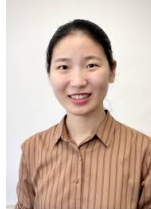


FINITE ELEMENT FORM-VALUED FORMS – TOWARDS AN EXTENDED PERIODIC TABLE –

Kaibo Hu

joint work with Ting Lin (Peking), Qian Zhang (Jilin)



Discrete Exterior Calculus workshop, IMSI, Chicago, Sep 2025



- ▶ KH, Ting Lin, Qian Zhang. *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM Journal on Applied Algebra and Geometry (2025). $\leq 3D$, lowest order
- ▶ KH, Ting Lin. *Finite element form-valued forms: Construction*. arXiv: 2503.03243 (2025) nD .
- ▶ Yakov Berchenko-Kogan, Evan Gawlik (2025). Finite element spaces of double forms. arXiv preprint arXiv:2505.17243.

MOTIVATION

stress, strain tensors, dislocation density, disclination density in continuum mechanics,
metric, curvature (scalar, Ricci, Weyl, Riemann, Cotton...), torsion in differential geometry etc.

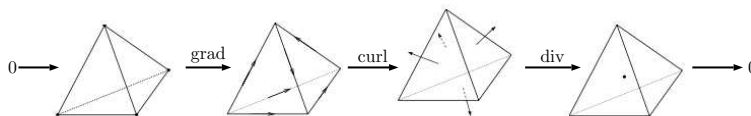
*Are there discrete analogues of such tensors with **symmetries** and **differential structures**?*

MOTIVATION

stress, strain tensors, dislocation density, disclination density in continuum mechanics,
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*Are there discrete analogues of such tensors with **symmetries** and **differential structures**?*

A special case: differential forms (fully skew-symmetric tensors), exterior derivatives



Whitney forms and higher order versions are well accepted as the canonical discretization for differential forms (skew-symmetric tensors). **k -forms discretized on k -cells, unisolvent with $\mathcal{P}^{-1} \Lambda^k$, conformity**

Raviart-Thomas (1977), Nédélec (1980) in numerical analysis

Bossavit (1988): differential forms and complex

Hiptmair (1999), Arnold, Falk, Winther (2006): systematic study, “Finite Element Exterior Calculus”



Pierre-Arnaud Raviart



Jean-Claude Nédélec



Franco Brezzi



Donatella Marini



Jim Douglas

OUTLINE

- 1 Continuous level: **form-valued forms and BGG** 4
- 2 Discrete level: **intrinsic finite elements** 15

CONTINUOUS LEVEL: FORM-VALUED FORMS AND BGG

1	Continuous level: form-valued forms and BGG	4
2	Discrete level: intrinsic finite elements	15

DIFFERENTIAL STRUCTURES IN ELASTICITY

Linear elasticity (Calabi, Kröner) complex

$$\text{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

DIFFERENTIAL STRUCTURES IN ELASTICITY

Linear elasticity (Calabi, Kröner) complex

embedding $\mathbb{R}^3 \rightarrow \mathbb{R}^3$

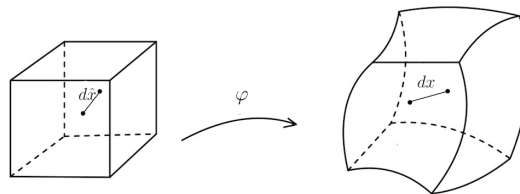
change of metric (strain)

$$\text{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

$$\varphi \longrightarrow e = (\hat{\nabla} \varphi) \cdot (\varphi \hat{\nabla}) - I$$

$e = 0$ iff φ is a rigid body motion.

Linearisation: $e = \text{sym grad } u$, in terms of displacement $u(\hat{x}) = \varphi(\hat{x}) - \hat{x}$.



$$dx - d\hat{x} = d\hat{x} \cdot e \cdot d\hat{x}$$

DIFFERENTIAL STRUCTURES IN ELASTICITY

Linear elasticity (Calabi, Kröner) complex

metric (strain)

Riemann curvature

$$\text{RM} \xrightarrow{\subset} \mathcal{C}^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} \mathcal{C}^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{inc}} \mathcal{C}^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{div}} \mathcal{C}^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

$$e \longrightarrow \text{Riem}(e)$$

Strain tensor (change of metric) $e = (\widehat{\nabla} \varphi) \cdot (\varphi \widehat{\nabla}) - I$ satisfies $\text{Riem}(e) = 0$.

Defect theory: Kröner et al. used violation of compatibility conditions to model defects and incompatibility

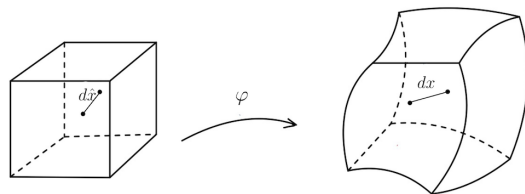
Linearisation: Saint-Venant compatibility condition $\text{inc } e := \nabla \times e \times \nabla = 0$.



Bernhard Riemann



Ekkehart Kröner



$$dx - d\hat{x} = d\hat{x} \cdot e \cdot d\hat{x}$$

DIFFERENTIAL STRUCTURES IN ELASTICITY

Linear elasticity (Calabi, Kröner) complex

curvature / stress

covector / force

$$\text{RM} \xrightarrow{\subset} \mathcal{C}^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} \mathcal{C}^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{inc}} \mathcal{C}^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{div}} \mathcal{C}^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

$$\sigma \longrightarrow \nabla \cdot \sigma$$

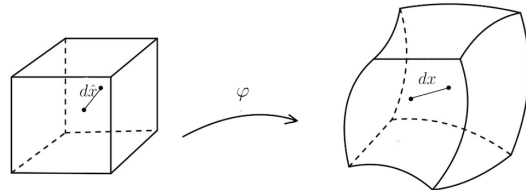
Cauchy stress tensor σ balances load $\text{div } \sigma = f$ with $\sigma = Ae$ (Hooke's law); incompatibility causes internal stress $\text{inc } e$.



Robert Hooke



Augustin-Louis Cauchy



$$dx - d\hat{x} = d\hat{x} \cdot e \cdot d\hat{x}$$

A NONLINEAR COMPLEX

cohomology not well defined, but exactness is. Exactness corresponds to important theorems.

Observations: KH, *Nonlinear elasticity complex and a finite element diagram chase* Springer INdAM Series (2024).

exactness: **rigidity**
two motions induce same metric iff up to RM

rigid body motion $\xrightarrow{\subset}$ map $\mathbb{R}^3$ to $\mathbb{R}^3$ $\xrightarrow{\varphi \mapsto \varphi^* g_0 - g_0}$ metric $\xrightarrow{\text{Ricci}}$ curvature

exactness: **fundamental thm of Riem geometry**
metric has vanishing curvature iff metric is Euclidean

Challenges for discretising nonlinear complex even in 1D:

$$0 \longrightarrow C^\infty \xrightarrow{u \mapsto u^2} C_+^\infty \longrightarrow 0 \quad \text{exact: } \forall w, \exists u = \sqrt{w}, \text{ s.t., } w = u^2.$$

not work for polynomials!

$$0 \longrightarrow \mathcal{P}_k \xrightarrow{u \mapsto u^2} \mathcal{P}_{2k}^+ \longrightarrow 0$$

Algebraic geometric issues. Relevant to preserving nonlinear constraints.

Question: tools for studying nonlinear complexes? discretisation?

COMPLEXES FROM COMPLEXES

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

► Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 1: connect two (or more) de Rham complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{curl}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0 \\ & & & & \nearrow S^0 & & \nearrow S^1 & & \nearrow S^2 & & \\ 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{curl}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0 \end{array}$$

$S^\bullet$: algebraic operators, connecting components of vectors/matrices

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Step 2: elimination

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{S} + \mathbb{K} & \xrightarrow{\text{curl}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0 \\
 & & & \nearrow -\text{mskw} & & \nearrow \mathbb{S} & & \nearrow 2\text{vskw} & & & \\
 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{curl}} & \mathbb{S} + \mathbb{K} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0
 \end{array}$$

$\mathbb{S}$: symmetric matrix, $\mathbb{K}$: skew-symmetric matrix

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 & & & \nearrow -\text{mskw} & & \nearrow \mathbb{S} & \nearrow 2 \text{ vskw} \\
 0 & \longrightarrow & \cancel{\mathbb{R}^3} & \xrightarrow{\text{grad}} & \cancel{\mathbb{R}^{3 \times 3}} & \xrightarrow{\text{curl}} & \mathbb{S} + \cancel{\mathbb{K}} \xrightarrow{\text{div}} \mathbb{R}^3 \longrightarrow 0
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Step 3: connect rows by zig-zag

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{sym grad}} & \mathbb{S} & \xrightarrow{\text{curl}} & \\ & & & & \searrow & & \\ & & & & \text{curl}^T & & \\ & & & & \longrightarrow & \mathbb{S} & \xrightarrow{\text{div}} \mathbb{R}^3 \longrightarrow 0. \end{array}$$

Conclusion: cohomology of the output (elasticity) is isomorphic to the input (de Rham)

Analytic results follow: Poincaré–Korn inequalities, Hodge decomposition, compactness...

Inspired by the Bernstein-Gelfand-Gelfand (BGG) construction (B-G-G 1975, Čap, Slovák, Souček 2001, Eastwood 2000, Arnold, Falk, Winther 2006, Arnold, KH 2021, Čap, KH 2023)

COMPLEXES FROM COMPLEXES

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

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Step 3: connect rows by zig-zag

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{sym grad}} & \mathbb{S} & \xrightarrow{\text{curl}} & \\ & & & & & \swarrow & \\ & & & & & \text{curl}^T & \\ & & & & & \searrow & \\ & & & & \mathbb{S} & \xrightarrow{\text{div}} & \mathbb{R}^3 \longrightarrow 0. \end{array}$$

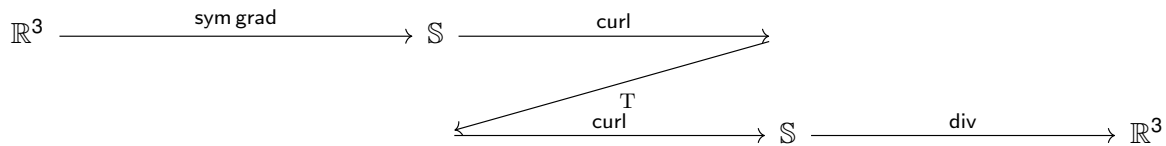
Conclusion: cohomology of the output (elasticity) is isomorphic to the input (de Rham)

Analytic results follow: Poincaré–Korn inequalities, Hodge decomposition, compactness...

