The magic of Exterior Calculus bridging Fundamental Physics to real Computations

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The Portwings Project

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RAM ROBOTICS AND MECHATRONICS

UNIVERSITY OF TWENTE.

Robotics And Mechatronics Lab, The Netherlands

Motivation

- In Physics **Symmetries** (conserved quantities) are at the very fundament of the description of nature and **Gauge Fields** are at the core of the description of interaction
- **Energy Is the Mathematical Embodiment** of the invariance of physical laws (more later)
- **Energy is at the core of the description of any Interaction**
- Mathematics is the **most powerful tool** Engineers have

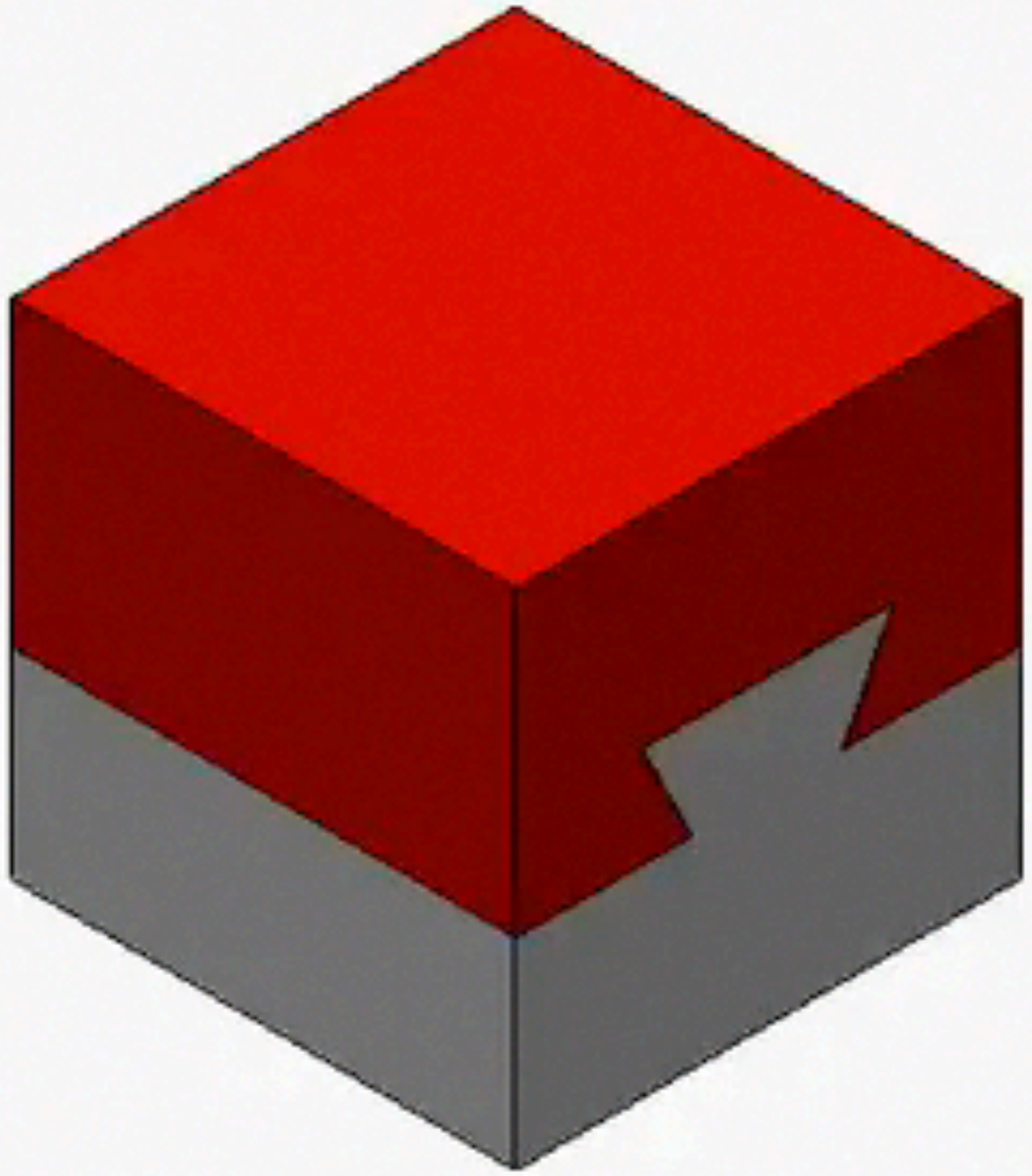
and yet,

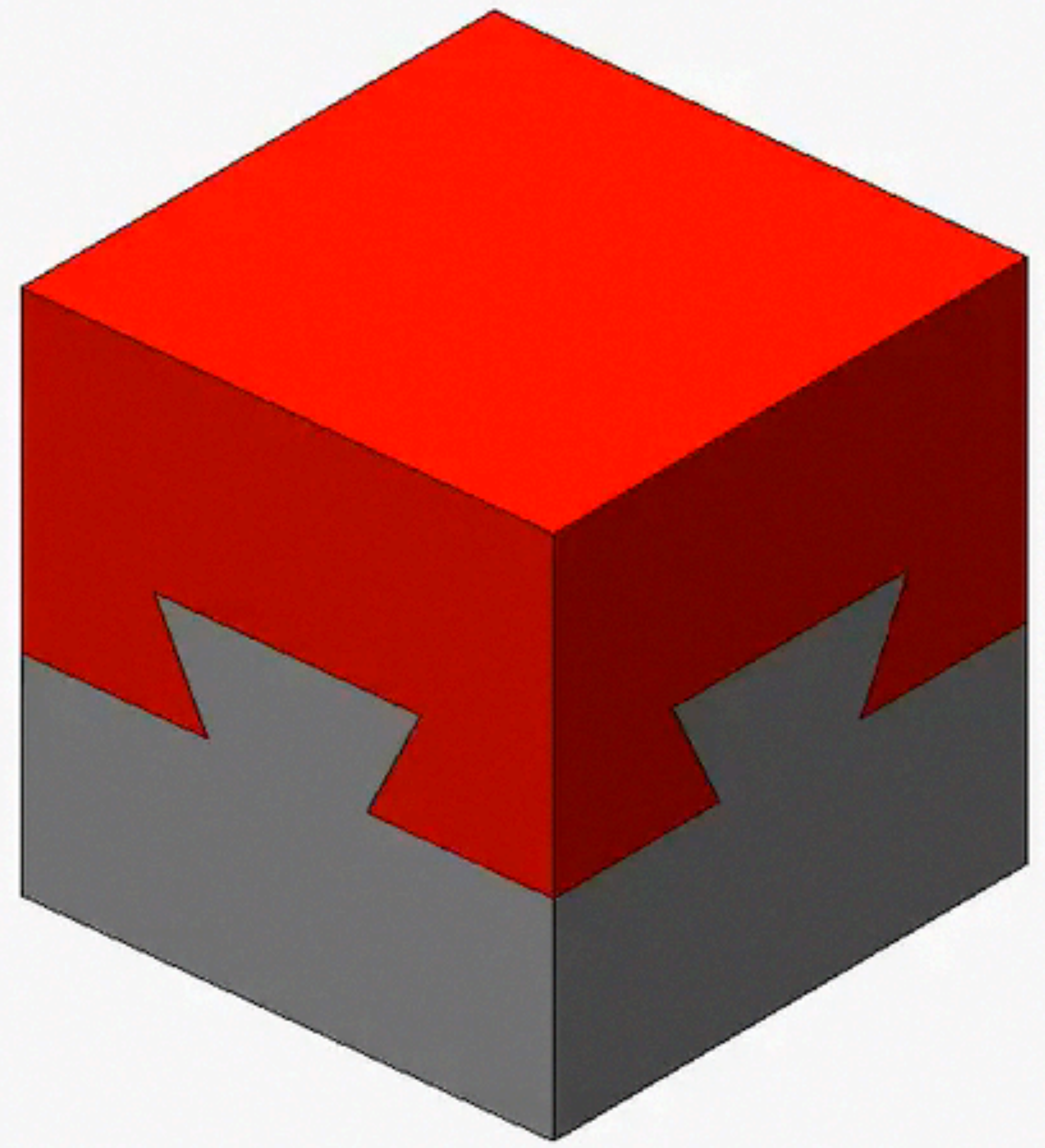
most of the time energy is ignored in the modelling and/or control of physical systems and physics and mathematics is not taken seriously.

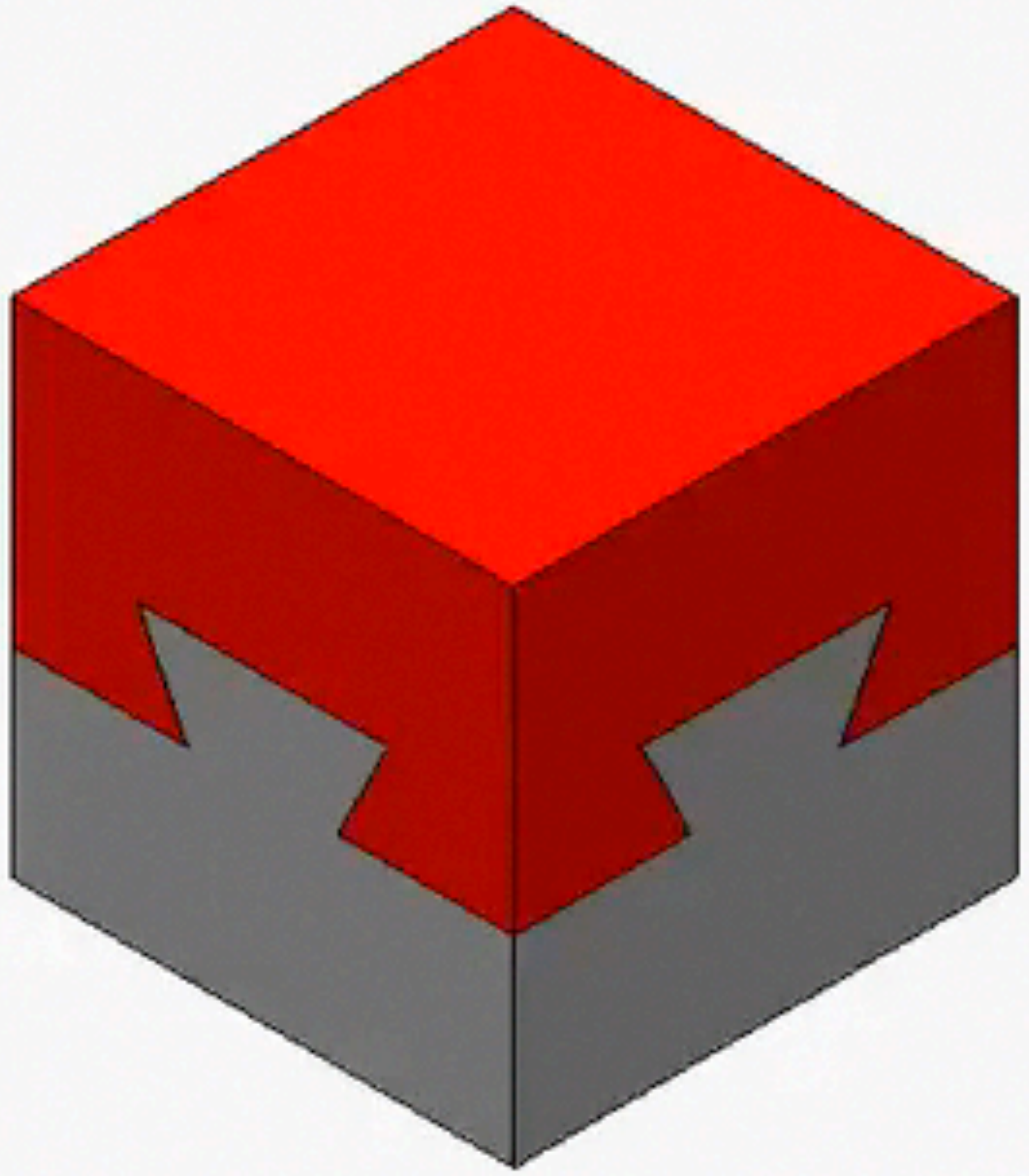
Goal: Give you an idea of why you may want to look into port-based concepts

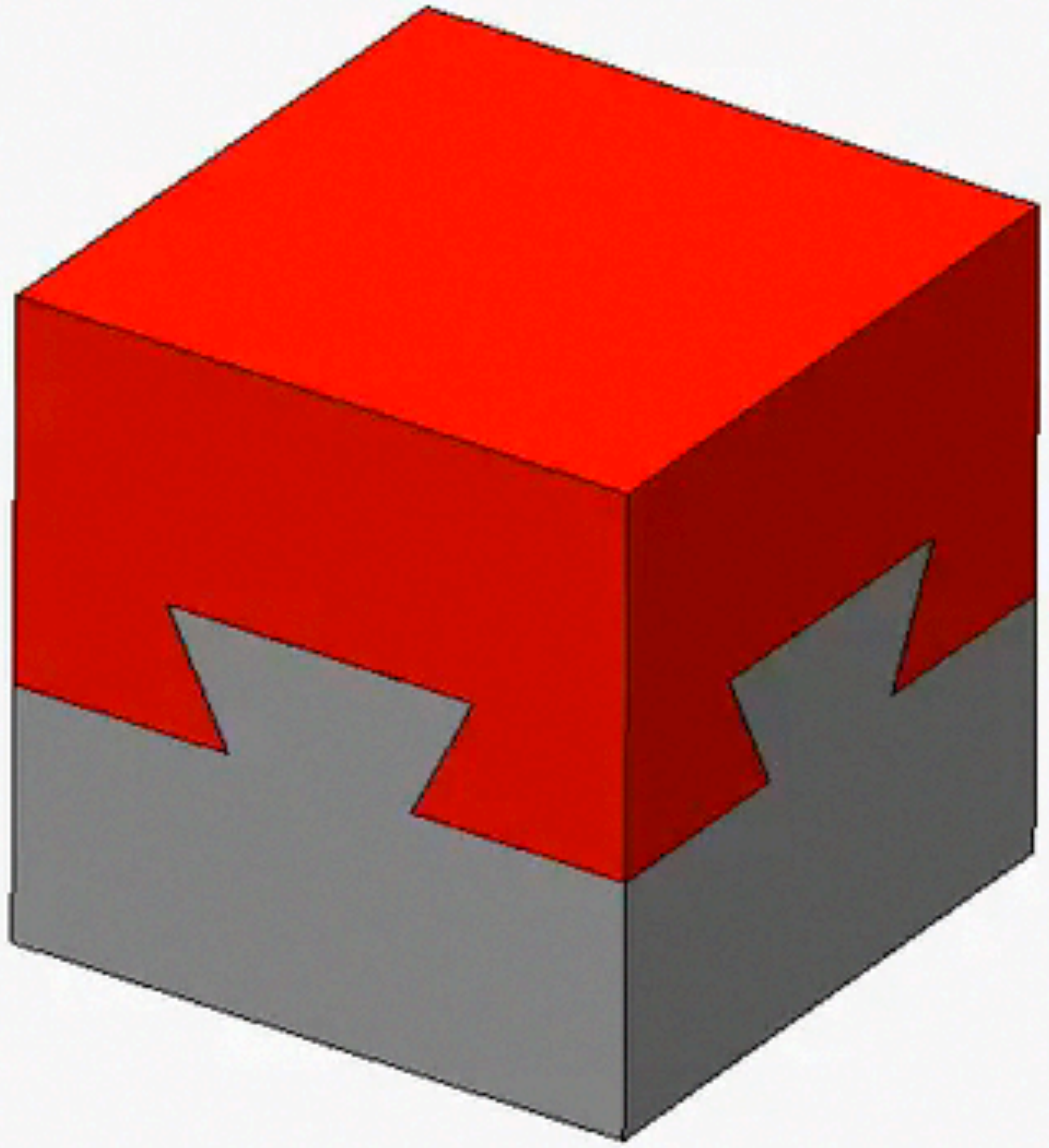
A `Tutorial like' way talk

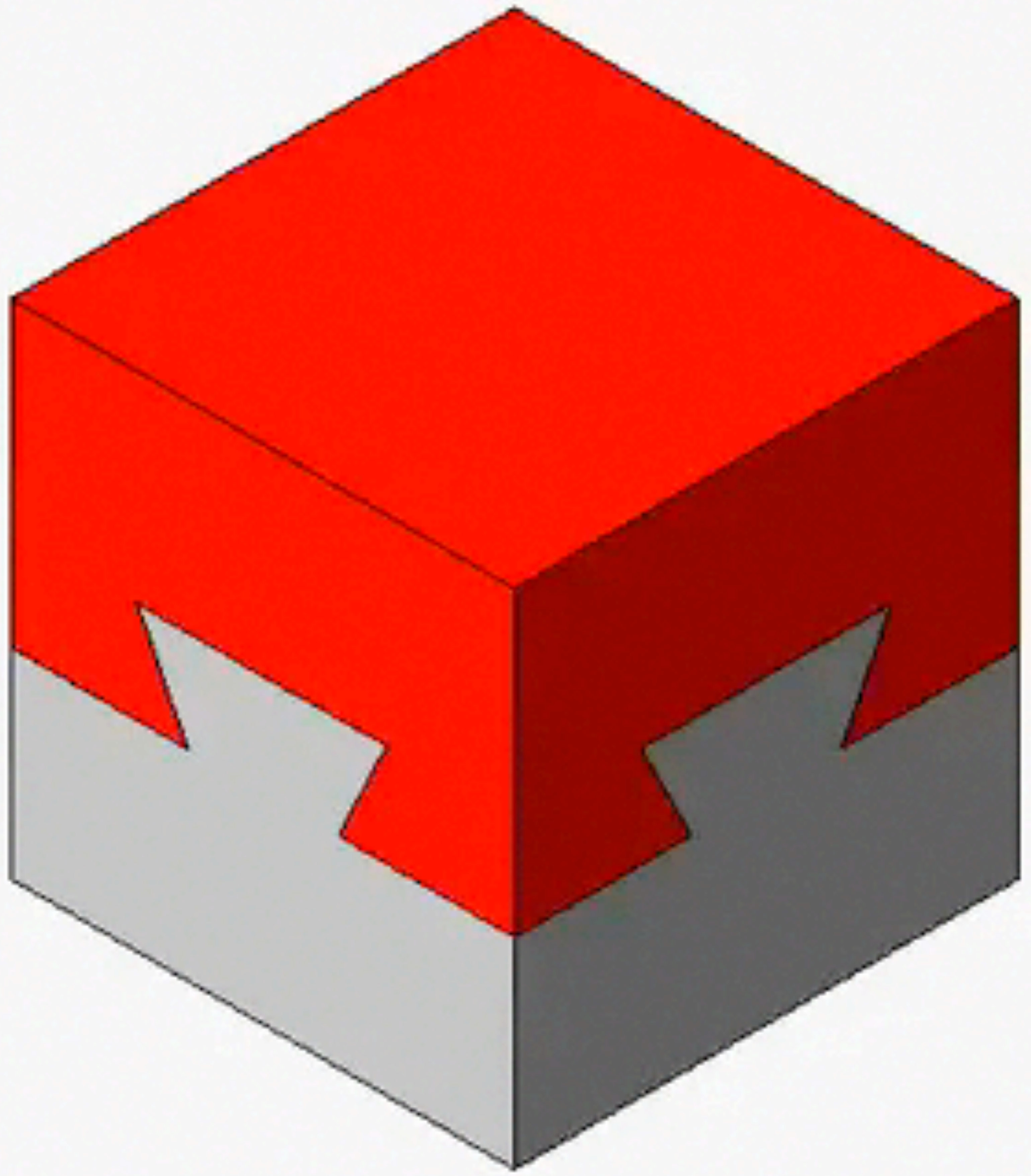
A game about paradigm shifts











Break your patterns: think out of the box (or aquarium :-))

I will try to let you think about Physical Modeling in a way you may not be familiar with, which could give you different perspectives: the Port-Based way


In all things of nature there is something of the marvelous.
Aristotle
quotefancy


Leonardo da Vinci
(1452-1519)
The Portwings Project
erc
Leonardo da Vinci, Codice sul volo degli uccelli. Biblioteca Reale di Torino, 1505.


