Blow-up Whitney forms

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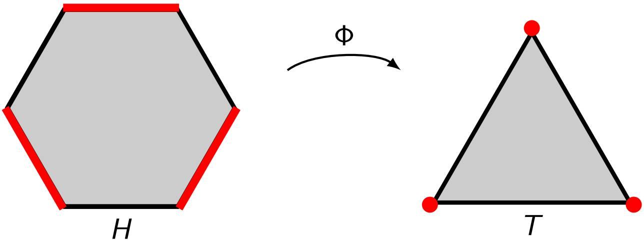
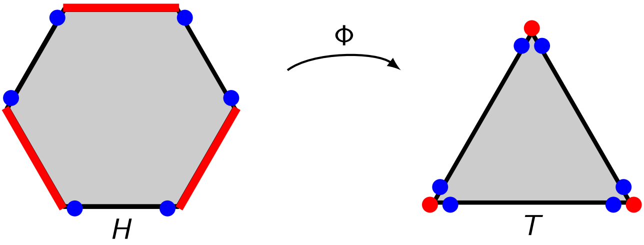
Motivating example



Problem

Find a continuous map $\Phi : H \to T$ that collapses the red edges and restricts to a diffeomorphism $\Phi|_{\mathring{H}} : \mathring{H} \to \mathring{T}$.

Motivating example

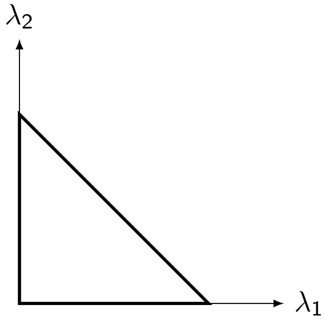


Problem

Find a continuous map $\Phi : H \to T$ that collapses the red edges and restricts to a diffeomorphism $\Phi|_{\mathring{H}} : \mathring{H} \to \mathring{T}$.

Observe: Φ^{-1} must be discontinuous at the vertices of T.

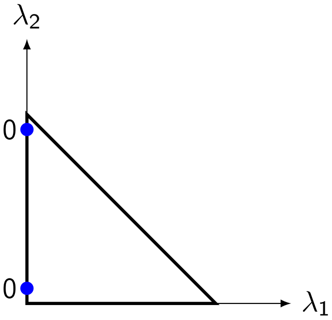
A discontinuous function



$$\psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0}$$

$$(\lambda_0 + \lambda_1 + \lambda_2 = 1)$$

A discontinuous function

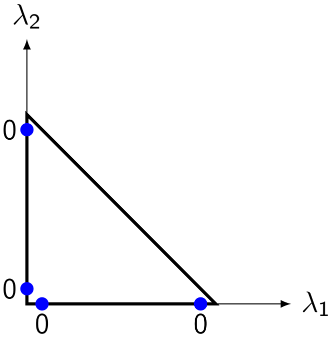


$$\psi_{120} = \frac{\lambda_1 \lambda_2}{\lambda_2 + \lambda_0}$$

$$(\lambda_0 + \lambda_1 + \lambda_2 = 1)$$

- Along the edge $\lambda_1 = 0$, $\psi_{120} = 0$.

A discontinuous function

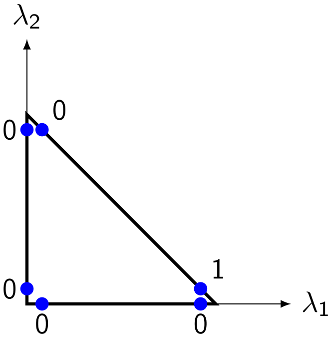


$$\psi_{120} = \frac{\lambda_1 \lambda_2}{\lambda_2 + \lambda_0}$$

$$(\lambda_0 + \lambda_1 + \lambda_2 = 1)$$

- Along the edge $\lambda_1 = 0$, $\psi_{120} = 0$.
- Along the edge $\lambda_2 = 0$, $\psi_{120} = 0$.