But, is it purely mathematical?

embedding/displacement

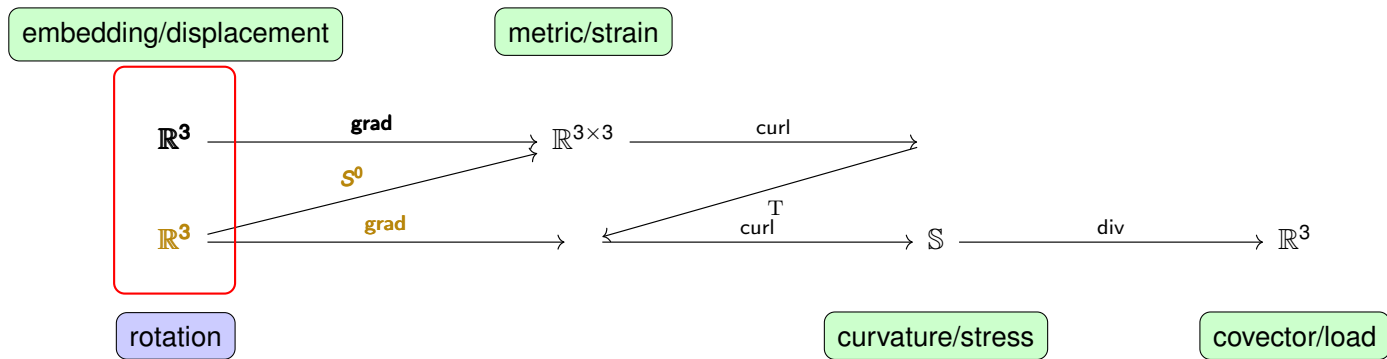
metric/strain



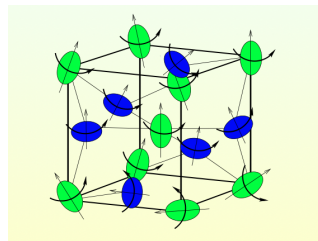
curvature/stress

covector/load

Riemann, Kröner, Cauchy, Hooke



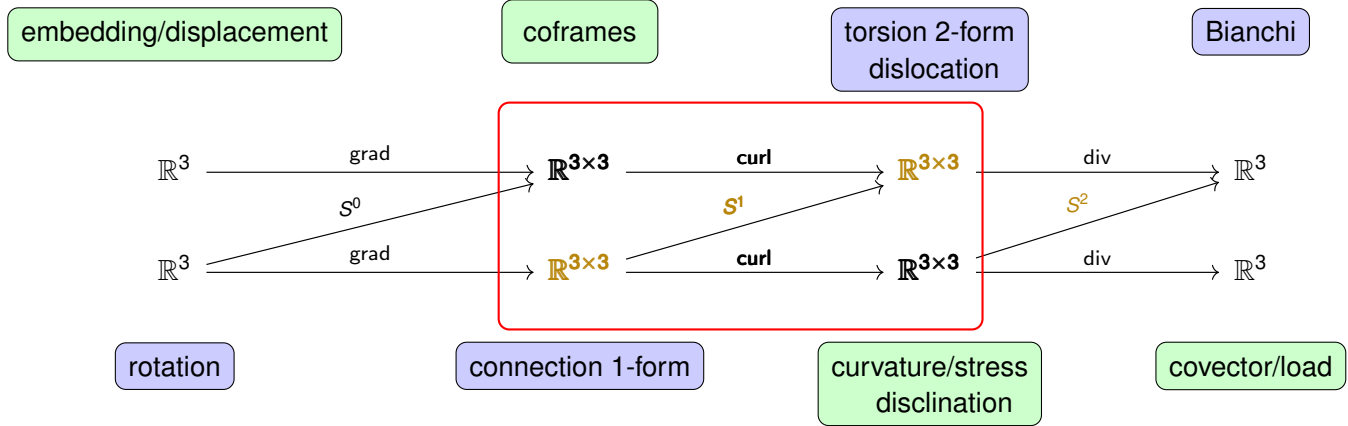
Cosserat brothers



Cosserat continua: microstructures (rotation, stretch etc.)

Observations: A. Čap & KH, *BGG sequences with weak regularity and applications*. FoCM (2024).

Leading to **parameter-robust scheme for Cosserat model**: A.Dziubek, KH, M.Karow & M. Neunteufel, SINUM (2025).



Élie Cartan

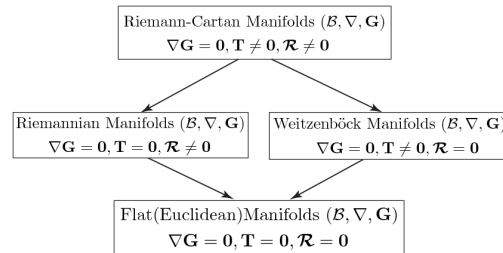


Arash Yavari



Alain Goriely

Cartan's bridge between Einstein and Cosserat brothers – torsion



Riemann-Cartan Geometry of Nonlinear Dislocation Mechanics,
 Yavari and Goriely, ARMA (2012)

Observations: Christiansen, KH, & Lin, *Extended Regge complex for linearized Riemann-Cartan geometry and cohomology*. arXiv (2023). **BGG construction is thus cohomology-preserving elimination of microstructures!**

PART OF A LARGER PICTURE... MECHANICS V.S. COMPLEXES V.S. GEOMETRY

Mechanics interpretation of BGG construction: eliminating microstructure variables (e.g., pointwise rotation) or torsion from twisted complexes via cohomology-preserving projections.

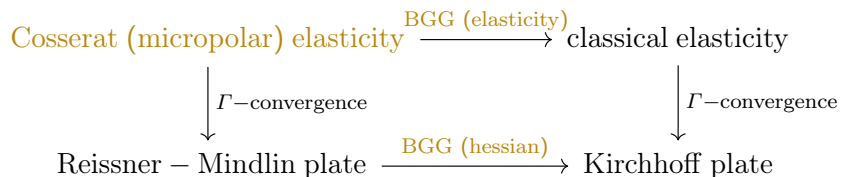
	twisted complex	BGG complex	
1D	Timoshenko beam	Euler-Bernoulli beam	(0-form Hodge Laplacian)
2D	Reissner-Mindlin plate	Kirchhoff-Love plate	
3D	Cosserat elasticity	classical elasticity	

twisted complex: complexes involving all components in diagram; **BGG complex:** after reduction

High order forms: continuum defect theory

Idea (Kröner, Nye etc.): strain in standard elasticity $e = \text{sym grad}(u)$ satisfying $\text{inc } e = 0$ (Saint-Venant compatibility). Defects lead to incompatibility: use e as a basic variable, and in general $\text{inc } e \neq 0$ describes defects.

Trace complexes: dimension reduction



Γ convergence: *The Reissner-Mindlin plate is the Γ -limit of Cosserat elasticity.* Neff, P., Hong, K. I., & Jeong, J. M3AS, (2010).

Mixed dimensional models: combinatorial manifolds, contact mechanics, porous media...

Čech-de Rham double complex (Jan Martin Nordbotten, Wietse M. Boon et al.)

RECALL: DIFFERENTIAL FORMS

On $\mathbb{R}^n$, a k -form is an element in $C^\infty(\Omega) \otimes \text{Alt}^k$.

► A general k -form: $\omega = \sum_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

► antisymmetric: $dx_i \wedge dx_j = -dx_j \wedge dx_i$, $dx_i \wedge dx_i = 0$.

$$f = f \Rightarrow \text{0-form, } \mathbb{R}$$

$$\omega = udx + vdy + wdz \Rightarrow \text{1-form, } \mathbb{V}$$

$$\eta = ady \wedge dz + bdz \wedge dx + cdx \wedge dy \Rightarrow \text{2-form, } \mathbb{V}$$

$$\mu = fdx \wedge dy \wedge dz \Rightarrow \text{3-form, } \mathbb{R}$$

► Differential operator: $d\omega = \sum_I \left(\sum_{j=1}^n \frac{\partial f_I}{\partial x_j} dx_j \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

$$d = \text{grad(0-form), curl(1-form), div(2-form), 0(3-form)}$$

► Traces $\iota_F^* \omega : \sum_I (f_I|_F) (dx_{i_1}|_F) \wedge \cdots \wedge (dx_{i_k}|_F)$.

$$\iota^* = \text{value(0-form), } t(1\text{-form}), \eta(2\text{-form}), 0(3\text{-form})$$

FORM-VALUE FORMS

On $\mathbb{R}^n$, a (k, ℓ) -form is an element in $\underbrace{(C^\infty(\Omega) \otimes \text{Alt}^k)}_{k \text{ forms}} \underbrace{\otimes \text{Alt}^\ell}_{\ell\text{-form valued}} =: C^\infty(\Omega) \otimes \text{Alt}^{k, \ell}.$

$$g = g_{ij} dx^i \otimes dx^j \quad (\text{sym } (1,1)\text{-form}), \quad R = R_{ijpq} dx^i \wedge dx^j \otimes dx^p \wedge dx^q \quad (\text{sym } (2,2)\text{-form}).$$

In three dimensions:

	dx	dy	dz			dx	dy	dz
dx	*	*	*		$dy \wedge dz$	*	*	*
dy	*	*	*		$dz \wedge dx$	*	*	*
dz	*	*	*		$dx \wedge dy$	*	*	*
(1,1) forms				(2,1) forms				
	$dy \wedge dz$	$dz \wedge dx$	$dx \wedge dy$		$dy \wedge dz$	$dz \wedge dx$	$dx \wedge dy$	
dx	*	*	*	$dy \wedge dz$	*	*	*	
dy	*	*	*	$dz \wedge dx$	*	*	*	
dz	*	*	*	$dx \wedge dy$	*	*	*	
(1,2) forms				(2,2) forms				

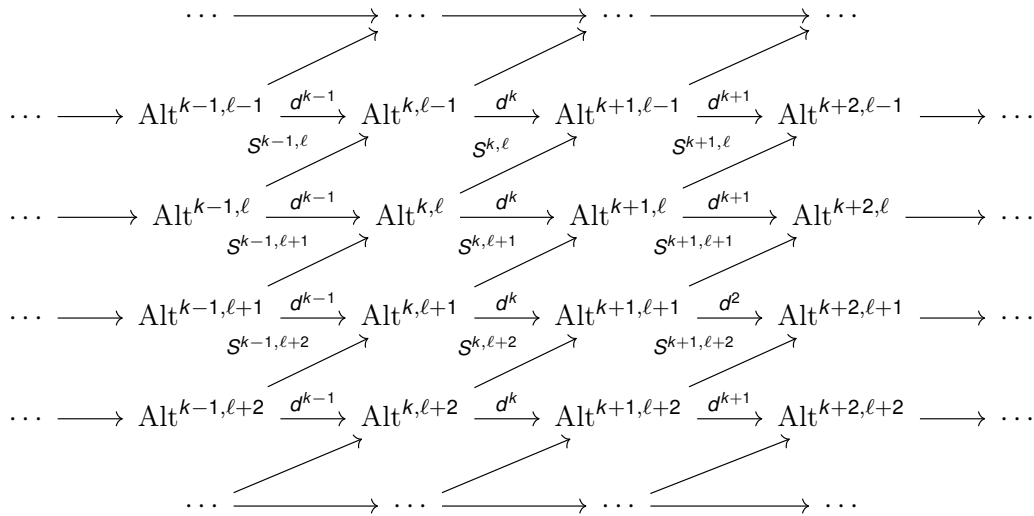
$$\begin{array}{ccccccc}
& \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
& & \nearrow & & \nearrow & & \nearrow & \\
\cdots & \longrightarrow & \text{Alt}^{k-1,\ell-1} & \xrightarrow{d^{k-1}} & \text{Alt}^{k,\ell-1} & \xrightarrow{d^k} & \text{Alt}^{k+1,\ell-1} & \xrightarrow{d^{k+1}} & \text{Alt}^{k+2,\ell-1} & \longrightarrow & \cdots \\
& & \nearrow S^{k-1,\ell} & & \nearrow S^{k,\ell} & & \nearrow S^{k+1,\ell} & & & & \\
\cdots & \longrightarrow & \text{Alt}^{k-1,\ell} & \xrightarrow{d^{k-1}} & \text{Alt}^{k,\ell} & \xrightarrow{d^k} & \text{Alt}^{k+1,\ell} & \xrightarrow{d^{k+1}} & \text{Alt}^{k+2,\ell} & \longrightarrow & \cdots \\
& & \nearrow S^{k-1,\ell+1} & & \nearrow S^{k,\ell+1} & & \nearrow S^{k+1,\ell+1} & & & & \\
\cdots & \longrightarrow & \text{Alt}^{k-1,\ell+1} & \xrightarrow{d^{k-1}} & \text{Alt}^{k,\ell+1} & \xrightarrow{d^k} & \text{Alt}^{k+1,\ell+1} & \xrightarrow{d^2} & \text{Alt}^{k+2,\ell+1} & \longrightarrow & \cdots \\
& & \nearrow S^{k-1,\ell+2} & & \nearrow S^{k,\ell+2} & & \nearrow S^{k+1,\ell+2} & & & & \\
\cdots & \longrightarrow & \text{Alt}^{k-1,\ell+2} & \xrightarrow{d^{k-1}} & \text{Alt}^{k,\ell+2} & \xrightarrow{d^k} & \text{Alt}^{k+1,\ell+2} & \xrightarrow{d^{k+1}} & \text{Alt}^{k+2,\ell+2} & \longrightarrow & \cdots \\
& & \nearrow & & \nearrow & & \nearrow & & & & \\
& \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots
\end{array}$$