Weight: 730 g
Payload: 20 g
Speeds +/- 80 Km/h
5 Beaufort wind conditions
Wings
EPP
expandable
polypropylene
Patented
Flapping and Gliding
Giulia

Best Paper Award RAM 2017

Reversed Von Kármán sheet generation

Strouhal number (0.2 < St < 0.4)

$$\mathrm{St} = \frac{fL}{U}$$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CLEANING KIT
RAIN GEAR

Why and **how** does it “**really**” work?

Only understanding that we can **systematically** improve it


To talk about robotic flapping (clearly) we must use the right tools and language:
physics and port-based / energy-aware methods
proper mathematics
What we cannot speak about (clearly)
we must pass over in silence.”
(Ludwig Wittgenstein 1889-1951)

Knowledge

Technē

(Useful/Engineering)

Epistēmē

(Theoretical/Science)

Empiricists

Sense-Experience as source

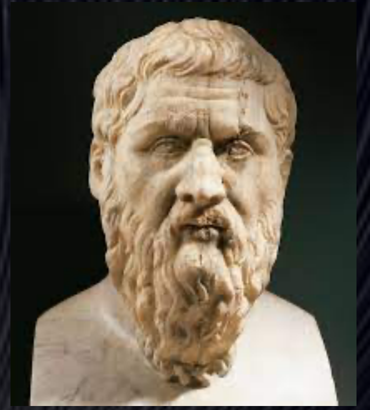
Aristotele

John Locke

Inductive Reasoning

Rationalists

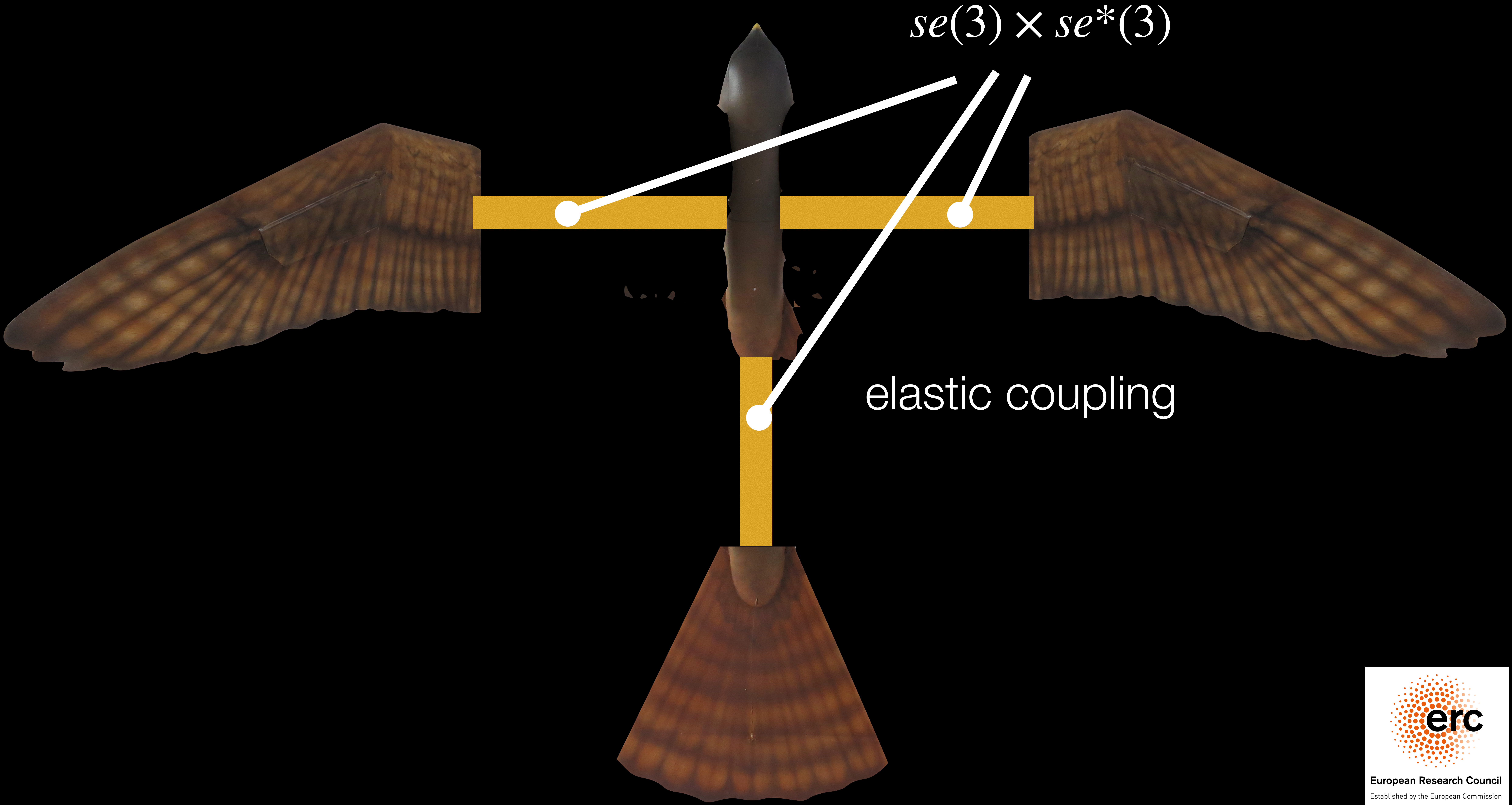
Reason as source



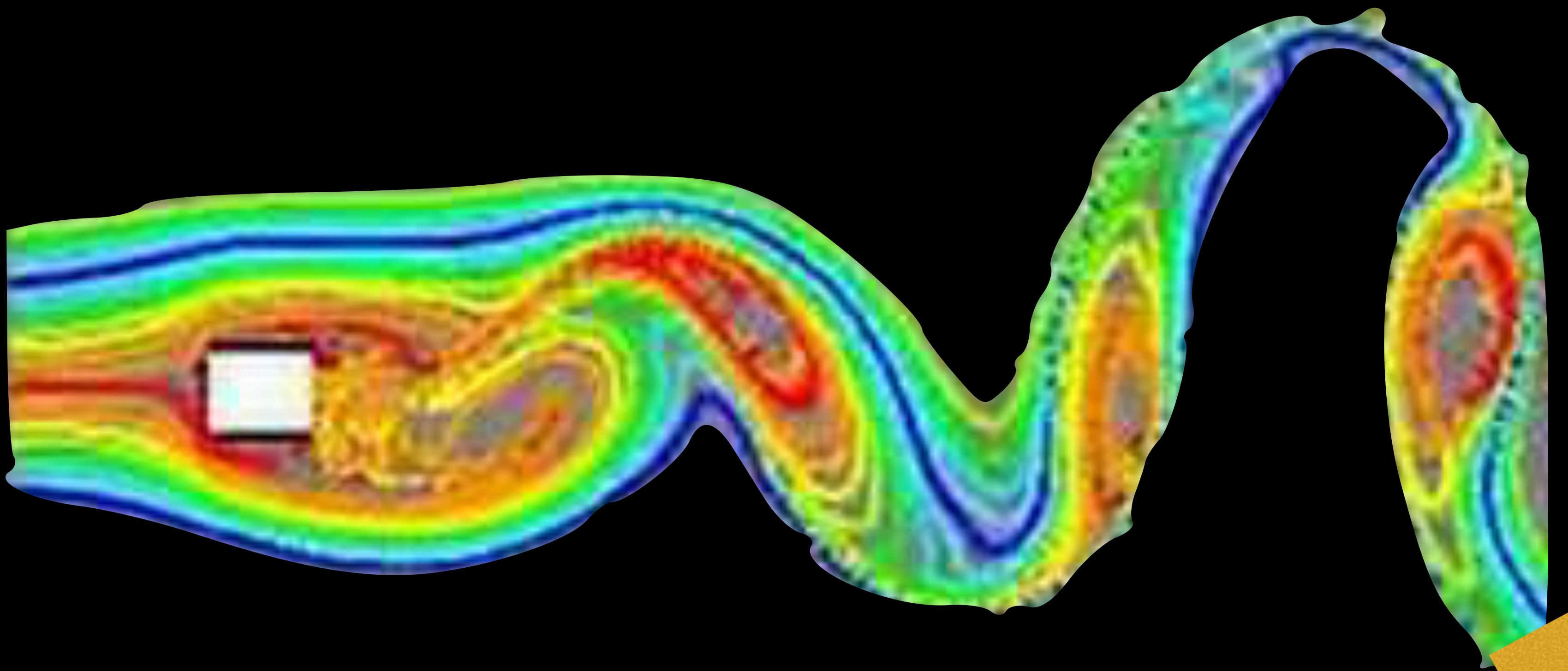
Plato

Rene Descartes

Deductive Reasoning



$se(3) \times se^*(3)$
elastic coupling



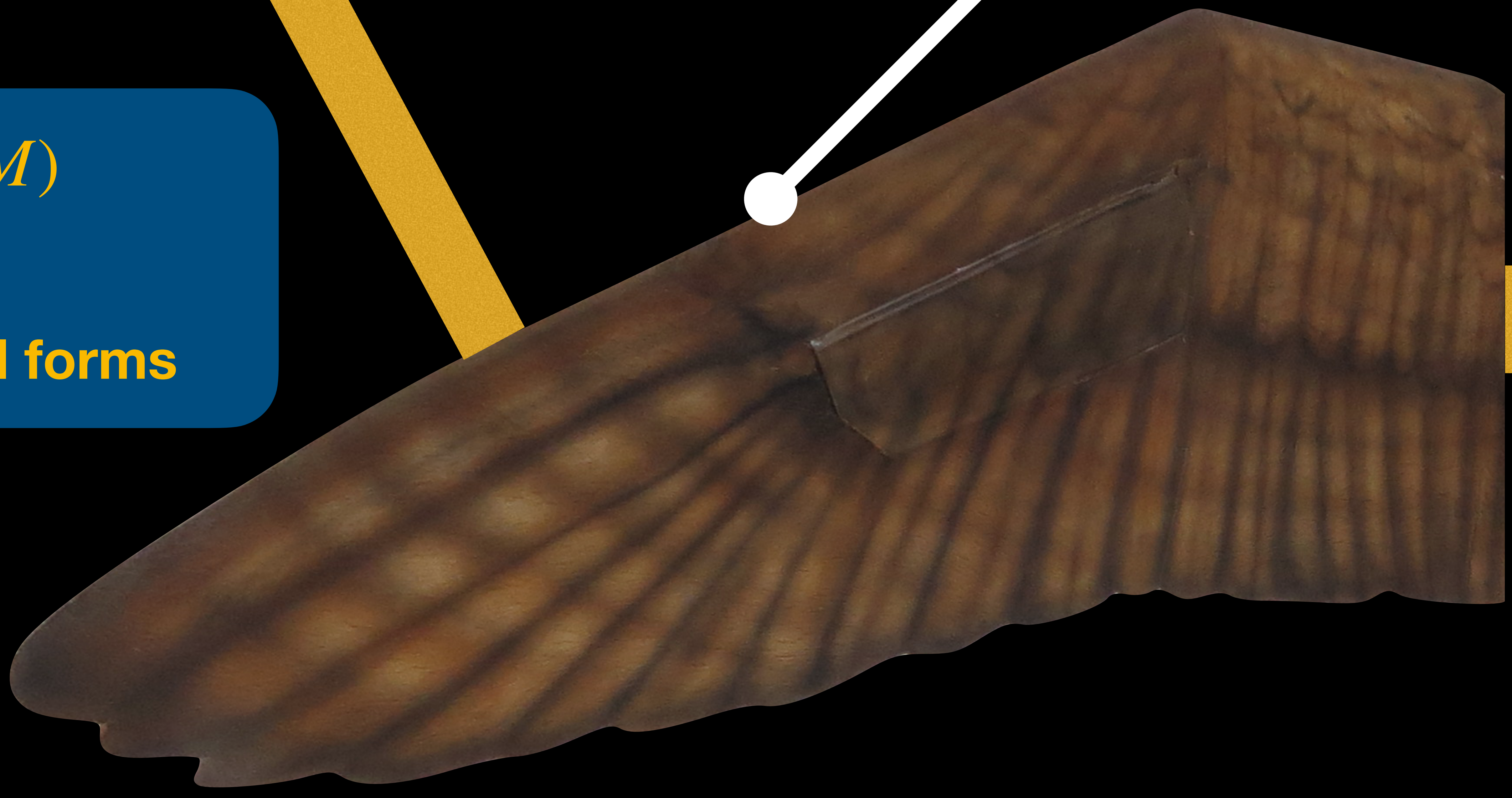
$$\rho\left(\frac{\partial v}{\partial t} + v \bullet \nabla v\right) = -\nabla p + \mu \Delta v + \delta F$$

Variable Boundary W

$\Omega^{n-1}(\partial M) \otimes \Omega^{1}(M)$

$\Omega^{0}(\partial M) \otimes \Gamma(TM)$

Tensor valued differential forms



$se(3) \times se^{*}(3)$

Structures Drive Progress in Mathematics and Science

$\mathbb{R}$ as a **set**

$(\mathbb{R}, +)$ as a **group**

$(\mathbb{R}, +, \bullet)$ as a **field**

$\mathbb{R}^n$ as a set

$(\mathbb{R}^n, \boxplus, \boxtimes, (\mathbb{R}, +, \bullet))$ as a **vector space**

$(\mathbb{R}^n, \boxplus, \boxtimes, (\mathbb{R}, +, \bullet), \mathrm{Id})$ as a **Hilbert space**

Be Aware of Coordinates

Indiscriminate use $\mathbb{R}^n$ $\sim$ Danger of not understanding needed structures

Understanding Concepts $\Leftrightarrow$ Understanding Needed Mathematical Structures

"A gentleman or a lady only chooses a basis if he/she must."

Mathematics is likely the **most beautiful creation of the human mind** and the **most powerful tool for engineers**

Abstraction:

get rid of what does **NOT** matter

J. H. POINCARE

"Mathematics is the art of giving the same name to different things."

