & 0 & 0 \\ 0 & [\mathbf{K}]_\times & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{W} \\ \mathbf{N} \\ \mathbf{M} \end{pmatrix}.$$

No dependence on the displacement in $\mathcal{J}$ but rather on the dual state

$$\mathcal{H} \partial_t \mathbf{x} = \mathcal{J}(\mathbf{x}) \mathbf{x}.$$

⁹Hodges, "Geometrically Exact, Intrinsic Theory for Dynamics of Curved and Twisted Anisotropic Beams"

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