where $\text{Alt}^{k,\ell} := \text{Alt}^k \otimes \text{Alt}^\ell$ ℓ -form-valued k -forms (k -forms taking value in ℓ -forms; (k, ℓ) -form)

$$S^{i,J}_\mu(v_0, \dots, v_i)(w_1, \dots, w_{J-1}) := \sum_{l=0}^i (-1)^l \mu(v_0, \dots, \widehat{v}_l, \dots, v_i)(v^l, w_1, \dots, w_{J-1}),$$

$$S^{i,J}_\dagger \mu(v_1, \dots, v_{i-1})(w_0, \dots, w_J) := \sum_{l=0}^i (-1)^l \mu(w_l, v_1, \dots, v_{i-1})(w_1, \dots, \widehat{w}_l, \dots, w_J),$$

$$\forall v_0, \dots, v_i, w_1, \dots, w_J \in \mathbb{R}^n.$$



Tensor symmetries encoded in diagrams: e.g.,

differential forms: $(k, 0)$ metric, strain: $(1, 1)$ curvature, stress: $(2, 2)$ torsion: $(2, 1)$

Example: Riemannian tensor

$\ker(\mathcal{S}^{2,2}) \subset \text{Alt}^{2,2}$: symmetry of Riemannian tensor (algebraic Bianchi identity)

$$R_{ab,cd} = -R_{ba,cd} = -R_{ab,dc}, \quad R_{ab,cd} = R_{cd,ab}, \quad R_{ab,cd} + R_{bc,ad} + R_{ca,bd} = 0.$$

$$\dim(\ker(\mathcal{S}^{2,2})) = \dim \text{Alt}^{2,2} - \dim \text{Alt}^{3,1} = \begin{cases} 1 & \text{in 2D} \\ 6 & \text{in 3D} \\ 20 & \text{in 4D} \\ \dots & \end{cases} \begin{array}{l} \text{Gauss curvature,} \\ \text{Ricci/Einstein,} \\ \text{Riemann} \end{array}$$

DERIVING COMPLEXES: COMPLEXES FROM COMPLEXES

Inspired by the Bernstein-Gelfand-Gelfand (BGG) construction. (B-G-G 1975, Čap,Slovák,Souček 2001, Eastwood 2000, Arnold,Falk,Winther 2006, Arnold,KH 2021, Čap,KH 2023)

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Alt}^{0,J-1} & \xrightarrow{d} & \text{Alt}^{1,J-1} & \xrightarrow{d} & \dots \xrightarrow{d} \text{Alt}^{n,J-1} \longrightarrow 0 \\
 & & & \nearrow S^{0,J} & & \nearrow S^{1,J} & & \nearrow S^{n-1,J} \\
 0 & \longrightarrow & \text{Alt}^{0,J} & \xrightarrow{d} & \text{Alt}^{1,J} & \xrightarrow{d} & \dots \xrightarrow{d} \text{Alt}^{n,J} \longrightarrow 0
 \end{array}$$

where $\text{Alt}^{i,J} := \text{Alt}^i \otimes \text{Alt}^J$ **J -form-valued i -forms** (i -forms taking value in J -forms; (i, J) -form)

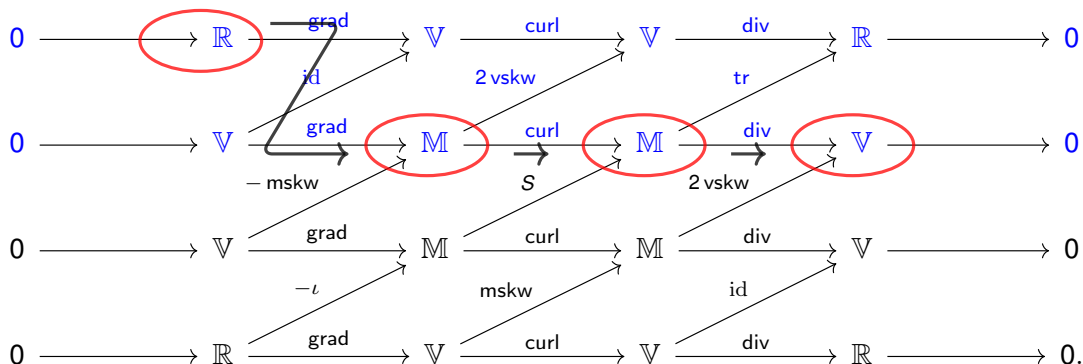
Output:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker(S_{\dagger}^{-1,J}) & \longrightarrow & \dots & \longrightarrow & \ker(S_{\dagger}^{J-2,J}) \xrightarrow{d} \\
 & & & & & & \nearrow S^{-1} \\
 & & & & & & \longleftarrow d \\
 & & & & & & \ker(S^{J,J}) \longrightarrow \dots \longrightarrow \ker(S^{n,J}) \longrightarrow 0.
 \end{array}$$

$S_{\dagger}$: adjoint of S

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix



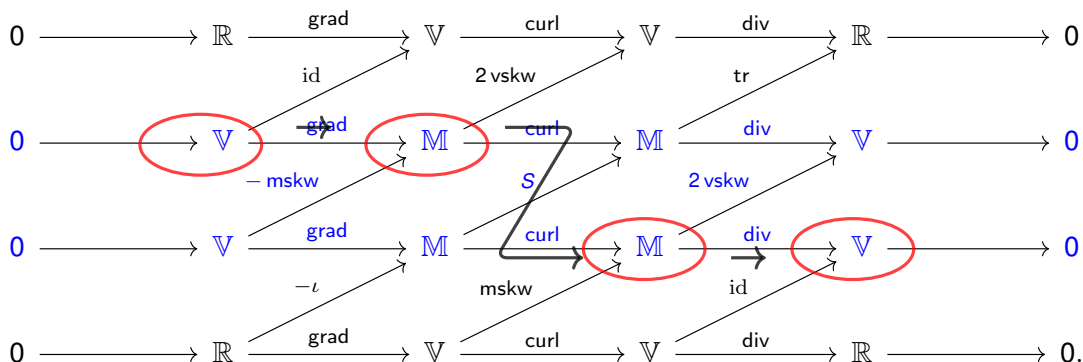
Hessian complex:

$$0 \longrightarrow C^\infty \xrightarrow{\text{hess}} C^\infty(\mathbb{S}) \xrightarrow{\text{curl}} C^\infty(\mathbb{T}) \xrightarrow{\text{div}} C^\infty(\mathbb{V}) \longrightarrow 0.$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix



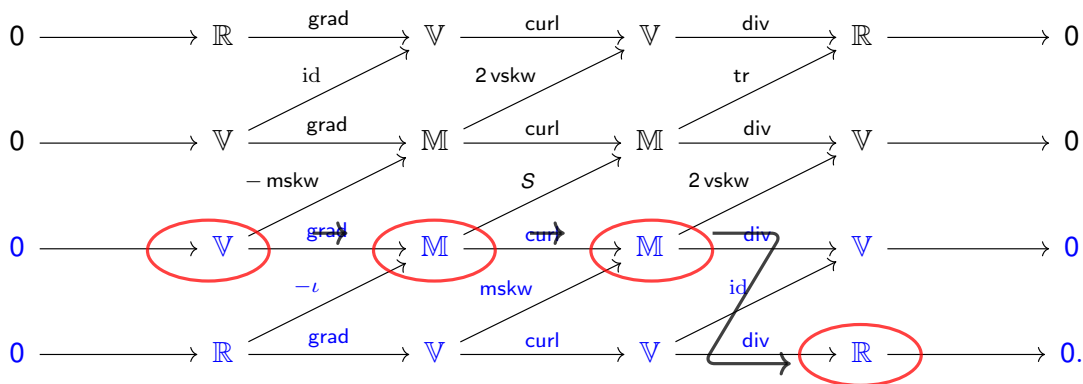
elasticity complex:

$$0 \longrightarrow C^\infty(\mathbb{V}) \xrightarrow{\text{def}} C^\infty(\mathbb{S}) \xrightarrow{\text{inc}} C^\infty(\mathbb{S}) \xrightarrow{\text{div}} C^\infty(\mathbb{V}) \longrightarrow 0.$$

elasticity, defects, metric, curvature

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix



divdiv complex:

$$0 \longrightarrow C^\infty(\mathbb{V}) \xrightarrow{\text{dev grad}} C^\infty(\mathbb{T}) \xrightarrow{\text{sym curl}} C^\infty(\mathbb{S}) \xrightarrow{\text{div div}} C^\infty \longrightarrow 0.$$

plate theory, elasticity

DISCRETE LEVEL: INTRINSIC FINITE ELEMENTS

1	Continuous level: form-valued forms and BGG	4
2	Discrete level: intrinsic finite elements	15

DISCRETIZATION OF COMPLEXES: FINITE ELEMENTS AND SPLINES

- ▶ **2D stress**: Arnold-Winther 2002, J.Hu-S.Zhang 2014, Christiansen-KH 2018,
- ▶ **2D strain**: Chen-J.Hu-Huang 2014 (Regge/HHJ), Christiansen-KH 2018 (conforming), Chen-Huang 2020, DiPietro-Droniou 2021 (polygonal meshes), KH 2023
- ▶ **3D elasticity**: various results on last part of complex, Hauret-Kuhl-Ortiz 2007 (discrete geometry/mechanics), Arnold-Awanou-Winther 2008, Christiansen 2011 (Regge), Christiansen-Gopalakrishnan-Guzmán-KH 2020, Chen-Huang 2021, J.Hu-Liang-Lin 2023, Gong-Gopalakrishnan-Guzmán-Neilan 2023
- ▶ **3D Hessian**: Chen-Huang 2020, J.Hu-Liang 2021, Arf-Simeon 2021 (splines)
- ▶ **3D divdiv**: Chen-Huang 2021, J.Hu-Liang-Ma 2021, Sander 2021 ($H(\text{sym curl})$, $H(\text{dev sym curl})$), J.Hu-Liang-Ma-Zhang 2022, J.Hu-Liang-Lin 2023, DiPietro-Hanot 2023
- ▶ **nD** : Chen-Huang 2021 (last two spaces), 2D arbitrary regularity: Chen-Huang 2022, Bonizzoni-KH-Kanschat-Sap 2023 (tensor product construction, nD , (k, ℓ) -forms)
- ▶ **conformal complexes**: KH-Lin-Shi 2023.