About Port Based Thinking and University curricula

NOW

(Bad) Linear Algebra

(Little) Mechanics

(Little) Electromagnetism

(Little) Fluid Dynamics

A MUCH MORE EFFICIENT AND EFFECTIVE WAY

Proper Linear Algebra

Basic Concepts of Differential Geometry & Exterior Calculus

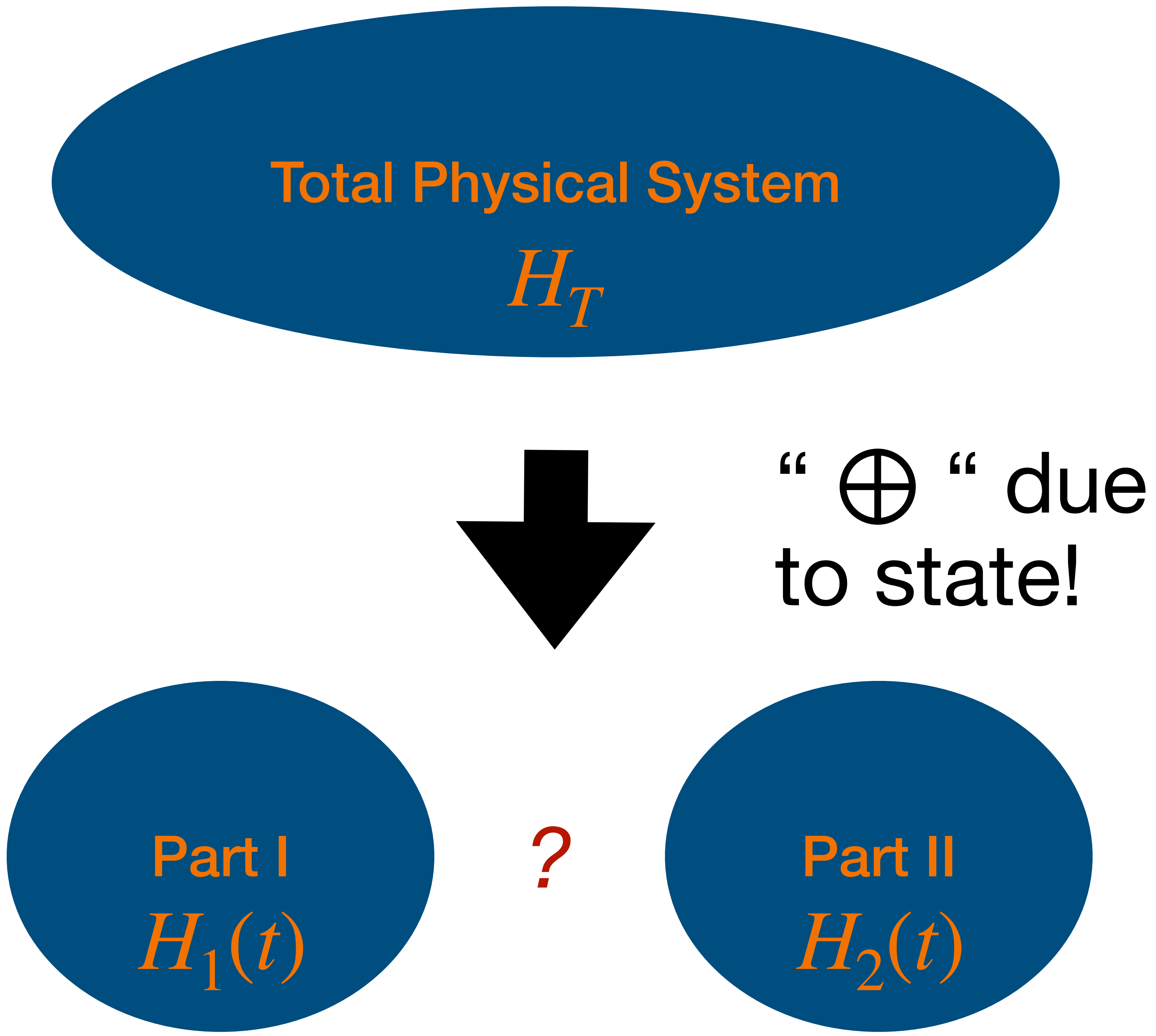
Port Based Modeling

Mechanics Electromagnetism Fluid Dynamics

Decomposition in Physics
and
Power Ports

Decomposition

Energy is the glue of physics: Energy Decomposition



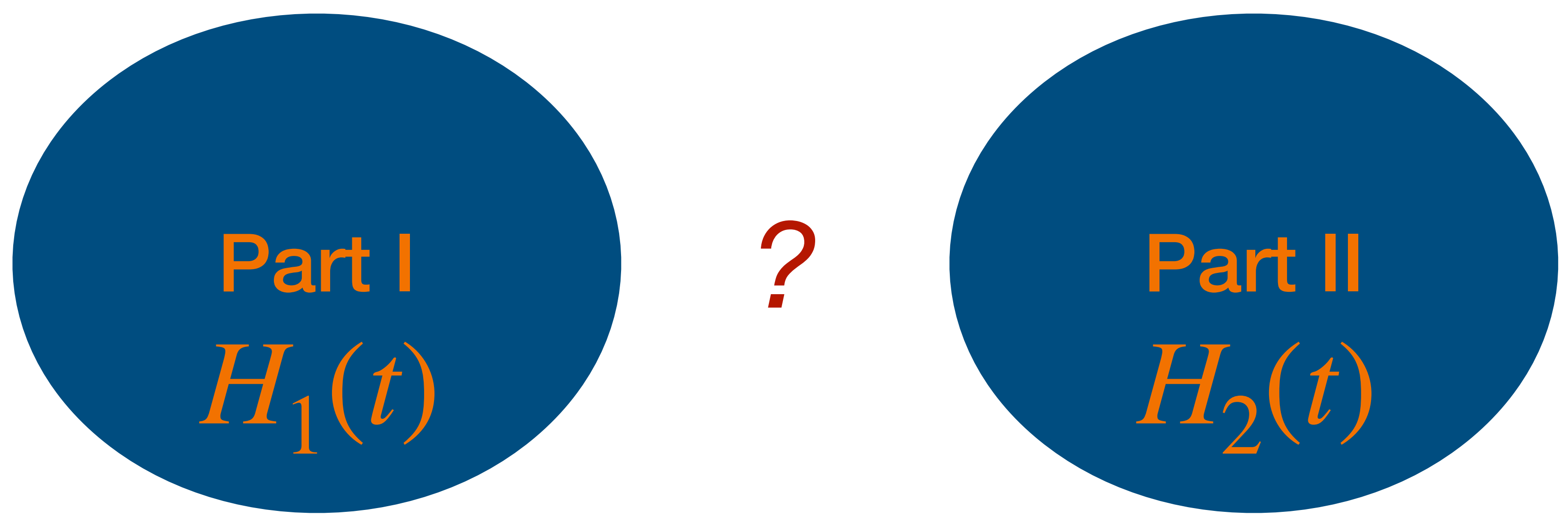
How to describe interconnection to ensure the requirement?

The (classical) Physical System under consideration will have an Energy $H(t) \in \mathbb{R}$ content at each instant of time, independent of its Physical nature! If the system is not interacting with anything else we have that $H(t) = H_T$ is constant.

Splitting the Physical System in two, each of the parts will have an Energy $H_1(t), H_2(t) \in \mathbb{R}$ and we want that for the first principle of thermodynamics, at all times $H_1(t) + H_2(t) = H_T$

Decomposition

Energy is the glue of physics: Energy Decomposition



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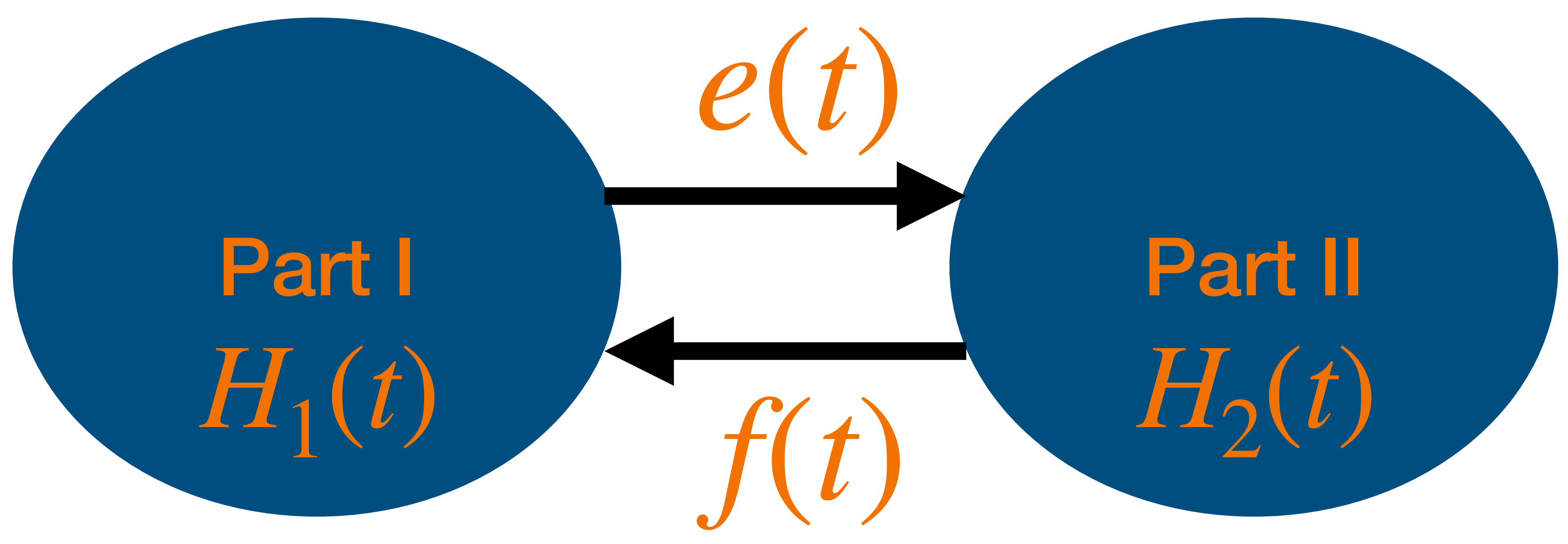
$$H_1(t) + H_2(t) = H_T \Rightarrow \boxed{\dot{H}_1(t) = -\dot{H}_2(t)}$$

Power Transfer!



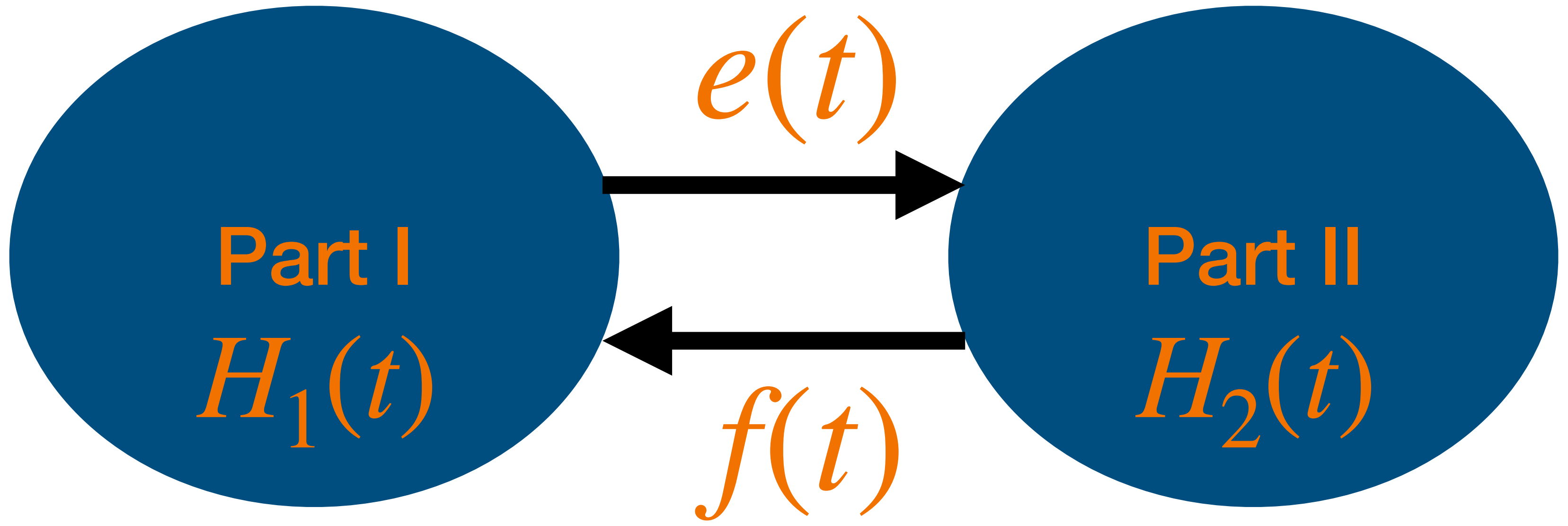
For Physical Reasons, the two parts will influence each other (bilaterally). Let's name $f(t)$ the influence of II on I and $e(t)$ the influence of I on II.

What is "the nature" of $e(t)$ and $f(t)$?

(Newton's third law: action-reaction)

Decomposition

Energy is the glue of physics: Energy Decomposition



$$H_1(t) + H_2(t) = H_T \Rightarrow \boxed{\dot{H}_1(t) = -\dot{H}_2(t)}$$

Power Transfer!

$f(t), e(t)$ should characterise the interaction at each instant of time t and should also take care that they represent a **scalar** power transfer as desired and some extra structure to satisfy **Kirchhoff's laws**.

Solution

$f(t), e(t)$ should have **the nature of a vector and a co-vector** and the **natural** application of one on the other $e(f)$ will result in a **scalar which will have the value of the power transfer** at that time!

$$f(t) \in \mathscr{V}, e(t) \in \mathscr{V}^* \Rightarrow e(f)(t) \in \mathbb{R}$$

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Finite Dimension

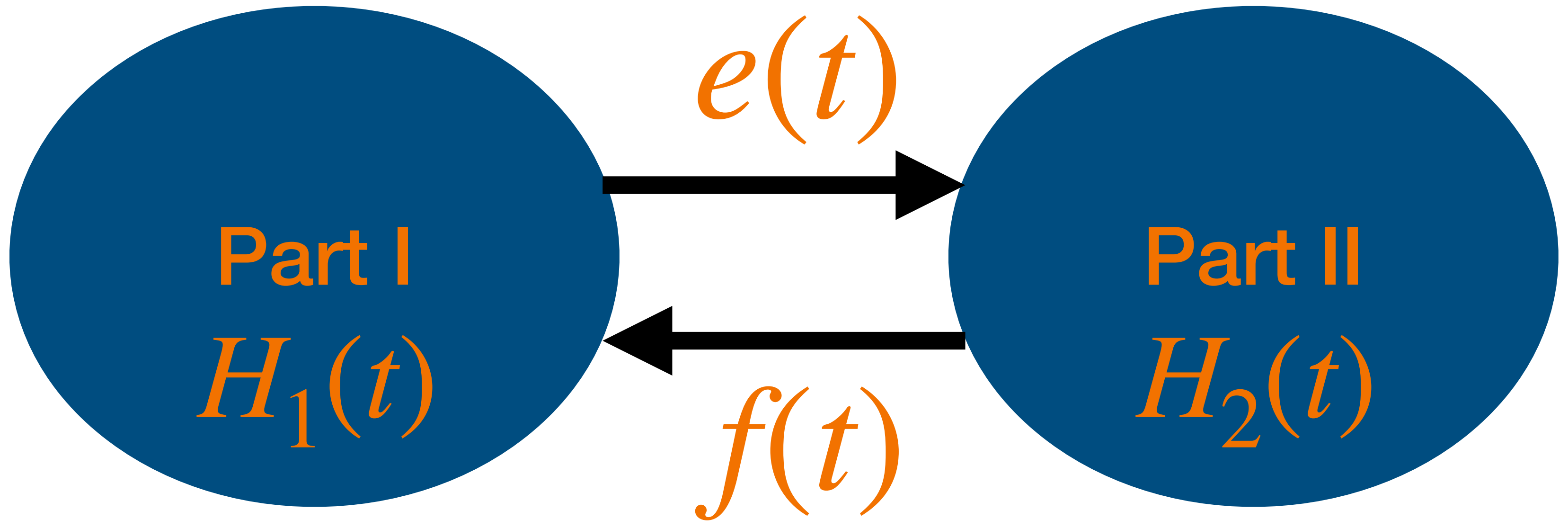
electrical circuits $\mathscr{V} \sim \mathbb{R}$

multibody dynamics $\mathscr{V} \sim se(3)$

Infinite Dimension

$\mathscr{V} \sim \Omega^k(M)$ **No strain**

$\mathscr{V} \sim \Omega^k(M; T^q_p M)$ **General Case of Continuous mechanics**



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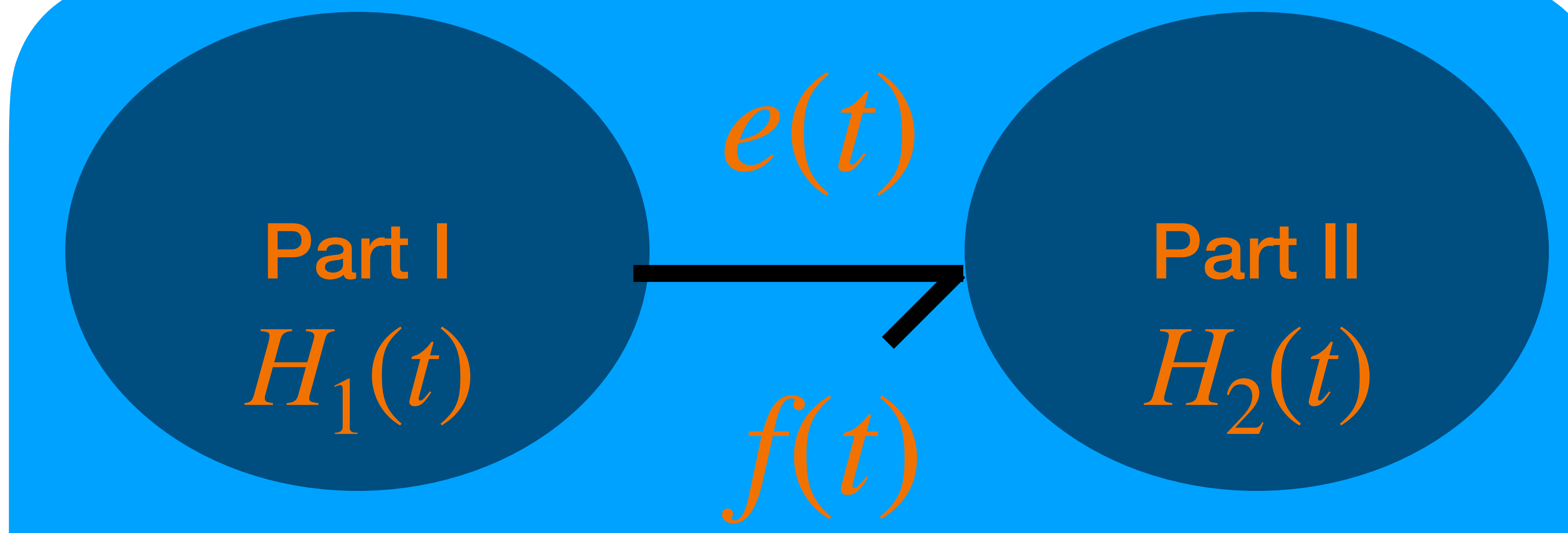


$$e_t := e(t), f_t := f(t)$$

$$P_{1\to 2}(t) = e_t(f_t)$$

And this is properly defined for ANY possible imaginable vector space!!

ANY field in Physics can be described therefore this way!!