A discontinuous function



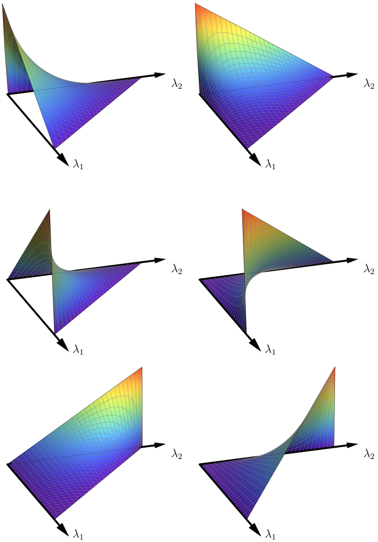
$$\psi_{120} = \frac{\lambda_1 \lambda_2}{\lambda_2 + \lambda_0}$$

$$(\lambda_0 + \lambda_1 + \lambda_2 = 1)$$

- Along the edge $\lambda_1 = 0$, $\psi_{120} = 0$.
- Along the edge $\lambda_2 = 0$, $\psi_{120} = 0$.
- Along the edge $\lambda_0 = 0$, $\psi_{120} = \lambda_1$.

Scalar-valued blow-up finite elements

$$b\mathcal{P}_1(T) = \text{span}\{\psi_{012}, \psi_{120}, \psi_{201}, \psi_{102}, \psi_{210}, \psi_{021}\}$$



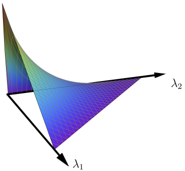
$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1 + \lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2 + \lambda_1},$$

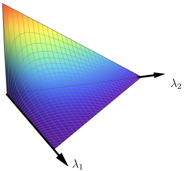
$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0 + \lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2 + \lambda_0},$$

$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0 + \lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1 + \lambda_0}.$$

Scalar-valued blow-up finite elements

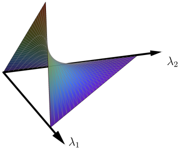
$$b\mathcal{P}_1(T) = \mathrm{span}\{\psi_{012}, \psi_{120}, \psi_{201}, \psi_{102}, \psi_{210}, \psi_{021}\} \supset \mathcal{P}_1(T)$$

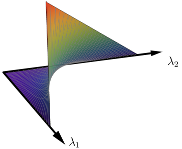




$$\psi_{012} = \frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}, \quad \psi_{021} = \frac{\lambda_0\lambda_2}{\lambda_2+\lambda_1},$$

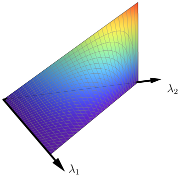
$$\psi_{012} + \psi_{021} = \lambda_0$$

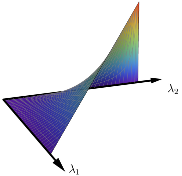




$$\psi_{102} = \frac{\lambda_1\lambda_0}{\lambda_0+\lambda_2}, \quad \psi_{120} = \frac{\lambda_1\lambda_2}{\lambda_2+\lambda_0},$$

$$\psi_{102} + \psi_{120} = \lambda_1$$





$$\psi_{201} = \frac{\lambda_2\lambda_0}{\lambda_0+\lambda_1}, \quad \psi_{210} = \frac{\lambda_2\lambda_1}{\lambda_1+\lambda_0}.$$

$$\psi_{201} + \psi_{210} = \lambda_2$$

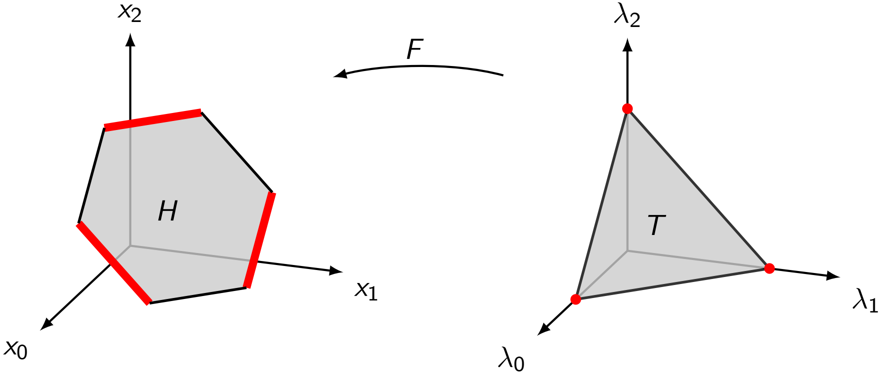
Scalar-valued blow-up finite elements

$$\psi_{ijk} = \frac{\lambda_i \lambda_j}{\lambda_j + \lambda_k} = \frac{\lambda_i \lambda_j}{(\lambda_i + \lambda_j + \lambda_k)(\lambda_j + \lambda_k)}$$