Question: Canonical finite elements, analogue of the Whitney forms?

correct cohomology, discrete topological/geometric structures...

GENERALIZING FINITE ELEMENTS: BACK TO DE RHAM'S CURRENTS

Use **currents** (measures, Dirac delta), rather than functions.

Geometric Measure Theory , **graphics**

(Codimensional geometry: A point cloud represents a probability measure; curve cloud, surface cloud...)

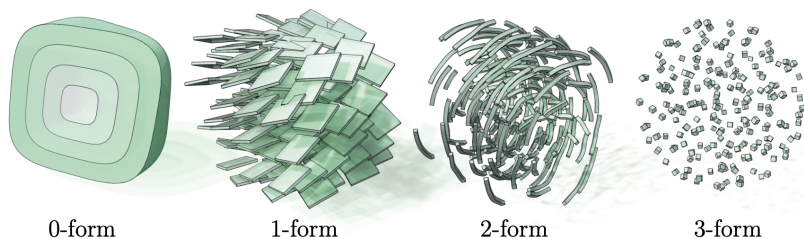
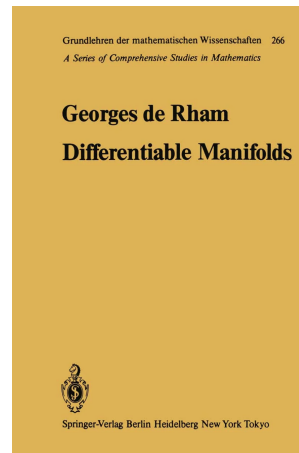


Figure 2.4 Differential k -forms can be represented by clouds of codimension- k geometries.

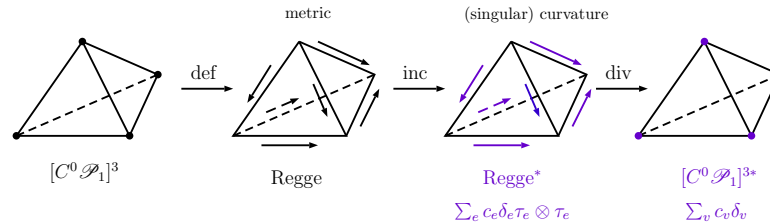
Figure: *Exterior Calculus in Graphics*, Stephanie Wang, Mohammad Sina Nabizadeh and Albert Chern; ACM SIGGRAPH 2023 courses.



GATHERING PIECES TO SOLVE THE PUZZLE

As the development of FEEC, several individual ingredients are already in the literature. Particularly, it is perhaps with little hesitation to accept Christiansen-Regge complex as the *canonical* discretization for the elasticity complex, due to **simple dofs, geometric interpretations, formal self-adjointness, correct cohomology**.

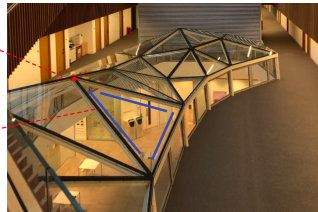
Christiansen 2011: Regge calculus = finite elements



Regge finite element v.s. Regge Calculus (‘*General relativity without coordinates*’, quantum gravity)

curvature:
Dirac measure on hinges

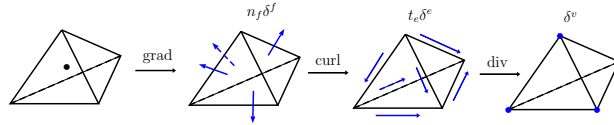
piecewise constant metric
represented by edge lengths



nD: Lizao Li (2018 Minnesota thesis), nonlinear curvature with Regge elements (Berchenko-Kogan, Gawlik 2022, Gopalakrishnan, Neunteufel, Schöberl, Wardetzky 2022, Gawlik, Neunteufel 2023)

Schöberl and collaborators systematically used finite elements + distributions to design numerical schemes.

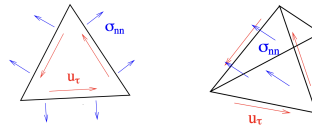
► Equilibrated residual error estimator



distributional finite elements, Braess-Schöberl 2008, Licht 2017

► Tangential Displacement Normal-Normal Stress (TDNNS) method for elasticity

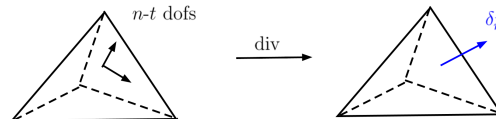
$$\sigma = -\varepsilon(\mathbf{u}), \quad \nabla \cdot \sigma = \mathbf{f}. \quad \sigma : \text{symmetric matrix}$$



div σ : tangential distribution Schöberl-Sinwel 2007

► Mass-Conserving mixed Stress (MCS) method for Stokes equations

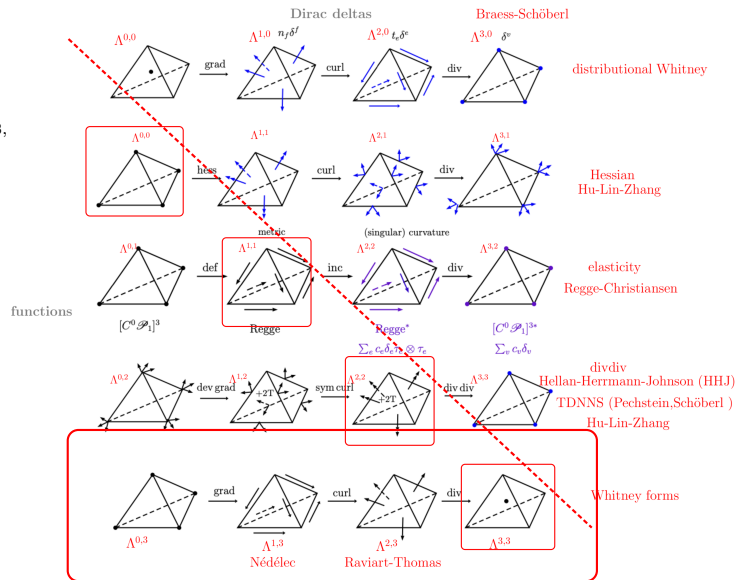
$$\sigma = -\nabla \mathbf{u}, \quad \nabla \cdot \sigma + \nabla p = \mathbf{f}. \quad \sigma : \text{trace-free matrix}$$



div σ : normal distribution Gopalakrishnan-Lederer-Schöberl 2020

KH, TING LIN. *Finite element form-valued forms (I): Construction*. ARXIV: 2503.03243 (2025)

KA, TING LIN. *Finite element form-valued forms (I): Construction*. ARXIV: 2503.03243 (2025)



Periodic Table of the Finite Elements

Nédélec, Raviart–Thomas, Whitney, Bossavit, Hiptmair, Arnold, Falk, Winther...

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS

KH, TING LIN. *Finite element form-valued forms (I): Construction*. ARXIV: 2503.03243 (2025)

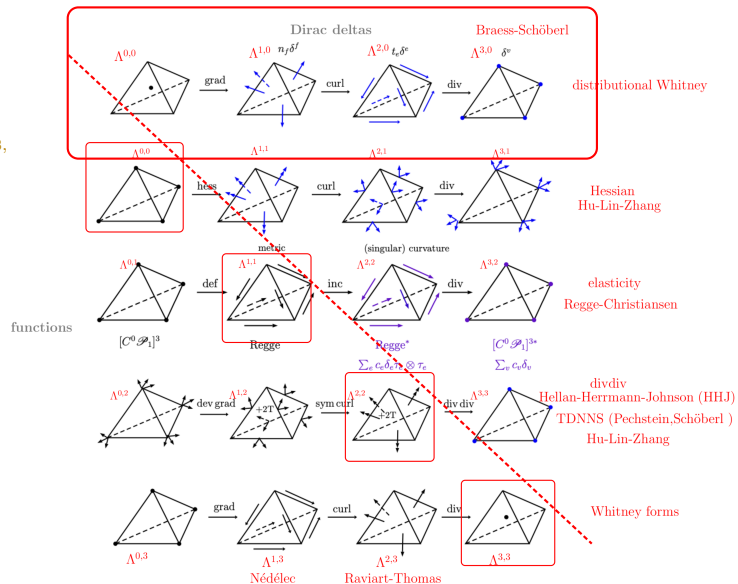
$$\text{Alt}^{-1,-1} \xrightarrow{\quad} \text{Alt}^{0,0} \longrightarrow \text{Alt}^{1,0} \longrightarrow \text{Alt}^{2,0} \longrightarrow \text{Alt}^3,$$

$$\begin{array}{ccccccc} \text{Alt}^{0,0} & \longrightarrow & & & & & \\ & \searrow & & & & & \\ & \longrightarrow & \text{Alt}^{1,1} & \longrightarrow & \text{Alt}^{2,1} & \longrightarrow & \text{Alt}^{3,1} \end{array}$$

$$\begin{array}{ccccccc} \text{Alt}^{0,1} & \longrightarrow & \text{Alt}^{1,1} & \longrightarrow & & & \\ & & \searrow & & & & \\ & & \longrightarrow & \text{Alt}^{2,2} & \longrightarrow & \text{Alt}^{3,2} \end{array}$$

$$\begin{array}{ccccccc} \text{Alt}^{0,2} & \longrightarrow & \text{Alt}^{1,2} & \longrightarrow & \text{Alt}^{2,2} & \longrightarrow & \\ & & & & \searrow & & \\ & & & & \longrightarrow & \text{Alt}^{3,3} \end{array}$$

$$\text{Alt}^{0,3} \longrightarrow \text{Alt}^{1,3} \longrightarrow \text{Alt}^{2,3} \longrightarrow \text{Alt}^{3,3} \xrightarrow{\quad} \text{Alt}^{4,4}$$



distributional de Rham complex (currents).