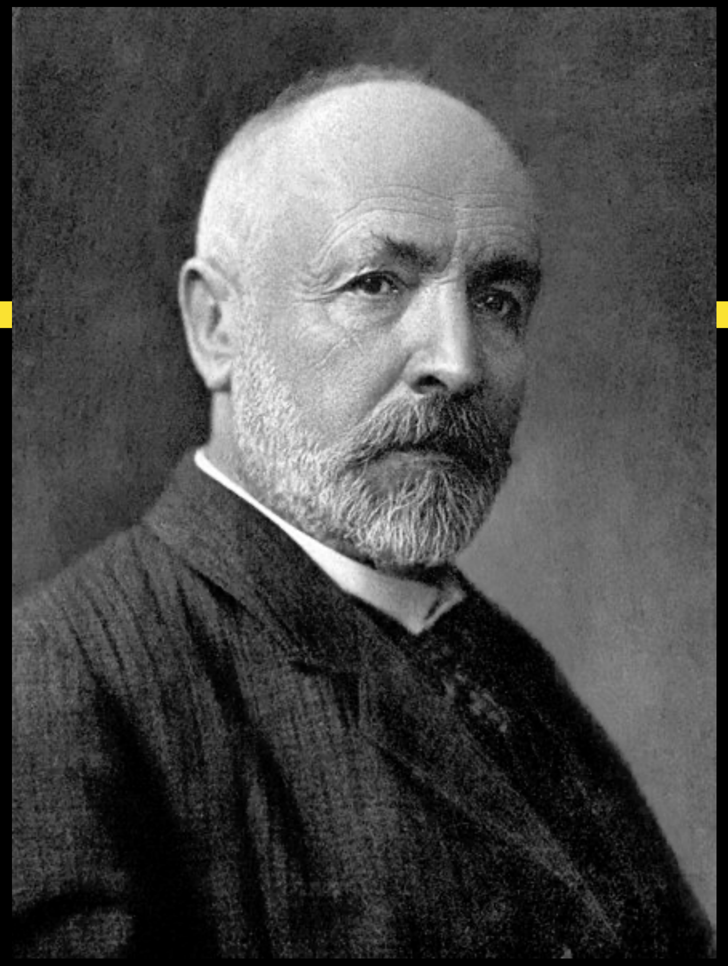
On Board
The geometry of inputs and outputs

On Board

- What is an **input**?
- What equations make sense for an **input of a n.l. system** ?
- The need for **extra structure**

The Mathematics Crisis in 1900

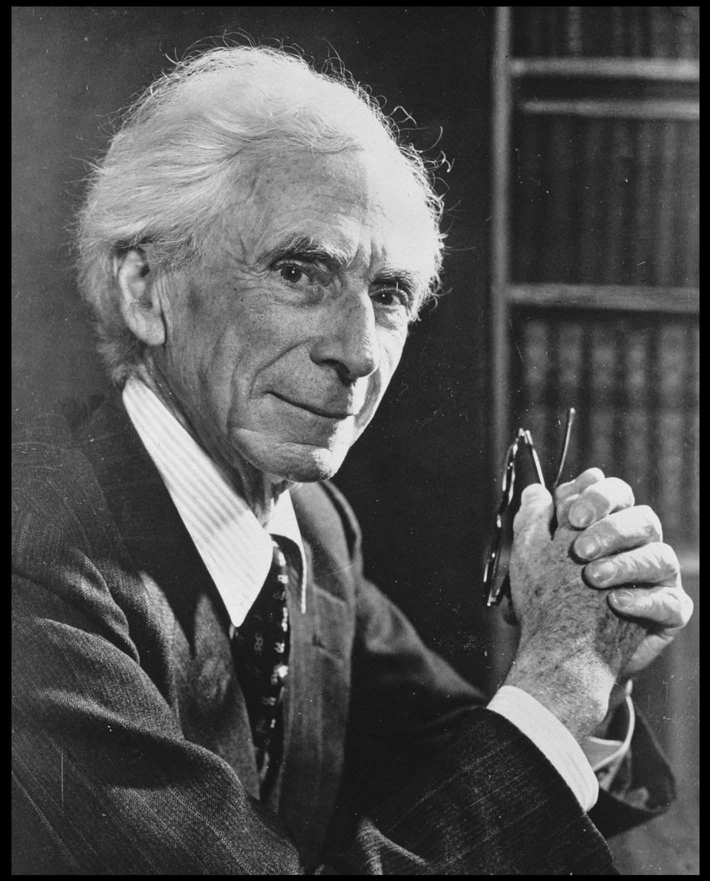
Naive
Set theory
and
Infinities



Georg Cantor

(1845-1918)

Paradox



Bertrand Russel

(1872 –1970)

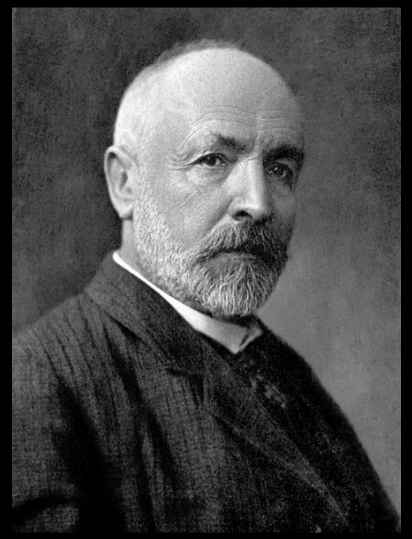
Vector Spaces

NO-Extra Structure

Duality -> Topology

Henri Poincaré
(1854-1912)

Intuitionist



Georg Cantor
(1845-1918)

1900
Mathematics Crisis

Vector Spaces

Additional Structure

Inner Product -> Geometry

David Hilbert
(1862-1943)

Formalist

“Duality” in Mathematics

Duality (for finite dimensional case)

$(\mathscr{V}, +, \bullet)$ Vector Space

$e_1, \ldots, e_n \in \mathscr{V}$ Basis

$(\mathscr{V}^*, \boxplus, \boxtimes)$ Dual vector Space

$e^1, \ldots, e^n \in \mathscr{V}^*$ Dual Basis

$$\mathbf{v} = \begin{pmatrix} e_1 & \ldots & e_n \end{pmatrix} \underbrace{\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}}_{\in \mathbb{R}^n}$$

$$\mathbf{F} = \underbrace{\begin{pmatrix} F_1 & \ldots & F_n \end{pmatrix}}_{\in \mathbb{R}^n} \begin{pmatrix} e^1 \\ \vdots \\ e^n \end{pmatrix}$$

Contravariant

Covariant

(Co=Variant with vectors)

$$\langle e^i \,|\, e_j \rangle := e^i(e_j) = \delta^i_j$$

INVARIANT!!!

$$\langle \mathbf{F} \,|\, \mathbf{V} \rangle := \mathbf{F}(\mathbf{V}) = F_i v^i$$

Bra Ket

Forces Transform <u>Differently</u> than velocities !

In General $A \neq (A^{-1})^T$!

No Inner Product

No Hilbert Structure

Vectors are different than Dual Vectors: they transform differently!

“Pairing” in Mathematics

Pairing and Duality

$\mathscr{A}$ $\mathscr{B}$ $p : \mathscr{A} \times \mathscr{B} \to \mathbb{R}$

Vector Spaces $\Rightarrow$ Dual Vector Spaces

$\mathscr{V} \Rightarrow \mathscr{V}^*$

DUALITY $\Rightarrow$ **NO Extra Structure**

$$< | >: \mathscr{V} \times \mathscr{V}^* \to \mathbb{R}; (f, e) \mapsto e(f)$$

Vector Spaces Vector Spaces

$\mathscr{V}$ $\mathscr{V}$

Extra Structure !!

$$< , >: \mathscr{V} \times \mathscr{V} \to \mathbb{R}; (v_1, v_2) \mapsto M(v_1, v_2)$$

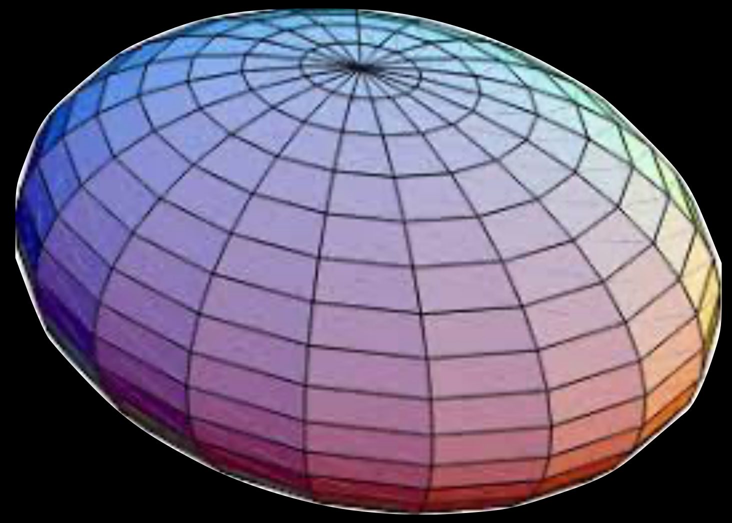
Hilbert Structure

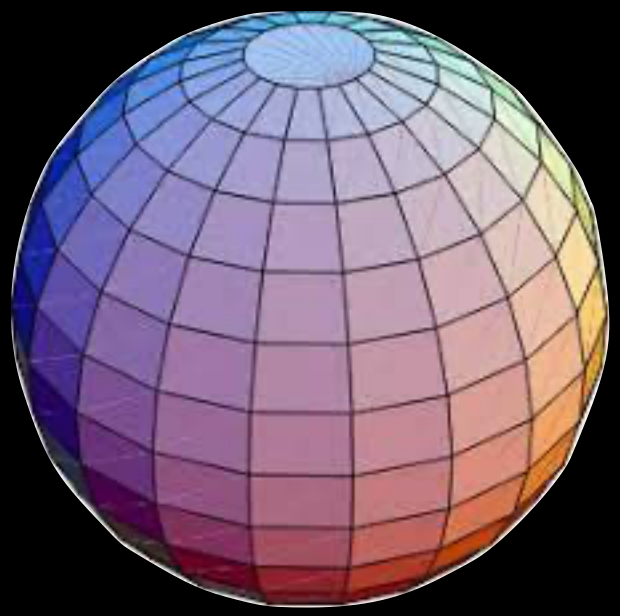
About Topology and Geometry

Topology Versus Geometry

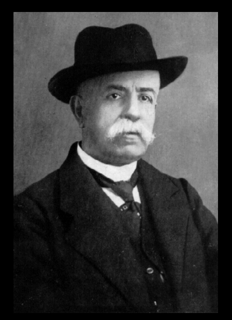
Topology

Neighbourhoods of points
No concept of Distance/Metric

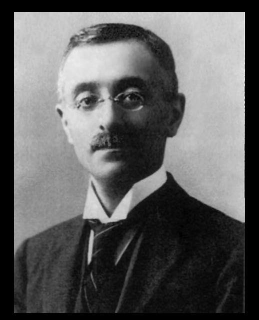
 = 

 = 

Pointcarre Conjecture
Ricci Flow: $\dot{g} = -2\text{Ric}$



Gregorio
Ricci-Curbastro



Tullio
Levi-Civita

Tensor Calculus

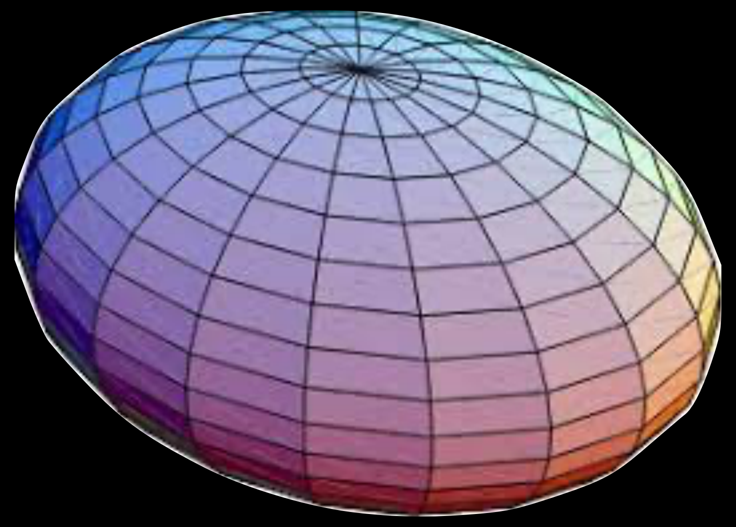
Shiing
Shen
Chern

Carl
Friedrich
Gauss

Pierre
Ossian
Bonnet

Geometry

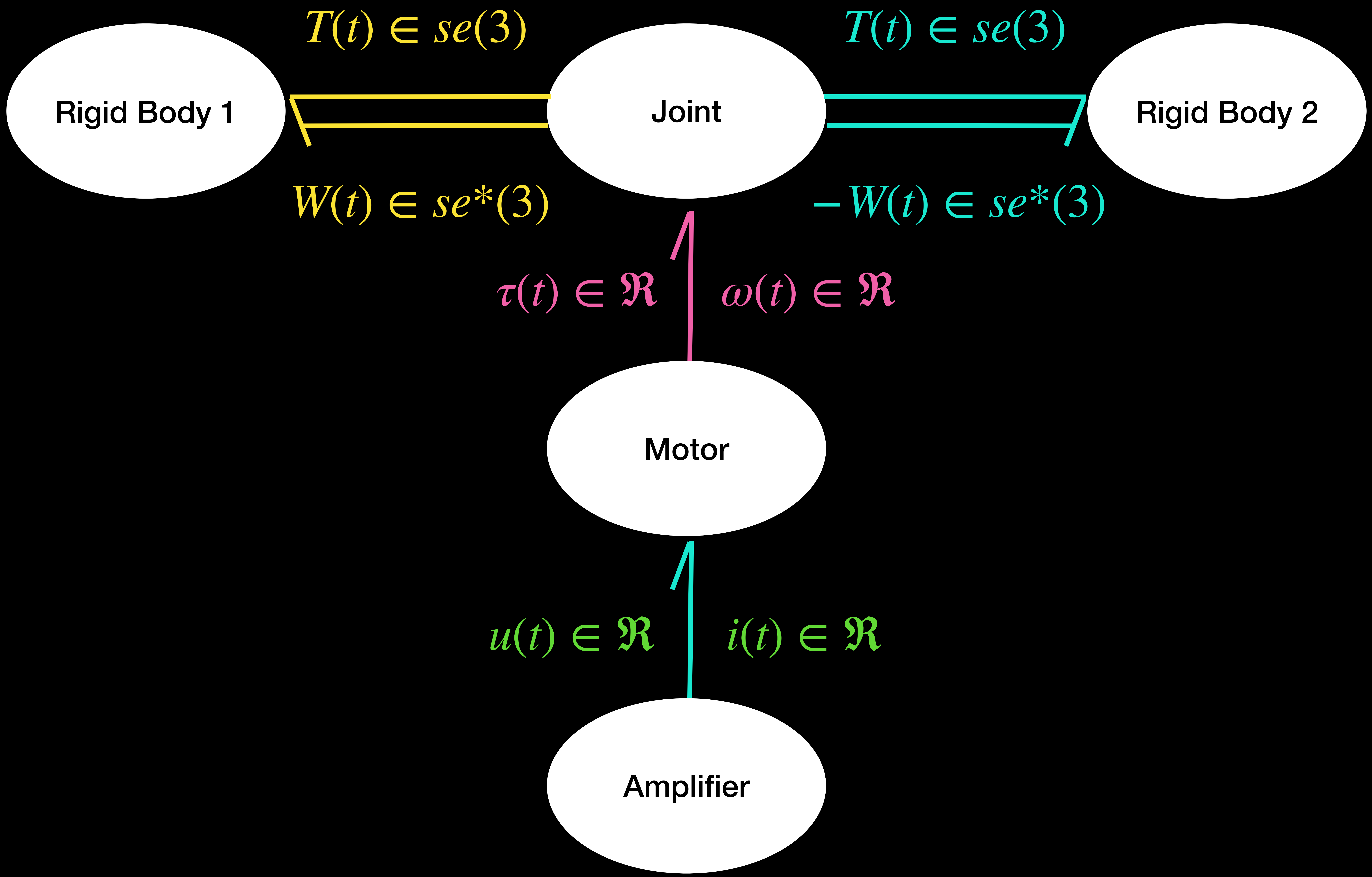
Shape
Concept of Distance/Metric present

 $\neq$ 

The Geometry & Topology of Power-Ports

Generalising “voltages and currents”

Example



Rigid Body 1
$T(t) \in se(3)$
Joint
$T(t) \in se(3)$
Rigid Body 2
$W(t) \in se^*(3)$
$-W(t) \in se^*(3)$
$\tau(t) \in \Re$
$\omega(t) \in \Re$
Motor
$u(t) \in \Re$
$i(t) \in \Re$
Amplifier

Finite Dimensional Case

ODEs and DAEs

Flows and Efforts are maps from time to a vector space and its dual respectively

$$f : T \to V \qquad\qquad e : T \to V^*$$

The power is then **WITHOUT any additional structure** $e(f)$

In multi-bodies V, V^* would be Lie-algebras representing Twist and Wrenches for example. **This is GENERAL and possible for ANY PHYSICAL finite dimensional system**

What about infinite dimensional Systems (fields)

in a covariant way which makes

EXPLICIT

what is geometry and topology ?

Infinite Dimensional Case

PDE and Fields

We have a domain on which fields are defined, a manifold $\mathscr{M}$, and its boundary $\partial\mathscr{M}$: we want to allow energy through the boundary: we have *power-port* ***densities***.

$$\partial\mathscr{M} = \partial\mathscr{M}$$

We need a "language" able to express all concepts:

- **in all dimensions**
- **coordinate-free**
- **applicable to all physical domains**
- **possibly in curved spaces**
- **making explicit what is topology and geometry**

Exterior Calculus
&
(T.V./L.A.V.) Differential Forms

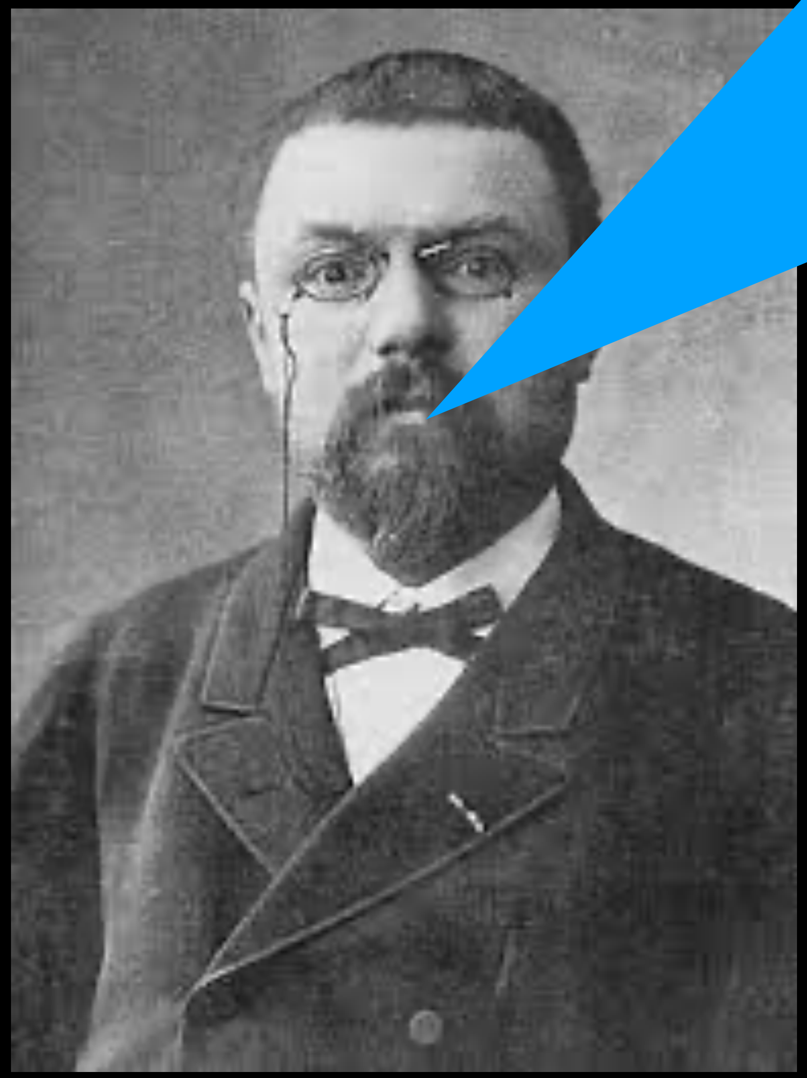
Integration is a **topological operation**:

it is NOT dependent on a metrical structure!

Riemann integration is a

“bad”/“old”

way to talk about integration!