Braess, Schöberl 2008: equilibrated residual error estimator

Licht 2017: double complex



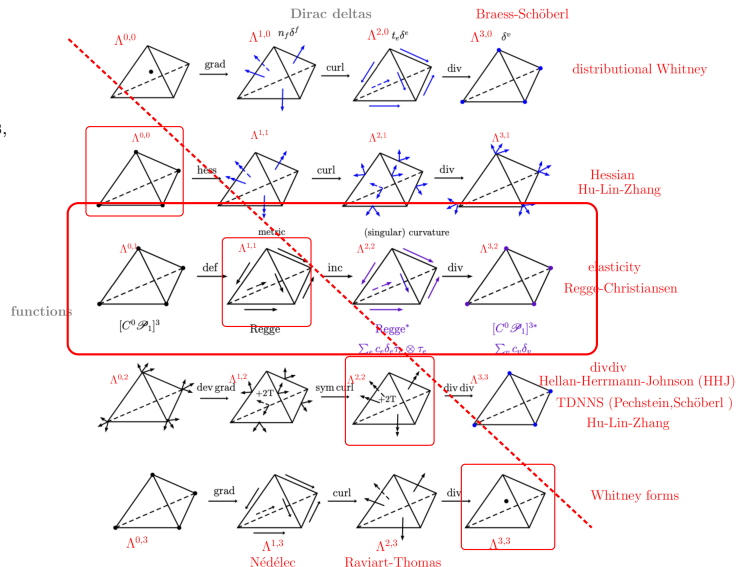
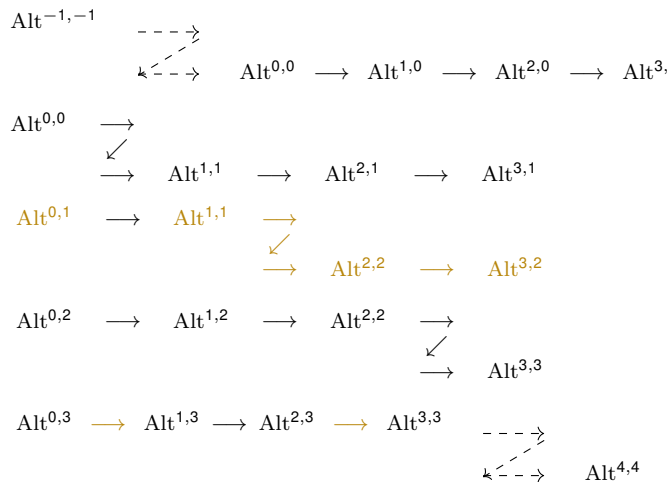
Dietrich Braess



Joachim Schöberl

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS

KH, TING LIN. *Finite element form-valued forms (I): Construction*. ARXIV: 2503.03243 (2025)



Christiansen's interpretation of **Regge calculus** as **finite elements**

Regge calculus (quantum & numerical gravity) :
edge length as metric, angle deficit as curvature

Regge finite element : piecewise constant symmetric tensor field



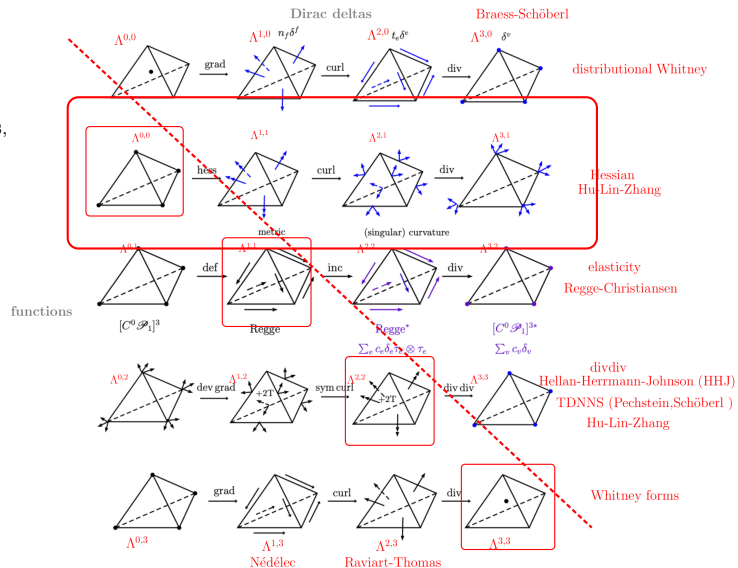
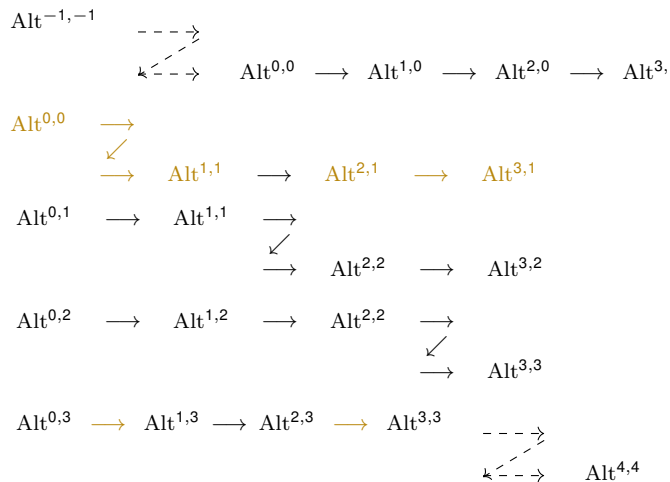
Tullio Regge



Snorre Christiansen

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS

KH, TING LIN. *Finite element form-valued forms (I): Construction*. ARXIV: 2503.03243 (2025)

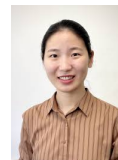


Hessian complex, unified structures identified.

Kaibo Hu, Ting Lin, Qian Zhang. *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM Journal on Applied Algebra and Geometry (2025).



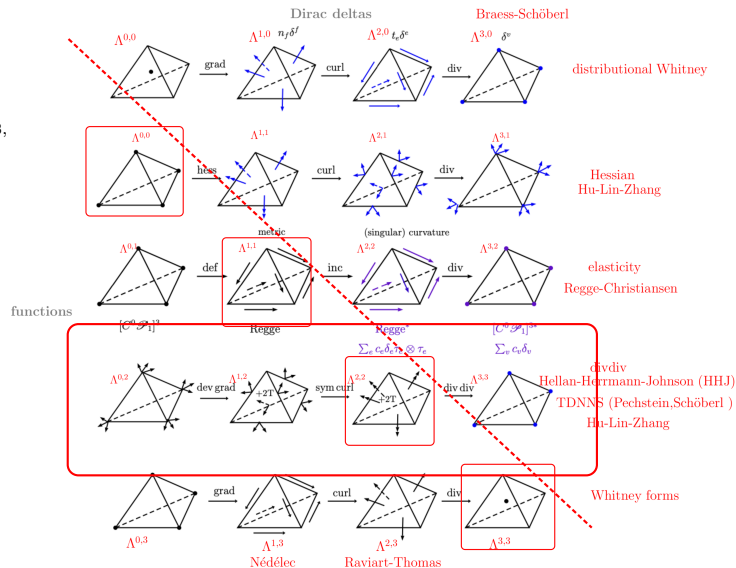
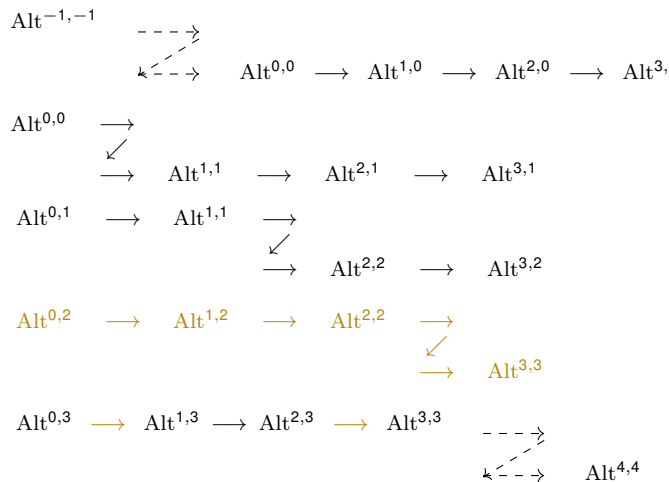
Ting Lin



Qian Zhang

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS

KH, TING LIN. *Finite element form-valued forms (I): Construction*. ARXIV: 2503.03243 (2025)



divdiv complex, dual to Hessian complex.

TDNNS for elasticity (Schöberl, Sinwel 2007), Hellan-Herrmann-Johnson (HHJ) element for plate.

Implemented by J.Schöberl in **NGSolve** with relativity applications

KH, Ting Lin, Qian Zhang. *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM Journal on Applied Algebra and Geometry (2025).



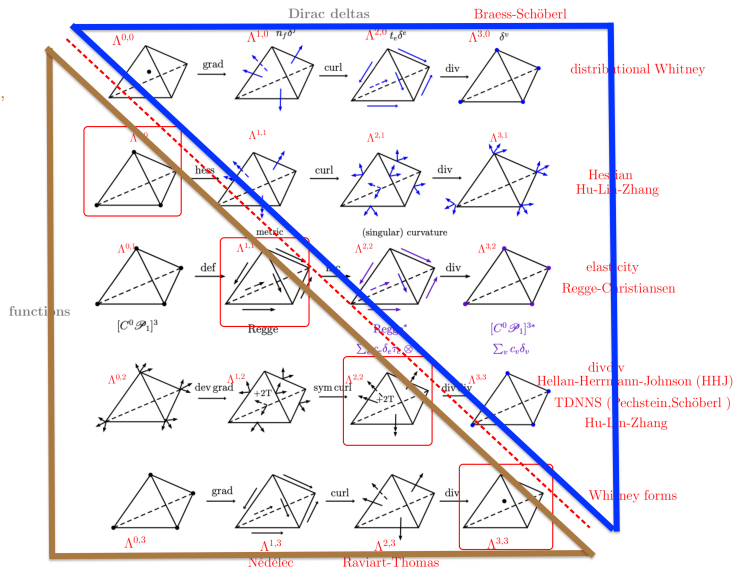
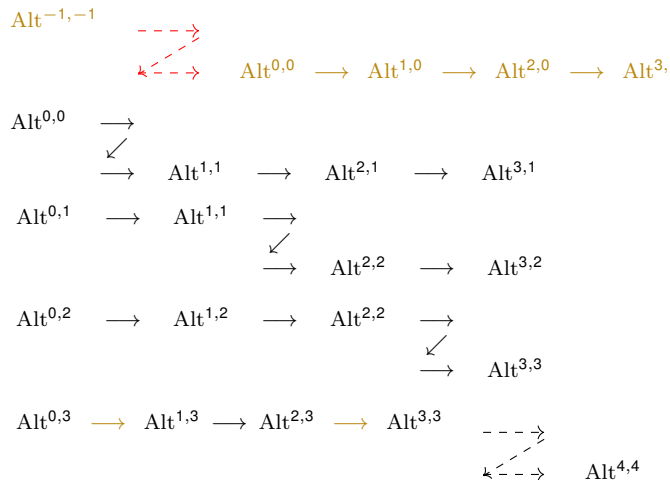
Astrid Pechstein



Joachim Schöberl

TOWARDS A FINITE ELEMENT PERIODIC TABLE FOR TENSORS

KH, TING LIN. *Finite element form-valued forms (I): Construction*. ARXIV: 2503.03243 (2025)



Patterns, Symmetries, Duality.

functions (classical finite elements) v.s. Dirac measures (currents). any dimension, any degree.



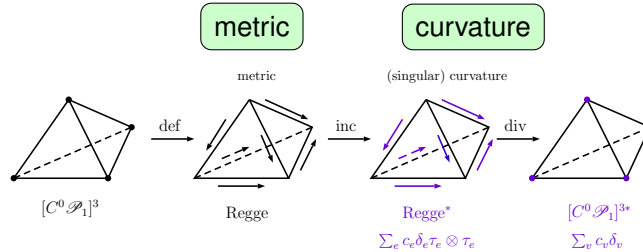
Georges de Rham

Classical Finite Element Periodic Table (last row) is the special case of the generalised Table where all spaces are piecewise polynomials.

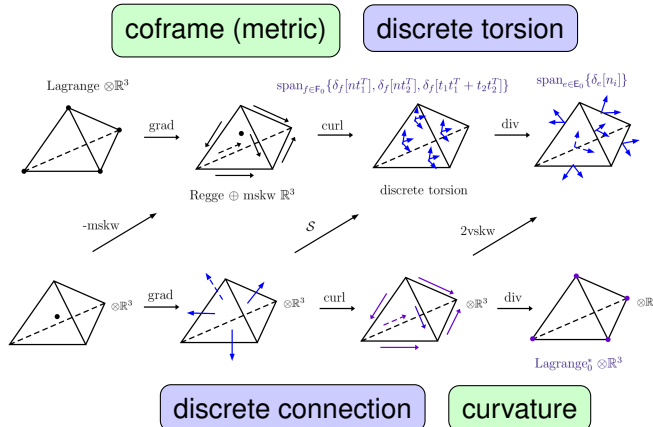
Cohomology in $\leq 3D$ proved (KH,Lin,Zhang 2025; Christiansen,KH,Lin 2023), isomorphic to continuous version (thus copies of de Rham cohomology via BGG).
 $> 3D$ ongoing. Dimension count holds, as a strong indication.

Extend Regge calculus/finite element

(S.Christiansen 2011, Regge for curvature)



to Riemann-Cartan geometry (S.Christiansen, KH, L.Ting 2023, extended Regge for curvature + torsion)



WHERE DOES MCS SHOW UP? JUMP TWO ROWS. $p = 2$

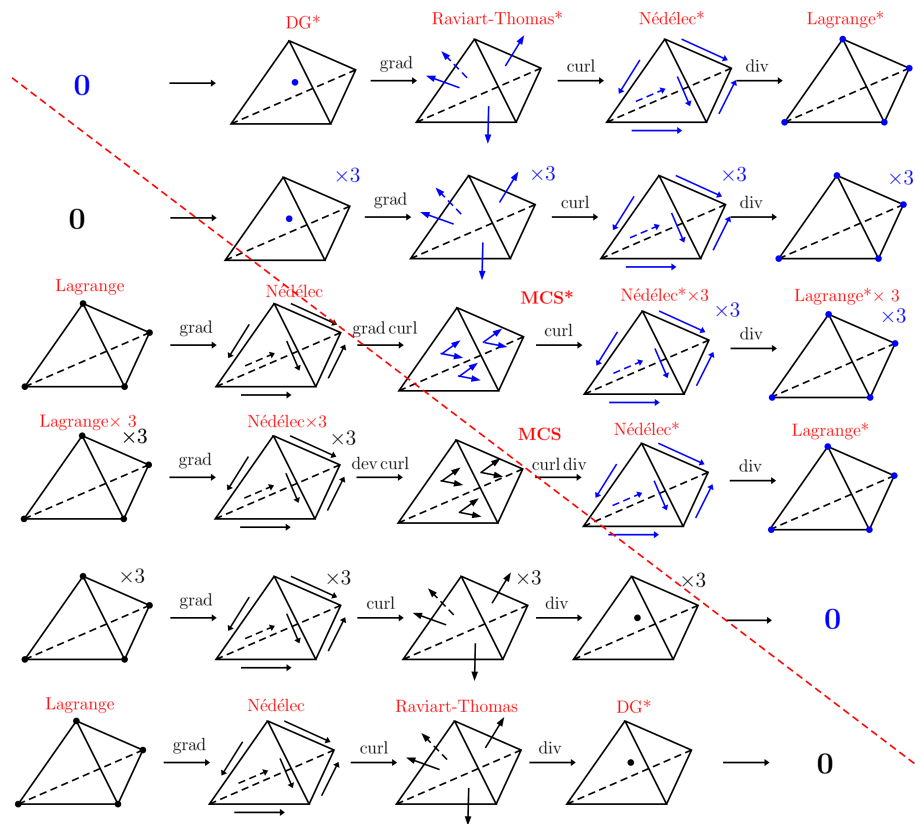
BGG happens not only between two consecutive rows...

$$\begin{array}{ccccccc}
 \text{Alt}^{0,k-1} & \text{Alt}^{1,k-1} & \text{Alt}^{2,k-1} & \dots & & & \\
 & \nearrow^{s_{[2]}^{0,k+1}} & \nearrow^{s_{[2]}^{1,k+1}} & & & & \\
 \text{Alt}^{0,k} & \text{Alt}^{1,k} & \text{Alt}^{2,k} & \dots & & & \\
 \text{Alt}^{0,k+1} & \text{Alt}^{1,k+1} & \text{Alt}^{2,k+1} & \dots & & & \\
 & \vdots & \vdots & \vdots & & \vdots & \\
 & \nearrow^{s_{[2]}^{0,\ell}} & \nearrow^{s_{[2]}^{1,\ell}} & & & & \\
 \text{Alt}^{0,\ell-1} & \text{Alt}^{1,\ell-1} & \text{Alt}^{2,\ell-1} & \dots & & & \\
 \text{Alt}^{0,\ell} & \text{Alt}^{1,\ell} & \text{Alt}^{2,\ell} & \dots & & &
 \end{array}$$

We can skip more rows.

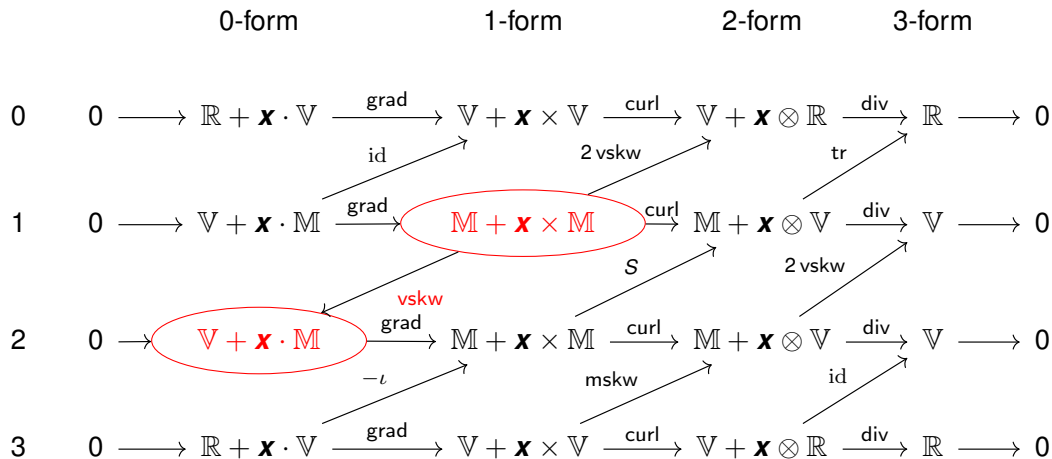
$$\begin{array}{ccccccc}
 0 \rightarrow \mathbb{R} & \xrightarrow{\text{grad}} & \mathbb{V} & \xrightarrow{\text{curl}} & \mathbb{V} & \xrightarrow{\text{div}} & \mathbb{R} \rightarrow 0 \\
 & & \nearrow^{\text{id}} & & \nearrow^{\text{tr}} & & \\
 0 \rightarrow \mathbb{V} & \xrightarrow{\text{grad}} & \mathbb{M} & \xrightarrow{\text{curl}} & \mathbb{M} & \xrightarrow{\text{div}} & \mathbb{V} \rightarrow 0 \\
 & & \nearrow^{\text{id}} & & \nearrow^{\text{id}} & & \\
 0 \rightarrow \mathbb{V} & \xrightarrow{\text{grad}} & \mathbb{M} & \xrightarrow{\text{curl}} & \mathbb{M} & \xrightarrow{\text{div}} & \mathbb{V} \rightarrow 0 \\
 & & \nearrow^{\text{id}} & & \nearrow^{\text{id}} & & \\
 0 \rightarrow \mathbb{R} & \xrightarrow{\text{grad}} & \mathbb{V} & \xrightarrow{\text{curl}} & \mathbb{V} & \xrightarrow{\text{div}} & \mathbb{R} \rightarrow 0
 \end{array}$$

DISCRETE LEVEL: INTRINSIC FINITE ELEMENTS



CONSTRUCTION OF SHAPE FUNCTIONS: ELIMINATION

Example: Regge



► Whitney (1,1) forms: $(\mathbb{V} + \mathbf{x} \times \mathbb{V}) \otimes \mathbb{V}$.

Whitney (0,2) forms: $(\mathbb{R} + \mathbf{x} \cdot \mathbb{V}) \otimes \mathbb{V}$.

► $\text{vskw} : \underbrace{\mathbb{M} + \mathbf{x} \times \mathbb{M}}_{\text{dim}=18} \rightarrow \underbrace{\mathbb{V} + \mathbf{x} \cdot \mathbb{M}}_{\text{dim}=12}$ is onto, and the kernel is $\mathbb{S}$.

TRACE AND GENERALIZED TRACE: CORRECT CONTINUITY & DOFS FOR $\text{Alt}^{k,\ell}$?

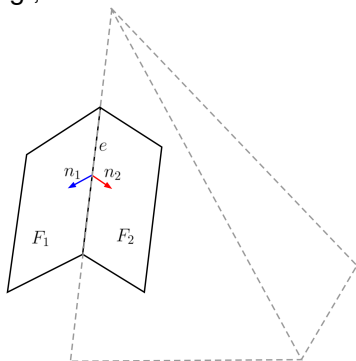
Trace of k -forms on j -cells: projection to the cell

$$\iota_F^* \omega(v_1, \dots, v_k) = \omega(\iota_* v_1, \dots, \iota_* v_k), \quad \omega \in C^\infty \text{Alt}^k(\Omega); v_1, \dots, v_k \in \mathcal{X}(\Omega).$$

trace of k -forms vanishes on ℓ -cells if $k > \ell$

Generalized trace j^* : *feeding tangent vectors as much as possible; allowing normal vectors if necessary.*

e.g., define trace of 2-forms on 1-cells. Feed 1 tangent vector and 1 normal ($n - 1$ choices) to the 2-form.



ω : 2-form, e : edge

Trace $\iota_e^* \omega = 0$

Generalized trace $j_e^* \omega \neq 0$,

project to F_1, F_2 .