Hermann Grassmann (1809-1877)

Élie Cartan (1869-1951)

Differential Forms

Generalising “evaluate a function in a point” to any dimension

The coordinate invariant way to talk about integration

Differential Forms

Intuitive, minimalistic ideas

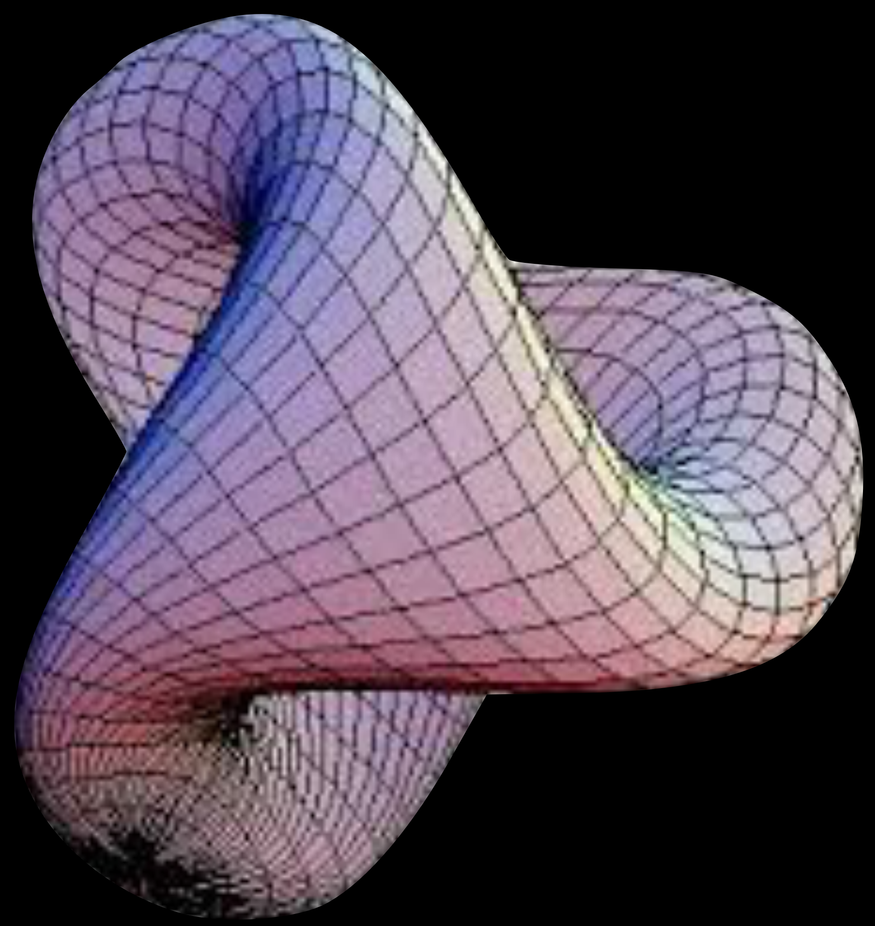
A k-form $\omega^k \in \Omega^k(\mathscr{M})$ is a **completely antisymmetric** k-linear field

$(k \in \mathbb{N})$

$$\omega^k_{x,t}(v_1, \ldots, v_k) \in \mathbb{R}$$

scalar valued

(For deformations and shear stress Tensor Valued forms are needed!!)



is a field in space-time

multilinear on tangent vector spaces at $x \in \mathscr{M}$

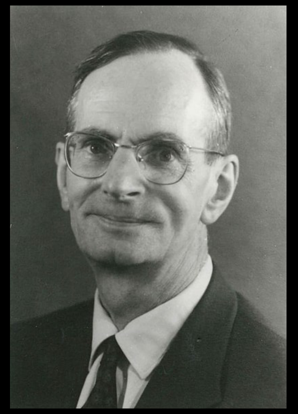
Forms: Measure of k-volumes

Due to the full antisymmetry:

k-forms ‘’measure k-volumes”of **k-dimensional** spaces **INTRINSICALLY**

Integration IS

Topological Pairing

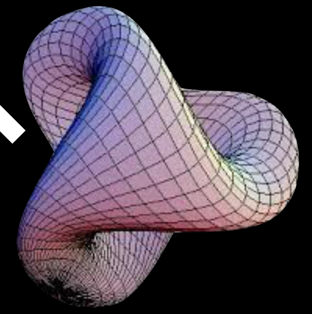


Georges de Rham
(1903-1990)

$$< \omega^k \,|\, \mathscr{M}^k >:= \int_{\mathscr{M}^k} \omega^k \quad \in \mathbb{R} \longleftarrow$$

scalar valued

Evaluate **k-Form** On **k dim-manifold**



Evaluate		**On**	
0-form	***function***	**point**	**0d-manifold**
1-form		**line**	**1d-manifold**
2-form		**surface**	**2d-manifold**
3-form		**volume**	**3d-manifold**

$\Rightarrow$

Generalisation of
evaluation of a function on a point
to ANY dimension

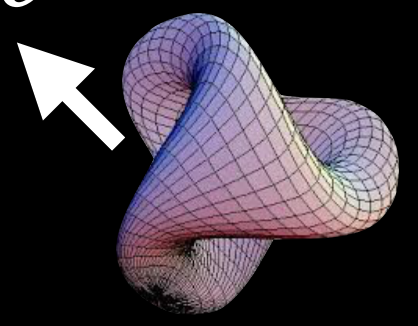
Forms Versus Pseudo-Forms

For dimensions higher than 0, orientation matter!

Topological Pairing

$$< \omega^k \,|\, \mathscr{M}^k >:= \int_{\mathscr{M}^k} \omega^k$$

Evaluate **n-Form** On **n d-manifold**



	Evaluate	On	
0-form	*function*	point	0d-manifold
1-form		line	1d-manifold
2-form		surface	2d-manifold
3-form		volume	3d-manifold

For manifold of dimensions > 0, we can define orientation (+1,-1)

$< \omega^k \,|\, -\mathscr{M}^k > = - < \omega^k \,|\, \mathscr{M}^k > \Rightarrow \omega^k$ *(**True**)* Form= **Straight =Inner Oriented**

$< \omega^k \,|\, -\mathscr{M}^k > = < \omega^k \,|\, \mathscr{M}^k > \Rightarrow \omega^k$ ***Pseudo*** Form= **Twisted** = **Outer Oriented**

Exterior Algebras Operators on Forms

- There is an operator $\wedge$ called the wedge-product which returns a (k+l)-form when applied to a k-form and a l-form:

$$\omega^k \wedge \omega^l \mapsto \omega^{k+l}$$

- There is a differential "**topological**" operator d (graded diff.) which maps a k form to a k+1 form

$$d(\omega^k) \mapsto \omega^{k+1}$$

$$\int_{\mathscr{M}^n} d\omega = \int_{\partial \mathscr{M}^{n-1}} \mathrm{tr}\ \omega$$

Stokes Theorem

- There is a "**metric**" operator $\star$ which maps a k **true**-form to a (n-k) **pseudo**-form.

$$\star(\omega^k) \mapsto \bar{\omega}^{n-k}$$

true-form $\mapsto$ pseudo-form
pseudo-form $\mapsto$ true-form

$$< \omega_1^k, \omega_2^k > := \int_{\mathscr{M}^n} \omega_1^k \wedge (\star \omega_2)^{n-k}$$

Hilbert Inner Product
positive definite!

Forms: Pairings

Metric/Energy Structure **Needed**

$$< \omega_1^k, \omega_2^k >:= \int_{\mathscr{M}^n} \omega_1^k \wedge (\star\omega_2)^{n-k}$$

Hilbert
Inner Product

No **Metric/Energy** Structure **Needed**

$$< \omega_1^k \,|\, \bar{\omega}_2^{n-k} >:= \int_{\mathscr{M}^n} \omega_1^k \wedge \bar{\omega}_2^{n-k}$$

Poincaré
Dual Product

Integration by Parts

One unique expression

$$\int_{\mathscr{M}^n} d\omega = \int_{\partial\mathscr{M}^{n-1}} \mathrm{tr}\ \omega$$

Stokes Theorem

Leibniz Rule for d

$$d(\omega^k \wedge \omega^l) = d\omega^k \wedge \omega^l + (-1)^k \omega^k \wedge \omega^l$$

$$k + l + 1 = n$$

$$\int_{\mathscr{M}^n} d(\omega^k \wedge \omega^l) = \int_{\partial\mathscr{M}^{n-1}} \mathrm{tr}\omega^k \wedge \mathrm{tr}\omega^l = \int_{\mathscr{M}^n} d\omega^k \wedge \omega^l + (-1)^k \int_{\mathscr{M}^n} \omega^k \wedge d\omega^l$$

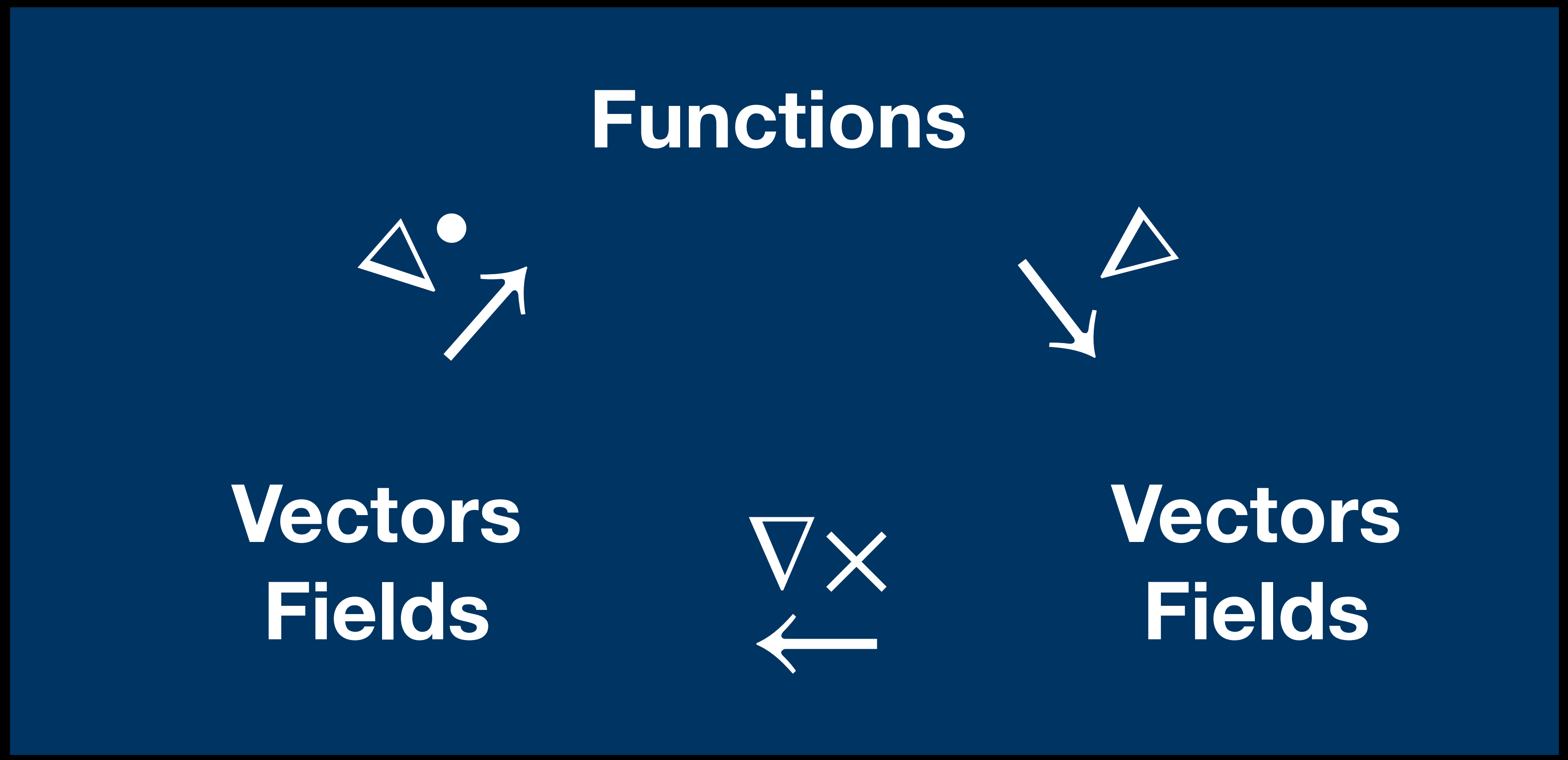
$$\int_{\mathscr{M}^n} d\omega^k \wedge \omega^l = (-1)^{k+1} \int_{\mathscr{M}^n} \omega^k \wedge d\omega^l + \int_{\partial\mathscr{M}^{n-1}} \mathrm{tr}\omega^k \wedge \mathrm{tr}\omega^l$$

Integration by Part

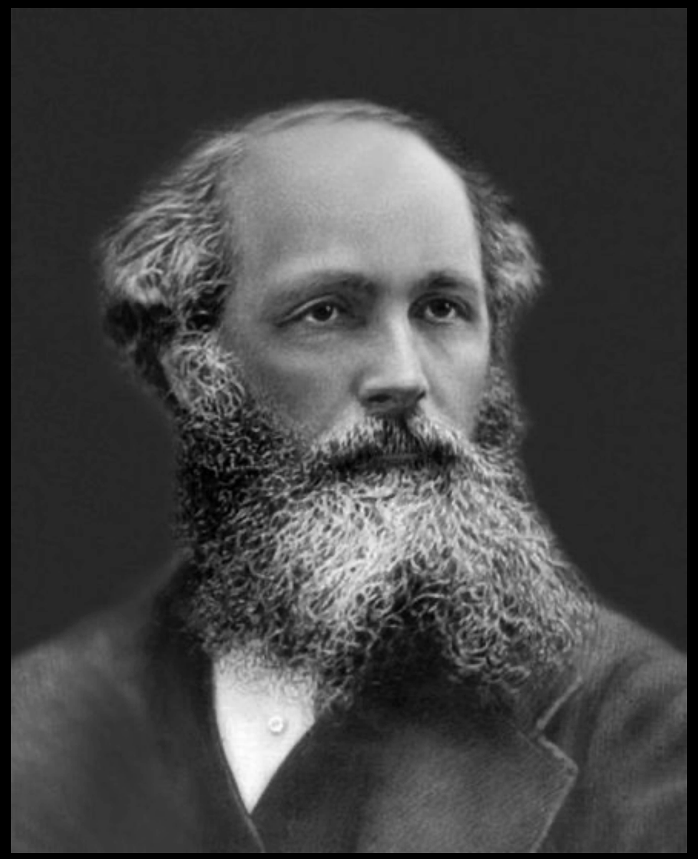
The Complex

The **Vector Calculus** Complex

$\mathcal{M}$ 3D Space



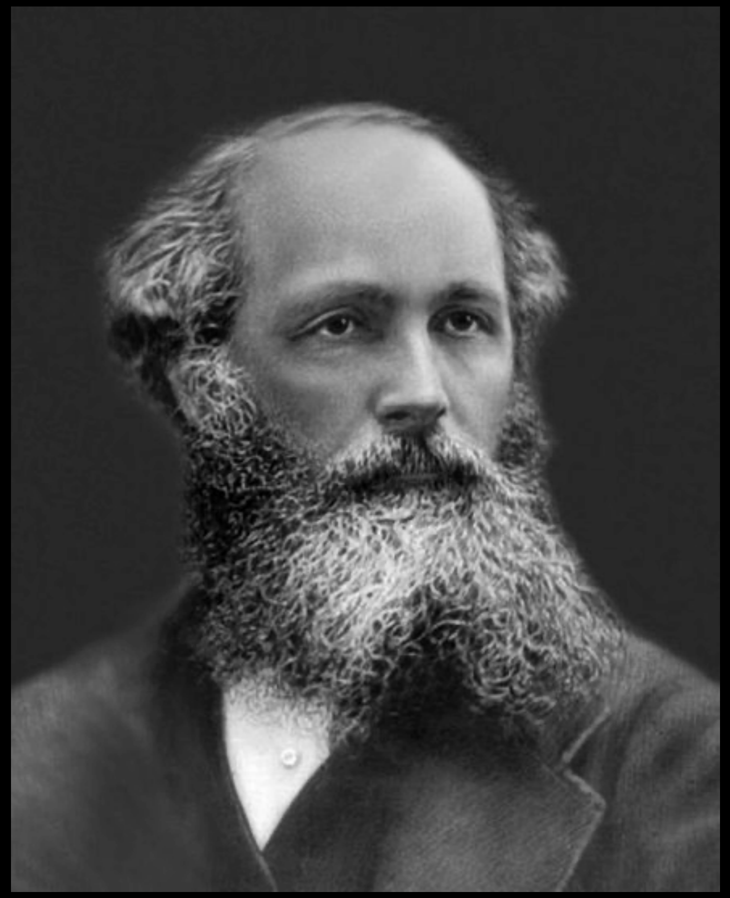
1873 reatise on Electricity and Magnetism



James Clerk Maxwell
(1831-1879)

Problems

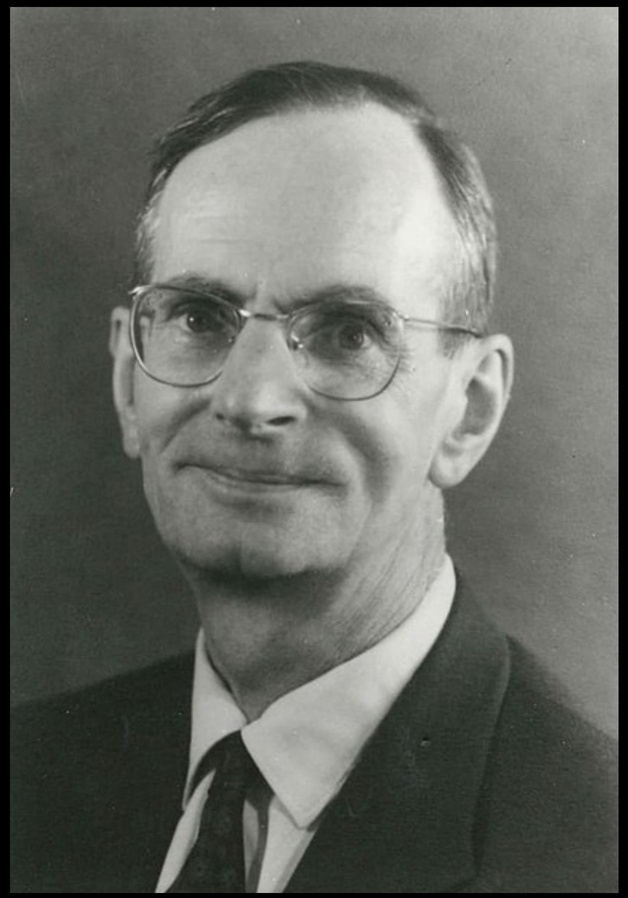
- No clear distinction between **topological properties** and **metrical properties**
- **Not** generalisable to **dimension higher than 3**
- **No** clear **information about orientation properties of objects**
- THE SAME operator d is represented with DIFFERENT operators (∇ , $\nabla \times$, $\nabla \bullet$)
- DIFFERENT objects are considered THE SAME: **Vector Fields or Functions**
- ..



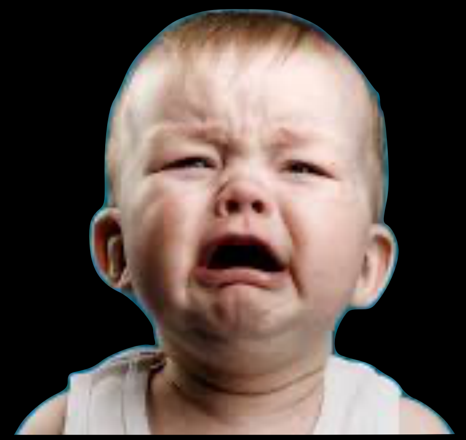
James Clerk Maxwell
(1831-1879)

Bernhard Riemann
(1826-1866)

Élie Cartan
(1869-1951)



Georges de Raham
(1903-1990)



We still

- teach Vector Calculus rather than Grassman/Cartan Exterior Calculus
- teach Riemann integration rather than de Raham Topological Pairing

1850 Time Line 2025

Hermann Grassmann
(1809-1877)

Henri Pointcarre
(1854-1912)

Vector Calculus — 1873 Maxwell Publication
"Treatise on Electricity and Magnetism"

2025-1873=152!

Riemann Integration — 1854 Riemann Lecture
Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe

2025-1854=171!

The Complex

The **Vector Calculus** Complex

3D Space

$$\begin{tikzcd} & \text{Functions} \arrow[dr, "\nabla"] & \\ \text{Vectors Fields} \arrow[ur, "\nabla\cdot"] & & \text{Vectors Fields} \arrow[ll, "\nabla\times"] \end{tikzcd}$$

$$\text{Functions} \xrightarrow{\nabla} \text{Vectors Fields} \xrightarrow{\nabla\times} \text{Vectors Fields} \xrightarrow{\nabla\cdot} \text{Functions}$$

The Complex

The Vector Calculus Complex

$\mathscr{M}$ 3D Space

$$\text{Functions} \xrightarrow{\nabla} \text{Vectors Fields} \xrightarrow{\nabla\times} \text{Vectors Fields} \xrightarrow{\nabla\bullet} \text{Functions}$$

$$\text{Functions} \xleftarrow{\nabla\bullet} \text{Vectors Fields} \xleftarrow{\nabla\times} \text{Vectors Fields} \xleftarrow{\nabla} \text{Functions}$$

The Complex

The Exterior Calculus Complex

$$\omega^k \in \Omega^k(\mathscr{M})$$

ANY Manifold
from which points, lines, .. are taken

Forms

Functions $\xrightarrow{\nabla}$ Vectors Fields $\xrightarrow{\nabla\times}$ Vectors Fields $\xrightarrow{\nabla\bullet}$ Functions

$$\Omega^0(\mathscr{M}) \xrightarrow{d} \Omega^1(\mathscr{M}) \xrightarrow{d} \Omega^2(\mathscr{M}) \xrightarrow{d} \Omega^3(\mathscr{M}) \xrightarrow{d} \cdots$$

Geometry $\downarrow \star_0$ $\downarrow \star_1$ $\downarrow \star_2$ $\downarrow \star_3$

$$\Omega^n(\mathscr{M}) \xleftarrow{d} \Omega^{n-1}(\mathscr{M}) \xleftarrow{d} \Omega^{n-2}(\mathscr{M}) \xleftarrow{d} \Omega^{n-3}(\mathscr{M}) \xleftarrow{d} \cdots$$

Topology

Pseudo Forms

Functions $\xleftarrow{\nabla\bullet}$ Vectors Fields $\xleftarrow{\nabla\times}$ Vectors Fields $\xleftarrow{\nabla}$ Functions

Example

Maxwell Equations

Faraday's law $\quad \dot{B} = -\,d\,E$

Ampère's law $\quad \dot{D} = d\,H \qquad (-J)$

Electric Energy Density (3 form) = $E \wedge D$

Magnetic Energy Density (3 form) = $H \wedge B$

Poynting Vector (2 form) = $E \wedge H$

Electric Field Intensity 1-form

E

Magnetic Field Induction 2-form

B

$$\begin{tikzcd} \Omega^1(\mathscr{M}) \arrow[r, "d"] \arrow[d, "\star_1"] & \Omega^2(\mathscr{M}) \arrow[d, "\star_2"] \\ \Omega^2(\mathscr{M}) & \Omega^1(\mathscr{M}) \arrow[l, "d"'] \end{tikzcd}$$

$\epsilon \qquad \mu$

Electric Field Induction 2-form

D

Magnetic Field Intensity 1-form

H

ELECTRIC **MAGNETIC**

Power flow in infinite dimensional space

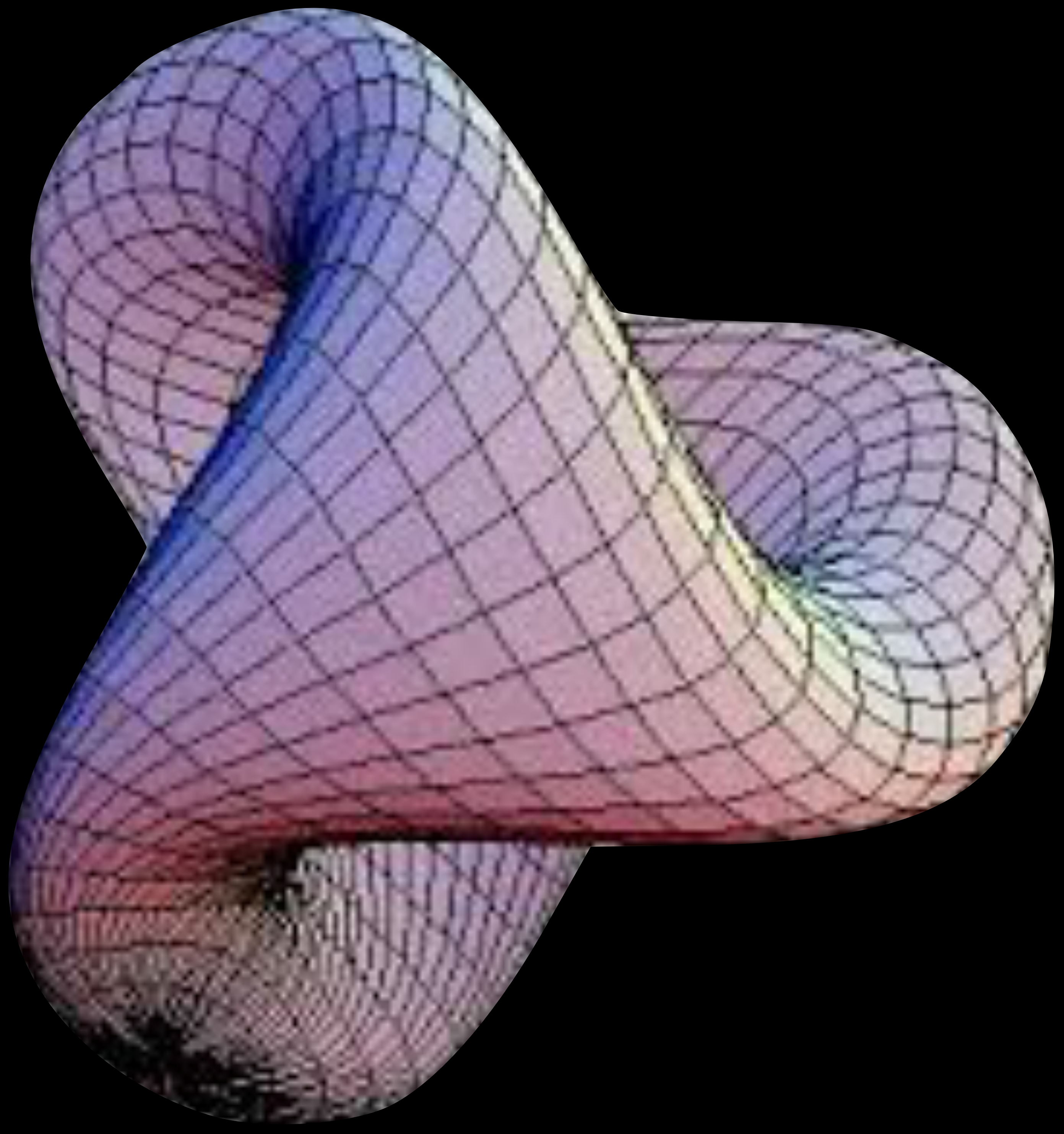
- In a distributed space, power needs to be integrated (for 3D $n = 3$):
  - Volume

$$P_V = \int_{\mathscr{M}^n} \omega_P^n$$

Power Densities

  - Surface

$$P_S = \int_{\partial \mathscr{M}^{n-1}} \omega_P^{n-1}$$



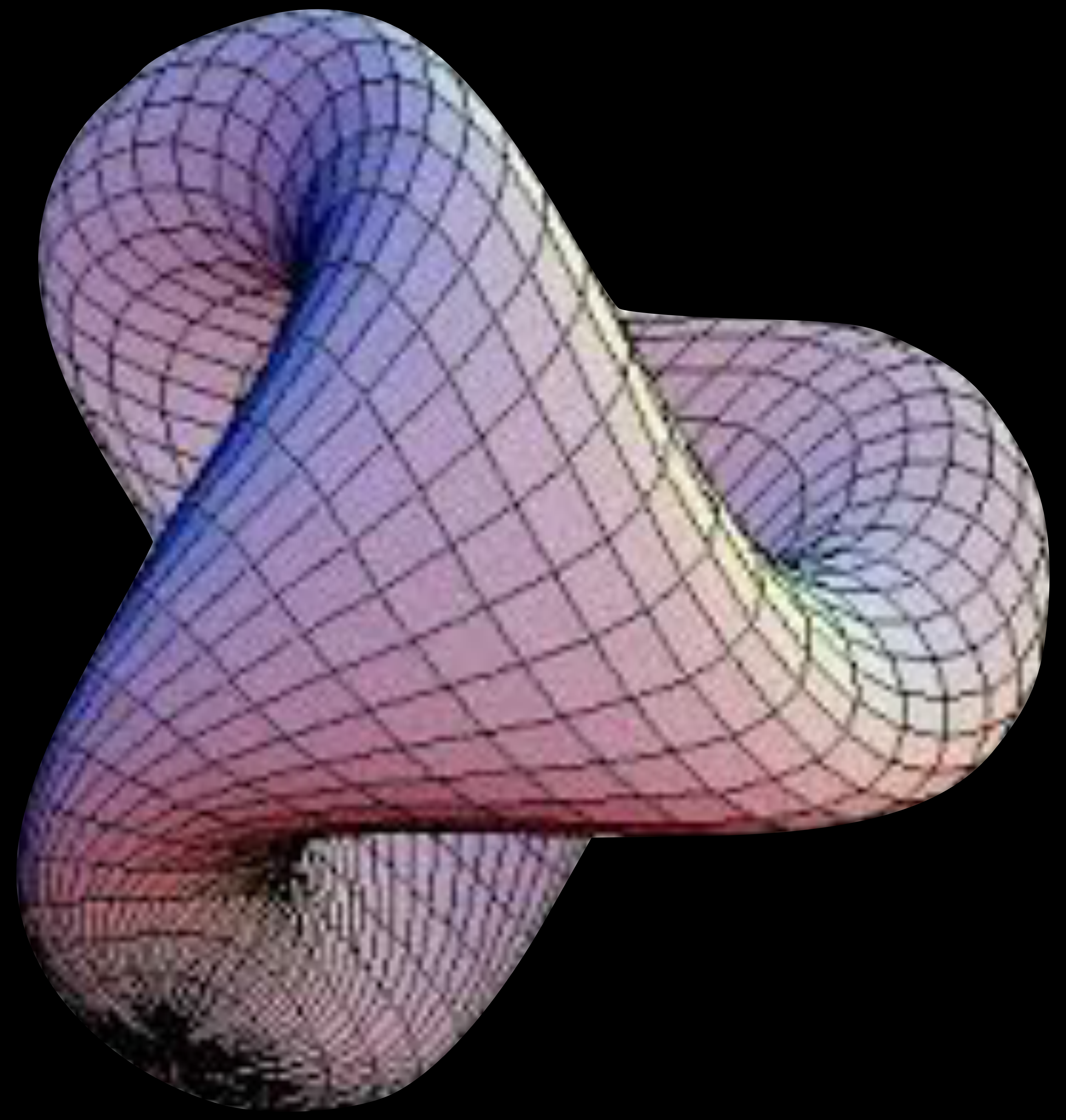
Efforts and Flows in infinite dimensional systems

$$P_V = \int_{\mathscr{M}^n} \omega_P^n = \int_{\mathscr{M}^n} \omega_e^k \wedge \omega_f^{n-k}$$

Efforts

Flows

$$P_S = \int_{\partial \mathscr{M}^{n-1}} \omega_P^{n-1} = \int_{\partial \mathscr{M}^n} \omega_e^k \wedge \omega_f^{n-1-k}$$



Natural Pairing, no extra structure needed!

Example: Poynting Vector in Electromagnetism $S^2 = E^1 \wedge H^1$

Example: Energy storage and continuity

$$\mathscr{H}^n(x^l(t))$$

Energy (density) n-form function of a **state l-form** function of time

$$H = \int_{\mathscr{M}^n} \mathscr{H}^n$$

Total energy contained in $\mathscr{M}^n$

$$\dot{H} = \int_{\mathscr{M}^n} \delta_x \mathscr{H}^{n-l} \wedge \dot{x}^l$$

Variational derivative

Energy Storage

Strong Form

$$\dot{\mathscr{H}}^n = d\Phi^{n-1} + \rho^n$$

Weak Form

$$\dot{H} = \int_{\mathscr{M}^{n-1}} \Phi^{n-1} + \int_{\mathscr{M}^n} \rho^n$$

Energy Continuity eq.

Example: Energy storage and motion

Energy Storage

$$\dot{H} = \int_{\mathscr{M}^n} \delta_x \mathscr{H}^{n-l} \wedge \dot{x}^l$$

Energy Continuity Equation

$$\dot{H} = \int_{\mathscr{M}^{n-1}} \Phi^{n-1} + \int_{\mathscr{M}^n} \rho^n$$

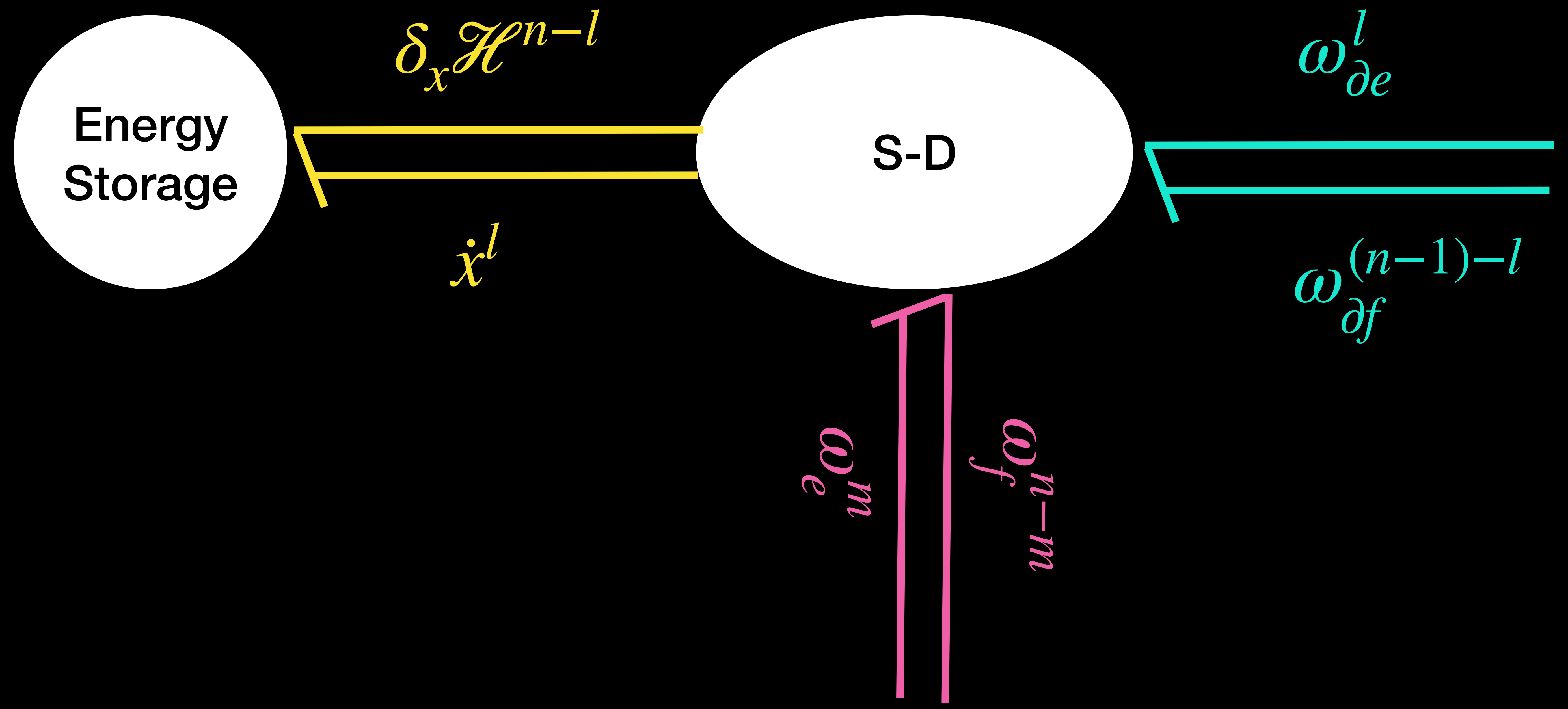
$$\Phi^{n-1} = \omega_{\partial e}^{k} \wedge \omega_{\partial f}^{(n-1)-k}$$

$$\rho^n = \omega_e^m \wedge \omega_f^{n-m}$$

$$\int_{\mathscr{M}^n} \delta_x \mathscr{H}^{n-l} \wedge \dot{x}^l = \int_{\mathscr{M}^{n-1}} \omega_{\partial e}^{l} \wedge \omega_{\partial f}^{(n-1)-l} + \int_{\mathscr{M}^n} \omega_e^m \wedge \omega_f^{n-m}$$

Stokes-Dirac Structure

$$\int_{\mathscr{M}^n} \delta_x \mathscr{H}^{n-l} \wedge \dot{x}^l = \int_{\mathscr{M}^{n-1}} \omega^l_{\partial e} \wedge \omega^{(n-1)-l}_{\partial f} + \int_{\mathscr{M}^n} \omega^m_e \wedge \omega^{n-m}_f$$



Working
on
Fluid Dynamics
and
Solid mechanics

We need more!