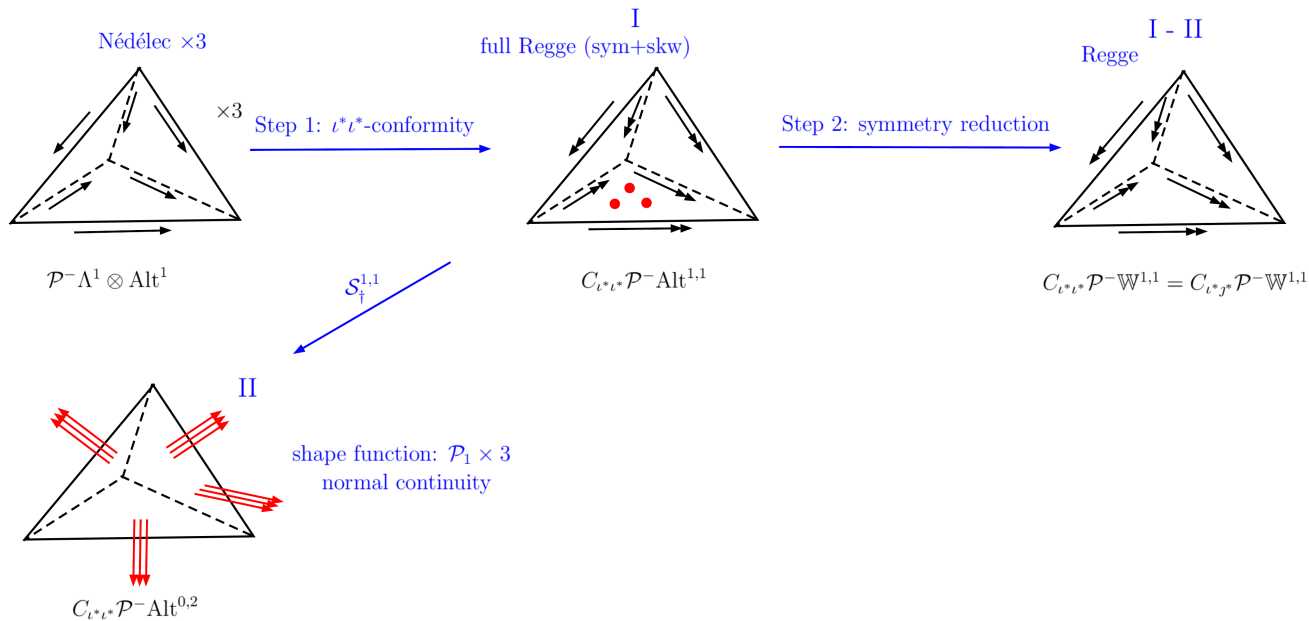
Trace ι^* (above) and generalized trace j^* (below) on k -forms in $\mathbb{R}^3$

$\begin{matrix} \text{dim}(e) \\ k \end{matrix}$	0	1	2
0	vertex value	edge value	face value
1	0	edge tangential	face tangential
2	0	0	face normal

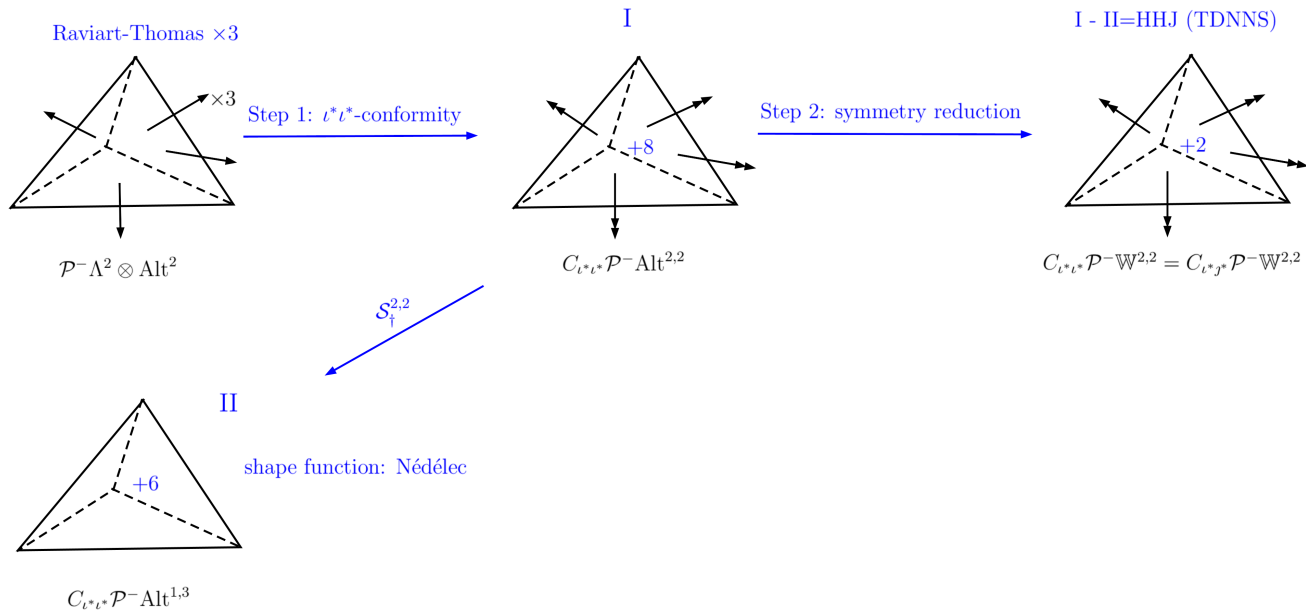
$\begin{matrix} \text{dim}(e) \\ k \end{matrix}$	0	1	2
0	vertex value	edge value	face value
1	vertex value	edge tangential	face tangential
2	vertex value	edge normal	face normal

Single-valuedness of the $\iota^* j^*$ -trace is the continuity we need for Hess, elasticity, divdiv etc.