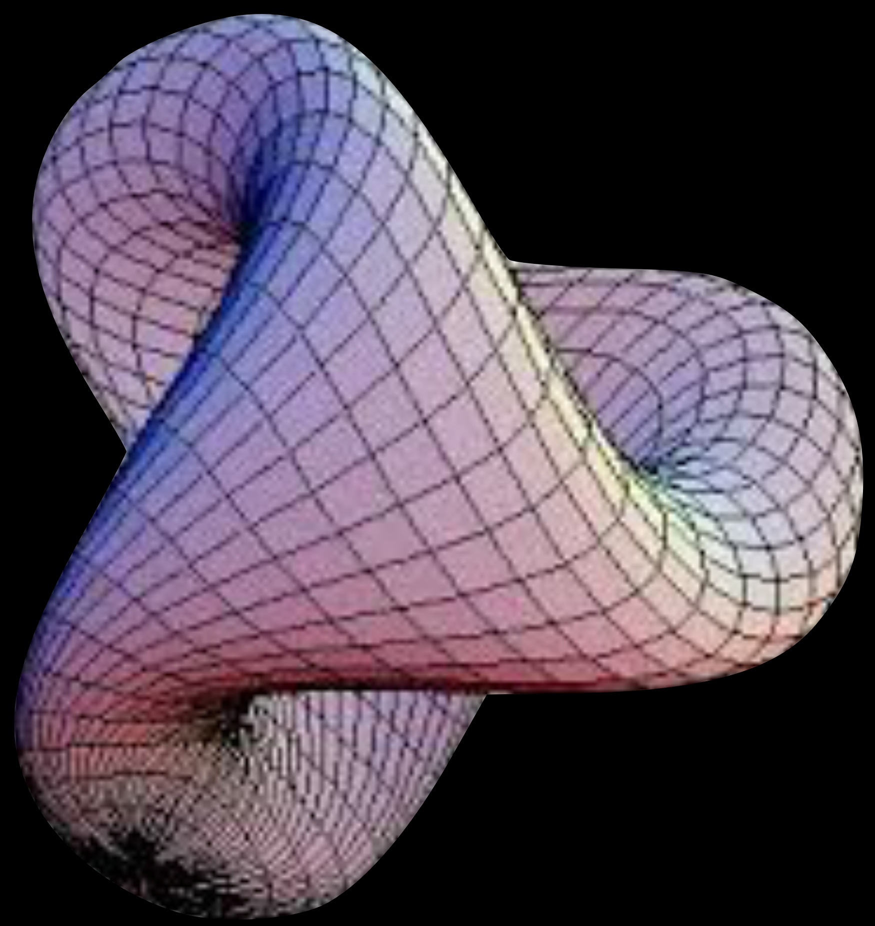
Differential Forms

Intuitive, minimalistic ideas

A k-form ω^k is a completely antisysimmetric k-linear field

$$\omega^k_{x,t}(v_1, \ldots, v_k) \in \Re$$

scalar valued



is a field in space-time

multilinear on tangent vector spaces at x

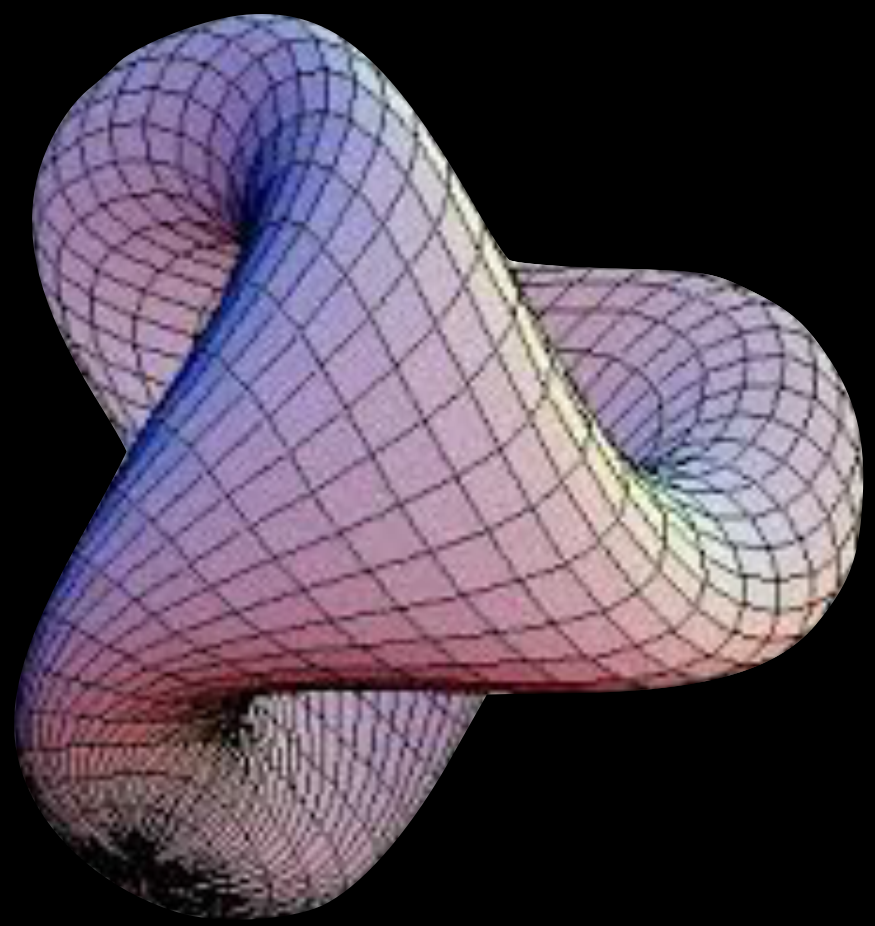
Tensor-Valued-Differential Forms

Intuitive, minimalistic ideas

A k-form ω^k is a completely antisysimmetric k-linear field

$$\omega^k_{x,t}(v_1, \ldots, v_k) \in T_x\mathscr{M}^j_i$$

tensor valued



is a field in space-time

multilinear on tangent vector spaces at x

Journal of Nonlinear Science (2023) 33:84
https://doi.org/10.1007/s00332-023-09945-7

Intrinsic Nonlinear Elasticity: An Exterior Calculus Formulation

Ramy Rashad[1] · Andrea Brugnoli[2] · Federico Califano[1] · Erwin Luesink[3] · Stefano Stramigioli[1]

Journal of Geometry and Physics 200 (2024) 105172

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journal homepage: www.elsevier.com/locate/geomphys

The principal bundle structure of continuum mechanics

Stefano Stramigioli

Journal of Nonlinear Science (2025) 35:35
https://doi.org/10.1007/s00332-025-10130-1

The Port-Hamiltonian Structure of Continuum Mechanics

Ramy Rashad[1,2] · Stefano Stramigioli[2]

The Geometry of Continua

(*its essence*)

Projectors

(We have **no concept of orthogonality** yet!)

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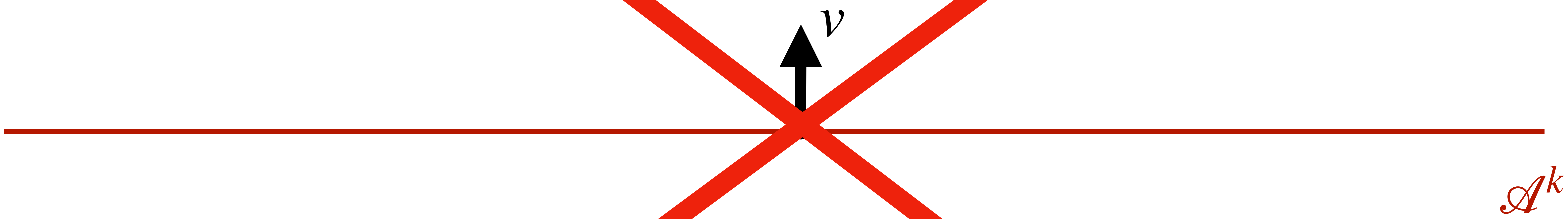
$\mathscr{V}^n$

~~What is the projection of v on $\mathscr{A}^k$?~~



This is a not well posed question!

Projection

Direct Sum Decomposition

Given:

- a Vector space $(\mathscr{V}^n, +, \bullet)$
- Two subspaces $\mathscr{A}^k, \mathscr{B}^{(n-k)} \subset \mathscr{V}^n$ s.t. $\mathscr{A}^k \cap \mathscr{B}^{(n-k)} = 0_{\mathscr{V}}$

We write

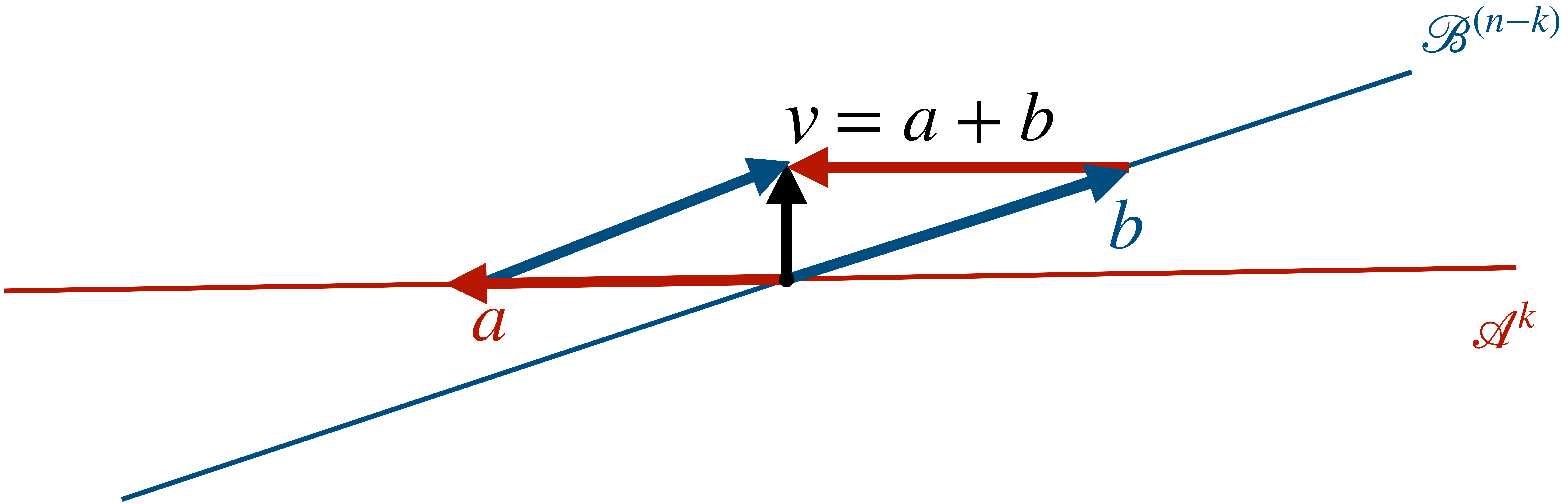
$$\mathscr{V}^n = \mathscr{A}^k \oplus \mathscr{B}^{n-k} \text{ (direct sum)}$$

Iff $\forall v \in \mathscr{V}, \exists!\, a \in \mathscr{A}, \exists!\, b \in \mathscr{B}, \quad$ s.t. $v = a + b$

We say that a is the projection of v along $\mathscr{B}$ and b is the projection of v along $\mathscr{A}$

$\mathscr{V}^n$

What is the projection of v on $\mathscr{A}^k$ **along $\mathscr{B}^{(n-k)}$?**



Answer: a

UNIVERSITY OF TWENTE.
The Geometry of Continua
Connection
$\nabla \hat{v}^\flat = \frac{1}{2} L_{\hat{v}} \hat{g} + \frac{1}{2} d\hat{v}^\flat$
All
Symm.
A-Symm.
$L_{\hat{v}} \hat{g}$
Rate of Strain
$\mathfrak{se}(3)$ Rigid Body Motions
Contents lists available at ScienceDirect
Journal of Geometry and Physics
journal homepage: www.elsevier.com/locate/geomphys
ELSEVIER
The principal bundle structure of continuum mechanics
Stefano Stramigioli
$\dot{c} \in T_c \mathscr{C}$
c
Deformation Space
$\mathscr{D} \xrightarrow{\mathscr{V}} \mathbb{R}$
Potential Energy
$\mathscr{D} \sim \mathscr{M}(\mathscr{B}) \ni \hat{g} = c^* g$
Matter
$T\mathscr{B}$
$\hat{v} \in \Gamma(T\mathscr{B})$
manifold $\mathscr{B}^n$
Matter $\hat{\mu}_m \in \Omega^n(\mathscr{B})$
constant structure
$\mathscr{C} := \mathrm{Emb}^\infty(\mathscr{B}, \mathscr{A})$
$c \in \mathscr{C}^\infty$
$T\mathscr{A}$
Ambient Space
$v \in \Gamma(T\mathscr{S})$
$\mathscr{S} := c(\mathscr{B}) \subset \mathscr{B}$
$\mathscr{A}^n$ manifold
$g \in \mathscr{M}(\mathscr{A})$ Metric
constant structure

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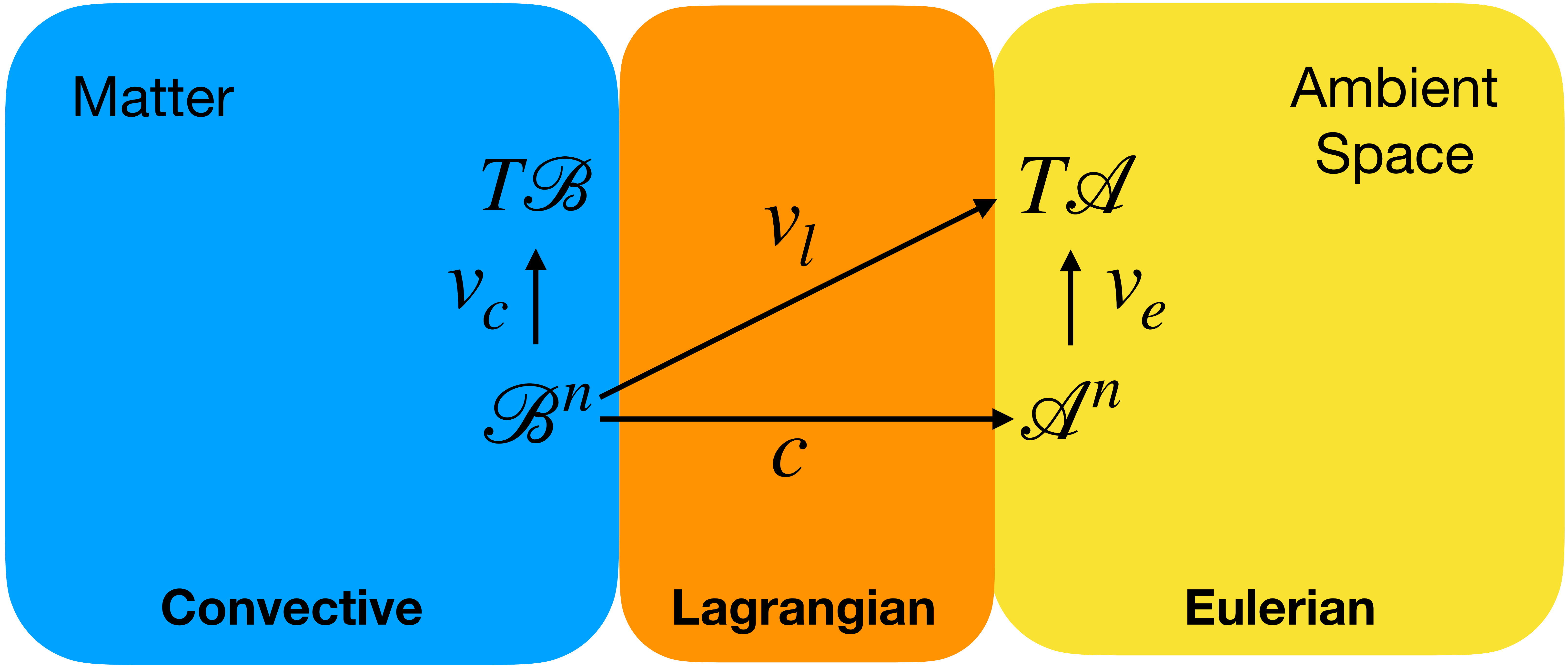


- Using the presented construction is possible to define precisely an **open port-based model for ANY elastic media**
- It can be shown that it is absolutely necessary to introduce **bundled valued** forms due to the **symmetric nature of the rate of strain**
- Introducing the right mappings of **metrics** it is possible to define **geometric connections** to do all operation invariantly like divergence.
- Differently than sometimes thought, **vorticity is not the source of dissipation**

$L_{v_c} g_c$
Rate of Strain

Energy storage and need for tensor-valued forms

Energy Storage

$$\dot{H} = \int_{\mathscr{M}^n} \delta_x \mathscr{H}^{n-l} \wedge \dot{x}^l$$

General Pairing

$$\dot{H} = \int_{\mathscr{M}^n} < \delta_q \mathscr{H}^{n-l} \,|\, \dot{q}^l >$$

$\mathscr{V}^*$ Dual $\mathscr{V}$

$$\underbrace{\nabla v_c^\flat}_{\text{All}} = \underbrace{\frac{1}{2} L_{v_c} g_c}_{\text{Symm.}} + \underbrace{\frac{1}{2} d v_c^\flat}_{\text{A-Symm.}}$$

$$\dot{H} = \int_{\mathscr{M}^n} \overbrace{\underbrace{\delta_q \mathscr{H}^{n-l}}_{\text{t.v. form}} \;\dot{\wedge}\; \underbrace{\dot{q}^l}_{\text{t.v. pseudo-form}}}^{n-form}$$

Energy storage and need for tensor-valued forms

$$\underbrace{\nabla v_c^\flat}_{\text{All}} = \underbrace{\frac{1}{2}L_{v_c}g_c}_{\text{Symm.}} + \underbrace{\frac{1}{2}dv_c^\flat}_{\text{A-Symm.}}$$

$$\dot{H} = \int_{\mathscr{M}^n} \overbrace{\underbrace{\delta_q \mathscr{H}^{n-l}}_{\text{t.v. form}} \;\dot{\wedge}\; \underbrace{\dot{q}^l}_{\text{t.v. pseudo-form}}}^{n-form}$$

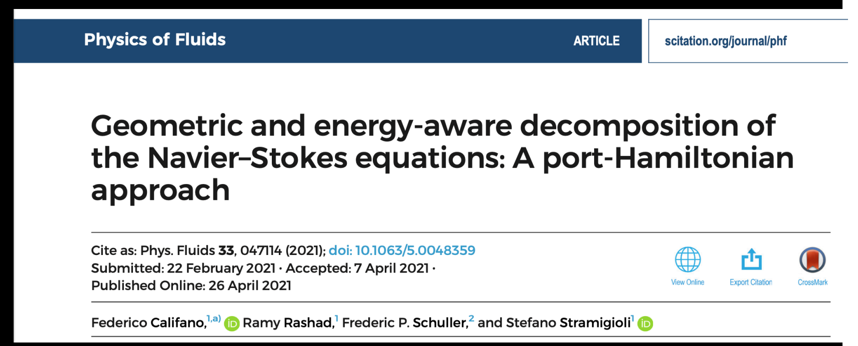
	Form-Leg (INTEGRATION)		**Value-Leg** (LOCAL PAIRING)
$E = e_F \otimes e_V \in$	$\Omega^k(M^n)$	$\otimes$	$\Gamma(T^p_l M^n)$
$F = f_F \otimes f_V \in$	$\Omega^{n-k}(M^n)$	$\otimes$	$\Gamma(T^l_p M^n)$

$$(e_F \otimes e_V)\,\dot{\wedge}\,(f_F \otimes f_V) := < e_v | f_v > \; e_F \wedge f_F \in \Omega^n(M^n)$$

$$\text{power} = \int_{M^n} E \,\dot{\wedge}\, F$$

Navier Stokes Equations

On-Shell Equations (barotropic case)