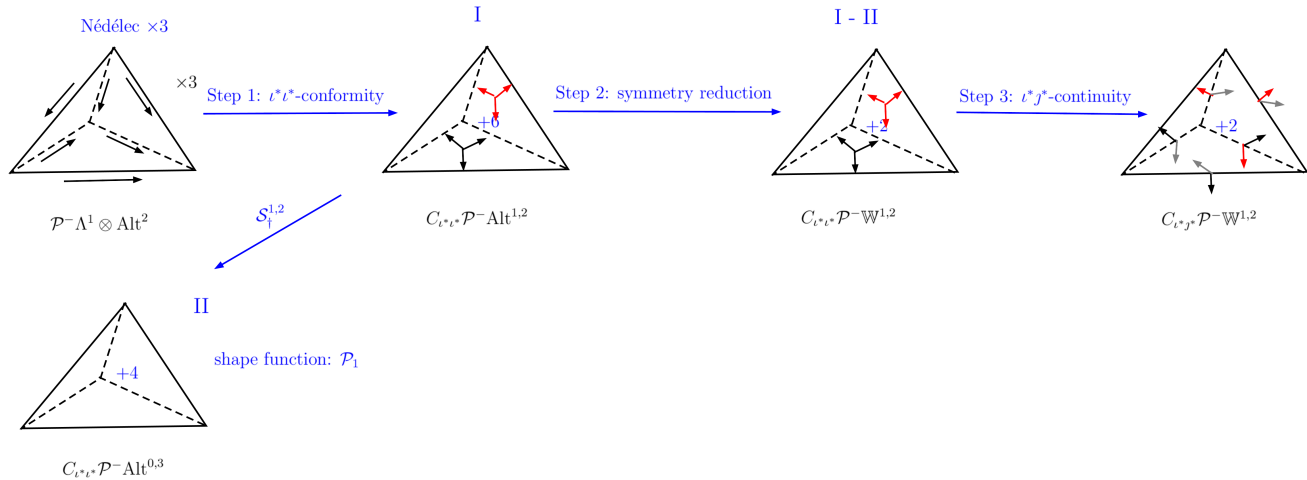
CONSTRUCTION THROUGH EXAMPLES: REGGE FROM NÉDÉLEC



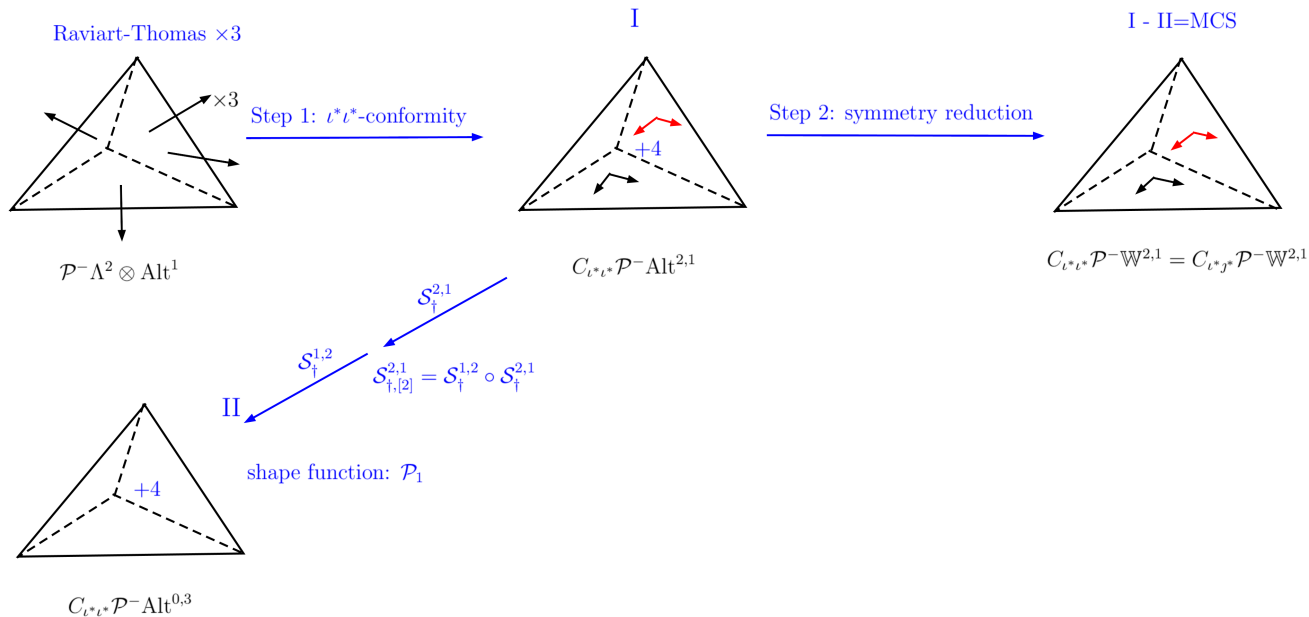
CONSTRUCTION THROUGH EXAMPLES: HHJ/TDNNS FROM RAVIART–THOMAS



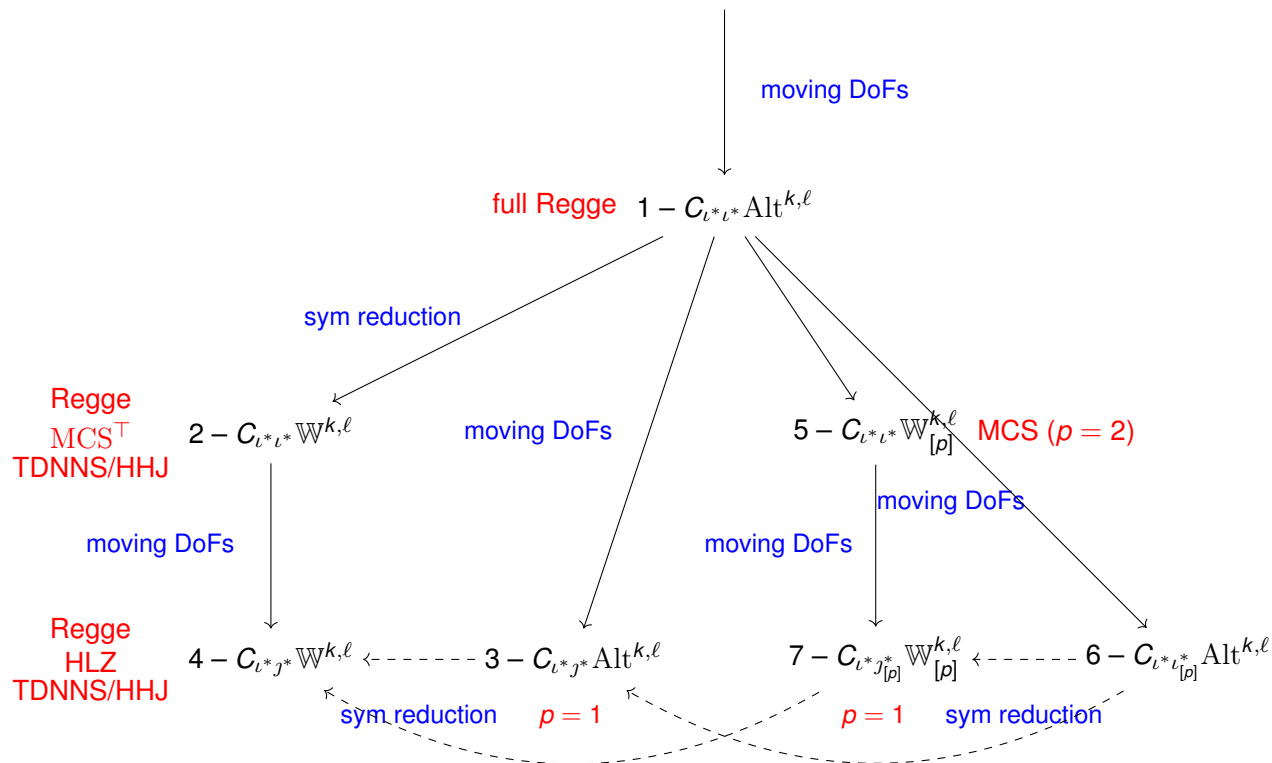
CONSTRUCTION THROUGH EXAMPLES: HLZ FROM NÉDÉLEC



CONSTRUCTION THROUGH EXAMPLES: MCS FROM RAVIART–THOMAS



Lagrange, Nédélec, Raviart–Thomas $C_{\iota^*} \text{Alt}^k = C_{\iota^* \rho^*} \text{Alt}^{k,\ell}$



SOME EXAMPLES IN 4D

► distributional curvature, 4D elasticity complex

$$0 \longrightarrow \mathbf{Lag} \otimes \mathbb{V} \xrightarrow{\text{sym grad}} \mathbf{Reg} \xrightarrow{\text{inc 4}} \bigoplus_{f \in \mathcal{T}_2^{\circ}} \delta_{mm}(f) \xrightarrow{\text{curl}} \bigoplus_{e \in \mathcal{T}_1^{\circ}} \delta_{mt}(e) \xrightarrow{\text{div}} \bigoplus_{v \in \mathcal{T}_0^{\circ}} \delta(v) \otimes \mathbb{K} \longrightarrow 0.$$

$\bigoplus_{f \in \mathcal{T}_2^{\circ}} \delta_{mm}(f)$: linear combination of functionals against (2,2)-forms on 2-cells.

(reminiscence of sectional curvature).

► curvature tensor (2,2)-form, dual of 4D elasticity complex

Side product: tensor decomposition from the BGG perspective

2-form valued 2-forms = 4-forms + 'skew (2, 2)-form' + Riemann-like tensor (algebraic curvature)

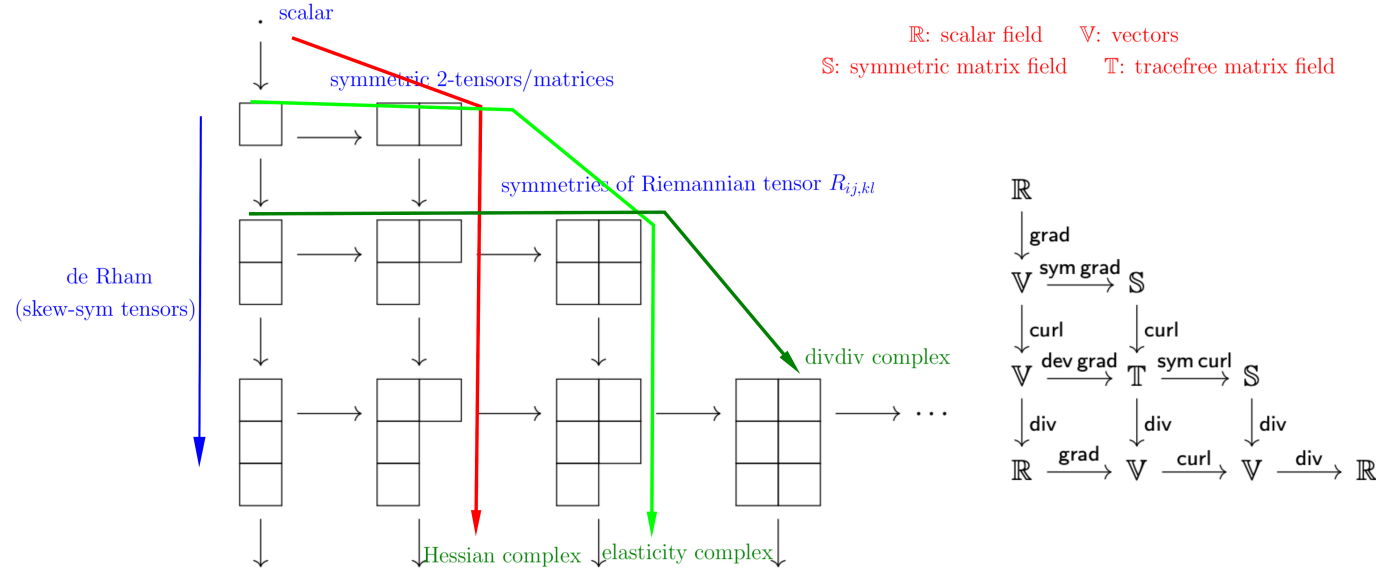
The diagram illustrates the decomposition of a 2-form valued 2-form (represented by two adjacent 2x2 grids) into three components: a 4-form (a single 4x1 grid), a skew (2,2)-form (two adjacent 2x1 grids), and a Riemann-like tensor (a 2x2 grid).

$$6 * 6 = 1 + 15 + 20$$

$$\text{Alt}^{2,2} = \mathcal{S}_{\dagger, [2]}^{2,2}(\text{Alt}^{2,2}) \oplus \left[\ker(\mathcal{S}_{\dagger, [2]}^{2,2}) \cap \mathcal{S}_{\dagger}^{2,2}(\text{Alt}^{2,2}) \right] \oplus \ker(\mathcal{S}_{\dagger}^{2,2})$$

Projections to the 1st and 3rd components: Bianchi symmetrization, Kulkarni–Nomizu operator

$$0 \longrightarrow \mathbf{Lag} \otimes \mathbb{K} \longrightarrow C_{\iota^* j^*} \mathcal{P}^- \mathbb{W}^{1,2} \longrightarrow C_{\iota^* \iota^*} \mathcal{P}^- \mathbb{W}^{2,2} \longrightarrow \bigoplus_{e \in \mathcal{T}_1^{\circ}} \delta_{tt}(e) \longrightarrow \bigoplus_{v \in \mathcal{T}_0^{\circ}} \delta(v) \otimes \mathbb{V} \longrightarrow 0.$$



Discretization: ongoing work with Jay Gangolekrihnan, Joachim Schöberl

[illegible]

$$\begin{array}{ccccccc}
\hat{H}^1 & \longrightarrow & \hat{H}(\text{curl}) & \longrightarrow & \hat{H}(\text{div}) & \longrightarrow & L_{2,\mathbb{R}} \\
\downarrow & \searrow * & \downarrow & \searrow * & \downarrow & \searrow * & \downarrow \searrow * \\
& H^{-1} & \longleftarrow & H^{-1}(\text{div}) & \longleftarrow & H^{-1}(\text{curl}) & \longleftarrow L_{2/\mathbb{R}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\hat{H}(\text{curl}) & \longrightarrow & \hat{H}_{\text{cc}} & \longrightarrow & \hat{H}_{\text{cd}} & \longrightarrow & \hat{H}_{\mathcal{RT}}^{-1}(\text{curl}) \\
\downarrow & \searrow * & \downarrow & \searrow * & \downarrow & \searrow * & \downarrow \searrow * \\
& H^{-1}(\text{div}) & \longleftarrow & H_{\text{dd}} & \longleftarrow & H_{\text{cd}\top} & \longleftarrow H(\text{div})/\mathcal{RT} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\hat{H}(\text{div}) & \longrightarrow & \hat{H}_{\text{cd}\top} & \longrightarrow & \hat{H}_{\text{dd}} & \longrightarrow & \hat{H}_{\mathcal{ND}}^{-1}(\text{div}) \\
\downarrow & \searrow * & \downarrow & \searrow * & \downarrow & \searrow * & \downarrow \searrow * \\
& H^{-1}(\text{curl}) & \longleftarrow & H_{\text{cd}} & \longleftarrow & H_{\text{cc}} & \longleftarrow H(\text{curl})/\mathcal{ND} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_{2,\mathbb{R}} & \longrightarrow & \hat{H}_{\mathcal{RT}}^{-1}(\text{curl}) & \longrightarrow & \hat{H}_{\mathcal{ND}}^{-1}(\text{div}) & \longrightarrow & \hat{H}_{\mathcal{P}_1}^{-1} \\
\downarrow & \searrow * & \downarrow & \searrow * & \downarrow & \searrow * & \downarrow \searrow * \\
& L_{2/\mathbb{R}} & \longleftarrow & H(\text{div})/\mathcal{RT} & \longleftarrow & H(\text{curl})/\mathcal{ND} & \longleftarrow H^1/\mathcal{P}_1
\end{array}$$

Gopalakrishnan, J., Hu, K., & Schöberl, J. (2025). A 2-complex containing Sobolev spaces of matrix fields. arXiv preprint arXiv:2507.11869.

PERSPECTIVES

Discretization of the entire BGG *diagram*; $\mathcal{P}_r^- \mathbb{W}_{[p]}^{k,\ell}$, $\mathcal{P}_r \mathbb{W}_{[p]}^{k,\ell}$, for any r, k, ℓ, p (BGG level), dimension n .

Intrinsic finite elements

Regge is an example, only using geometric information of the triangulation, not the ambient space. Allowing flexible and complicated geometries (mixed dimension etc.), and lack of information about embedded spaces.

Interactions between finite elements + Discrete Differential Geometry + PDEs

What is canonical discretization for geometric objects? Graphics, materials such as origami etc.

DEC version?



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Patrick Farrell, *Oxford U.*

Tammy Kolda, *MathSci.ai*

Buyang Li, *Hong Kong Poly.*

Weiqing Ren, *NUS Singapore*

J.M. Sanz-Serna, *U. Carlos III, Madrid*

Aretha Teckentrup, *Edinburgh U.*

Xiaochuan Tian, *UC San Diego*

Richard Turner, *Cambridge U.*

Scientific Committee: W. Bao (Singapore), Q. Du (Columbia), E. Faou (Rennes), B. Leimkuhler (Edinburgh), M. Leok (San Diego), C. Lubich (Tübingen), A. Ostermann (Innsbruck), L. Petzold (Santa Barbara), T. Tang (Hong Kong), C. Woodward (Liverpool)

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