Physics of Fluids ARTICLE scitation.org/journal/phf

Geometric and energy-aware decomposition of the Navier–Stokes equations: A port-Hamiltonian approach

Cite as: Phys. Fluids **33**, 047114 (2021); doi: 10.1063/5.0048359
Submitted: 22 February 2021 · Accepted: 7 April 2021 · Published Online: 26 April 2021

View Online Export Citation CrossMark

Federico Califano,[1,a)] Ramy Rashad,[1] Frederic P. Schuller,[2] and Stefano Stramigioli[1]

Coordinate dependent NS equations

$$\partial_t(\rho v^i) + \partial_m(\rho v^m v^i) = -\partial^i p + \partial_m \tau^{mi}$$

Static Pressure

$$\partial_t \rho + \partial_m(\rho v^m) = 0 \qquad \text{Mass Continuity}$$

$$\tau^{ij} := \lambda(\partial_m v^m)\delta^{ij} + \kappa(\partial^i v^j + \partial^j v^i)$$

Bulk Shear

$$p = \rho^2 \frac{\partial U}{\partial \rho}(\rho)$$

Barotropic (for simplicity and easily extendable)

Strictly Euclidean => hides geometry

Any n-dimensional Rimannian manifold (Differential Geometry Background needed!)

$$\nu := g(v, \cdot) \in \Omega^1(M)$$

co-vector representation of flow

$$\mu := \star\rho \in \Omega^n(M)$$

mass top-form

$$\mathscr{T} \in \Omega^1(M) \dot{\otimes} \Omega^{n-1}(M)$$

viscous stress tensor

$$\dot{\wedge} : (\Omega^1(M) \otimes \Omega^l(M)) \times (\Gamma(TM) \otimes \Omega^k(M)) \to \Omega^{l+k}(M)$$

pairing

$$P_{\mathscr{T}} = \int_S \mathscr{T} \dot{\wedge} v$$

power dissipated

$$\mathrm{d}_\nabla : \Omega^1(M) \dot{\otimes} \Omega^{n-1}(M) \to \Omega^1(M) \otimes \Omega^n(M)$$

exterior covariant derivative

geometrical

$$(\mathrm{d}_\nabla \mathscr{T}) \dot{\wedge} v = \mathrm{d}(\mathscr{T} \dot{\wedge} v) - \mathscr{T} \dot{\wedge} \nabla v$$

topological

$$(\mathscr{L}_v g)_{ij} = \nabla_i \nu_j + \nabla_j \nu_i$$

rate of strain (**symmetric!**)

ON- SHELL EQUATIONS

Momentum Balance

$$\dot{\nu} + \mathrm{d}\left(\frac{1}{2}\star(\nu \wedge \star\nu)\right) + \iota_v \mathrm{d}\nu = -\frac{\mathrm{d}p}{\star\mu} + \frac{\star_2 \mathrm{d}_\nabla \mathscr{T}}{\star\mu}$$

$$\mathscr{T}_\lambda := \lambda(\mathrm{div}(v)\mu_{\mathrm{vol}}), \qquad \text{Bulk}$$

$$\mathscr{T}_\kappa := \kappa(\star_2 \mathscr{L}_v g), \qquad \text{Shear}$$

Mass Continuity

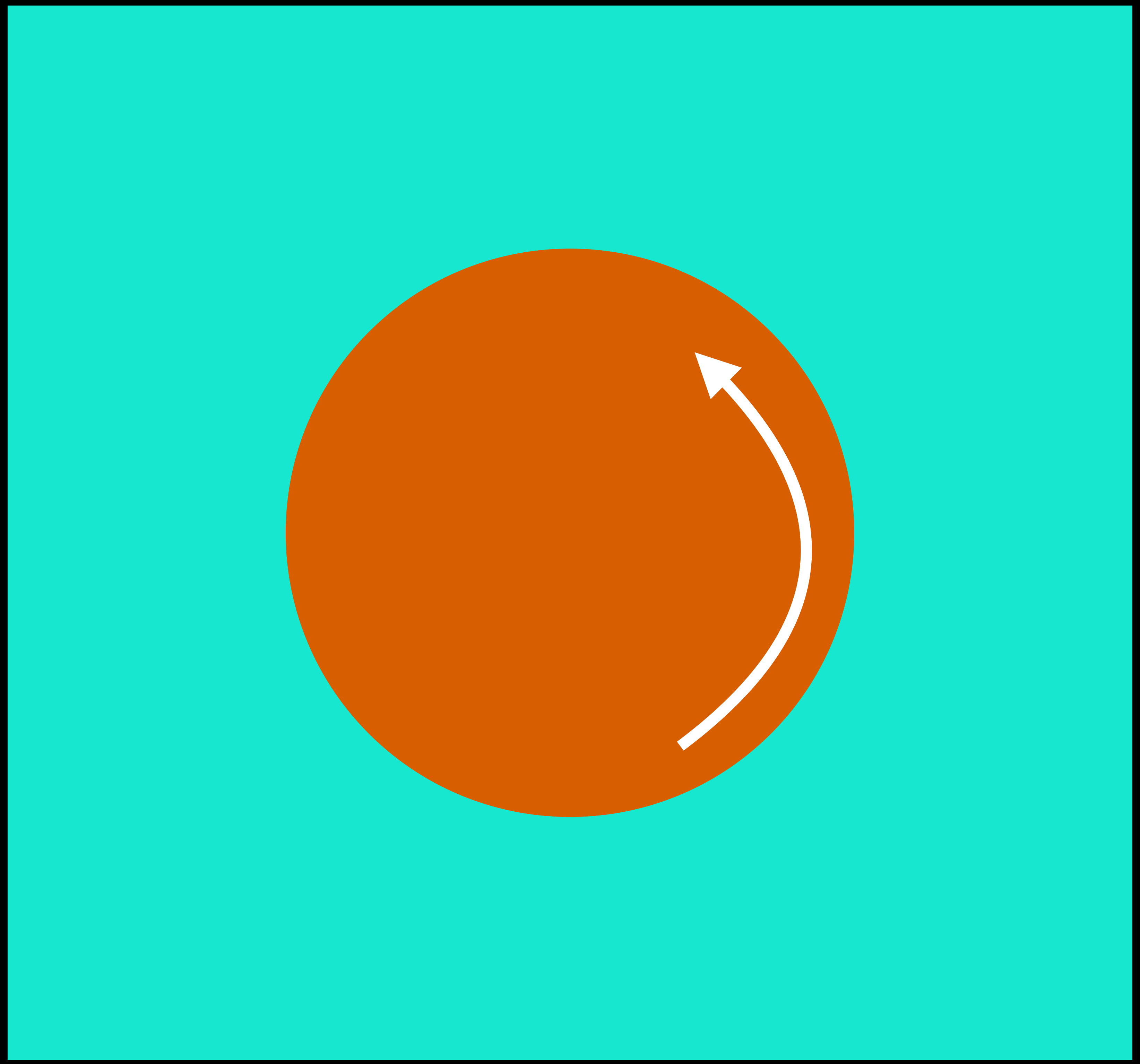
$$\dot{\mu} = -\mathrm{d}(\star\mu \star \nu)$$

Valid in any curved space of any dimension and completely coordinate invariant
NOTHING HIDES THE PHYSICS

A. D. Gilbert and J. Vanneste, "A geometric look at momentum flux and stress in fluid mechanics," (2019), arXiv:1911.06613

T. Frankel, The geometry of physics: an introduction (Cambridge university press, 2011)

Other intuitive reasons for natural use of t.v. forms



Rotating disc in fluid
with no slip condition

Traction force = surface integral of **Stress**

Stress = map from plane to force = **force values 2-form**

The restriction of Scalar Forms on the boundary would NOT be able to express a tangential force inducing flow in the fluid

Journal of Nonlinear Science (2023) 33:84
https://doi.org/10.1007/s00332-023-09945-7

Nonlinear Science

Check for updates

Intrinsic Nonlinear Elasticity: An Exterior Calculus Formulation

Ramy Rashad[1] · Andrea Brugnoli[2] · Federico Califano[1] · Erwin Luesink[3] · Stefano Stramigioli[1]

Received: 3 March 2023 / Accepted: 11 July 2023 / Published online: 24 July 2023

Abstract
In this paper, we formulate the theory of nonlinear elasticity in a geometrically intrinsic manner using exterior calculus and bundle-valued differential forms. We represent kinematics variables, such as velocity and rate of strain, as intensive vector-valued forms, while kinetics variables, such as stress and momentum, as extensive covector-valued pseudo-forms. We treat the spatial, material and convective representations of the motion and show how to geometrically convert from one representation to the other. Furthermore, we show the equivalence of our exterior calculus formulation to standard formulations in the literature based on tensor calculus. In addition, we highlight two types of structures underlying the theory: first, the principal bundle structure relating the space of embeddings to the space of Riemannian metrics on the body and how the latter represents an intrinsic space of deformations and second, the de Rham complex structure relating the spaces of bundle-valued forms to each other.

Communicated by Alain Goriely.

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Springer

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https://doi.org/10.1007/s00332-025-10130-1

Nonlinear Science

Check for updates

The Port-Hamiltonian Structure of Continuum Mechanics

Ramy Rashad[1,2] · Stefano Stramigioli[2]

Received: 1 April 2024 / Accepted: 15 January 2025

The most comprehensive and insightful description of continuous media and the need for tensor value forms

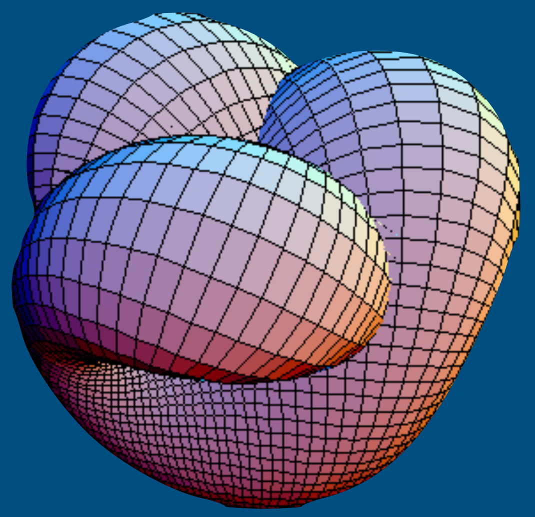
Going to Discretisation

Dual Field method

Intro to Mimetic (Topology Preserving) Discretisation

In the context of Exterior Calculus

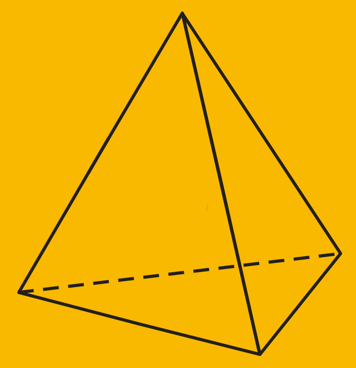
Physics ω^k



Continuous Field

$\downarrow \mathscr{R}$ (Reduction) $\quad \Pi$

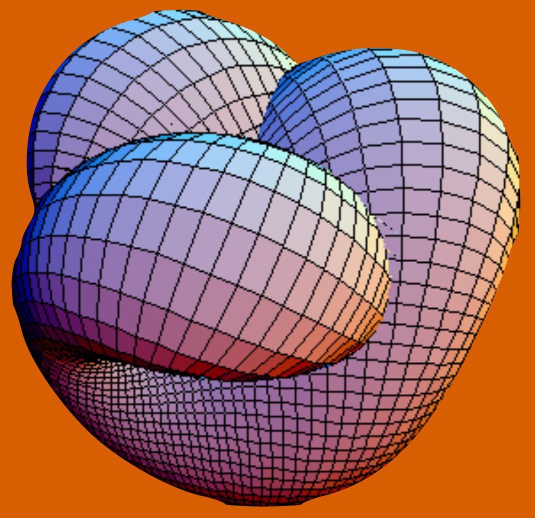
Numerics C^k



Values on Discrete Chains

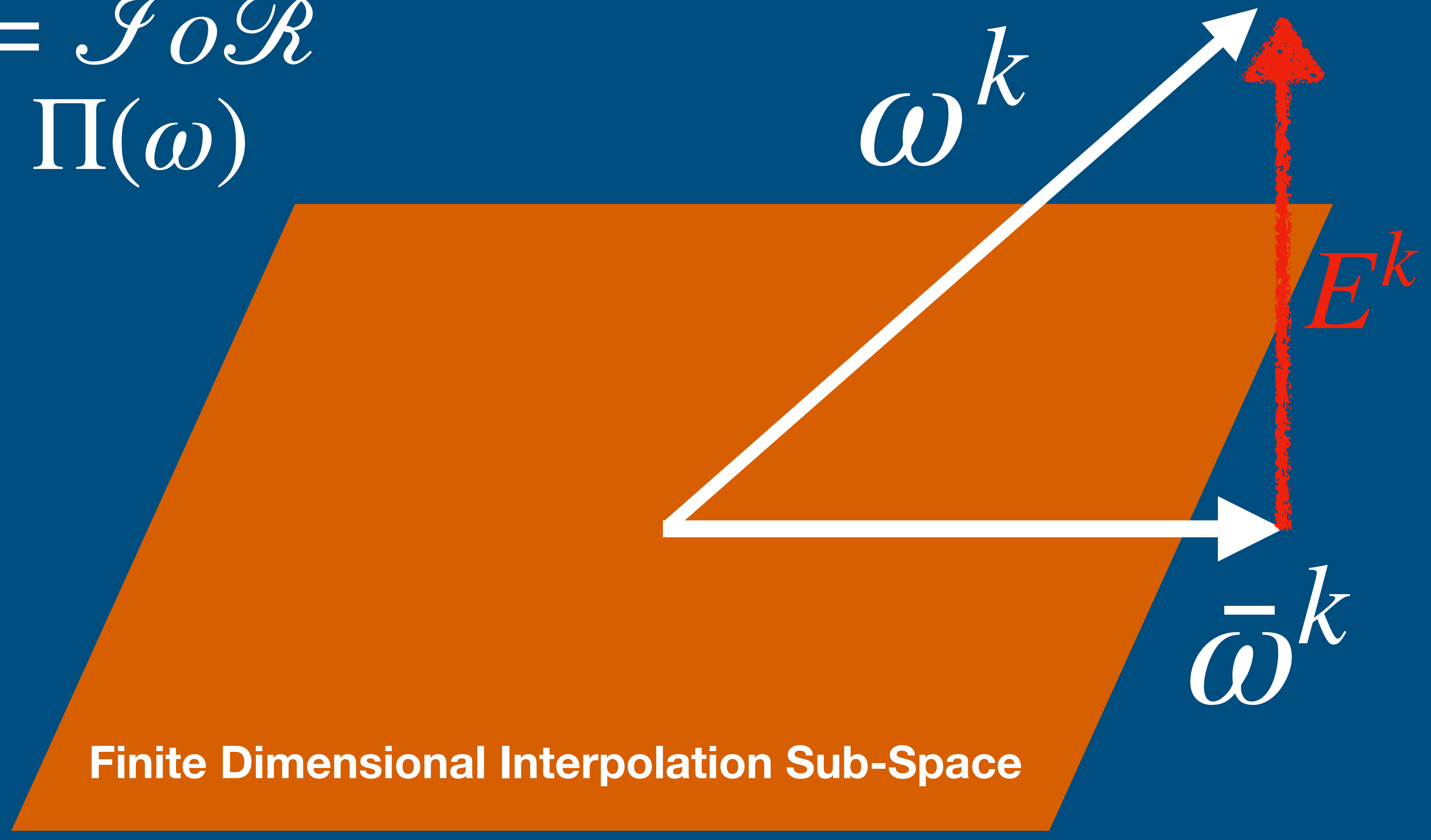
$\downarrow \mathscr{I}$ (Interpolation)

"Projected" Physics $\bar{\omega}^k$



Continuous Field

$$\Pi := \mathscr{I} o \mathscr{R}$$
$$\bar{\omega} = \Pi(\omega)$$



$$d\Pi = \Pi d \quad \Rightarrow \quad \Pi$$ **Structure=Topology Preserving!**

Co-chain Property (FEEC)

$\Uparrow$

$\mathscr{R}d = \partial\mathscr{R}$ **(De Rham map)**

$d\mathscr{I} = \mathscr{I}\partial$ **(Conforming Operator property)** **Whitney Forms**

Dual Fields Method

Both systems and use them along…

$$\dot{\omega} = \ldots \star\, \omega \ldots$$

$$\underbrace{\bar{\omega}}_{\text{Dual Field}} := \star\, \underbrace{\omega}_{\text{Primary Field}}$$

$$\dot{\omega} = \ldots \bar{\omega} \ldots$$

Primary System

- One Initialisation
- Use Each Other states
- ⋆ Disappeared !

$$\dot{\bar{\omega}} = \ldots (-1)^{l} \omega \ldots$$

Dual System

Dual Field Method

It could be seen

- This tools brings to a **spatial discretisation** of PDE which **outperforms any non-structure preserving method**
- This can be used together with **symplectic time integration**
- It has **open boundaries** !
- Can then be used for **model reduction, learning and Control** !

Upcoming work: GR & QFT

- To solve a specific problem we need the right structures and abstractions
- I am finalizing a way to show how Gravity can appear from interaction of Quantum Fields and exterior calculus plays a crucial role.
- This has been an open problem for many years and is due to the fact that excellent scientists have not been using the right abstractions and mathematical tools.

Conclusions

True knowledge comes with deep understanding of a topic and its inner workings.

Albert Einstein

Conclusions

- Port-Based methods is a fantastic way to **understanding the physics of continua (and flapping flights)**
- To handle geometric strain **new tools needed to be used** which showed issues previously not known in the literature.
- **Port-based thinking** is the driving force behind all this.
- Such insights have brought to **new challenges which are being tackled in theoretical physics (QFT and GR and their connection)**

My Personal Advise

I am convinced that the great breakthroughs can only be achieved by

- Deeply understanding the **right mathematical structures** needed to understand a problem: ***not too much***, ***not too little*** structure
- **Let the mathematics structure guide you** along the way by understanding the operation that you can or you cannot do!
- **Be critical on each step** you do and ask you always why ?
- And yet, trust the signals **your intuition and vision give you** but always **check with proper coordinate invariant mathematics**

Epistemological growth is not about solving a problem, but understanding "why and how" to solve it.

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Thanks for listening