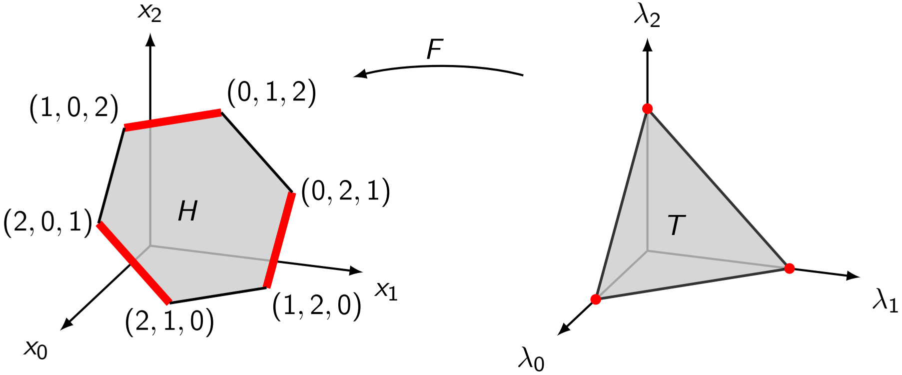
Key property

$$\lim_{\lambda_{j'} \to 0} \lim_{\lambda_{k'} \to 0} \psi_{ijk} = \begin{cases} 1 & \text{if } j = j' \text{ and } k = k', \\ 0 & \text{otherwise.} \end{cases}$$

Mapping a triangle to a hexagon



Mapping a triangle to a hexagon



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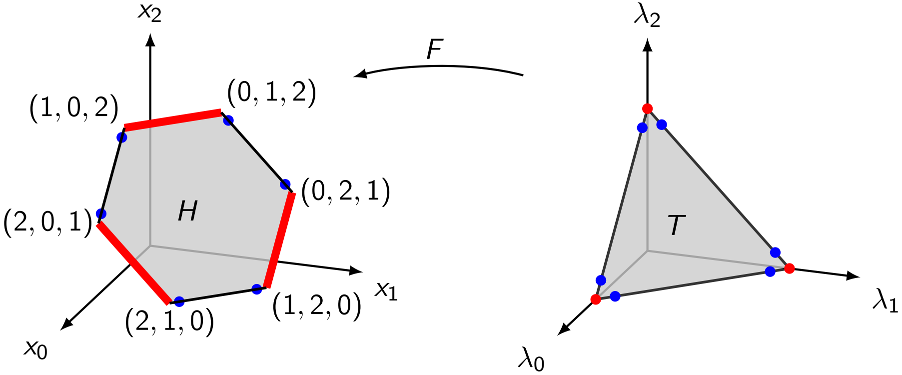
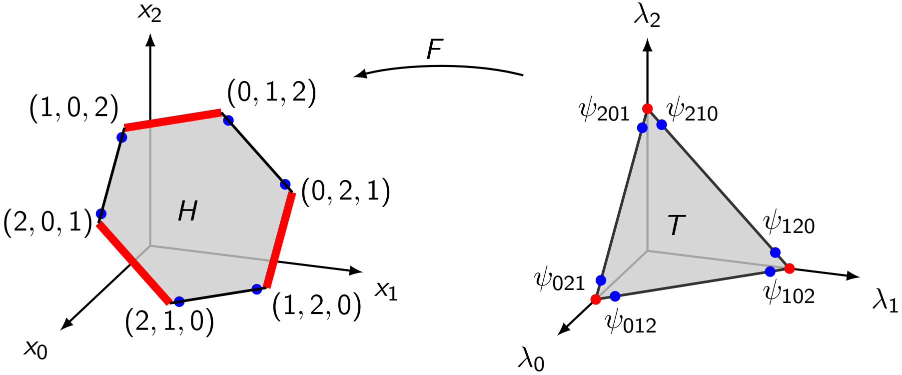
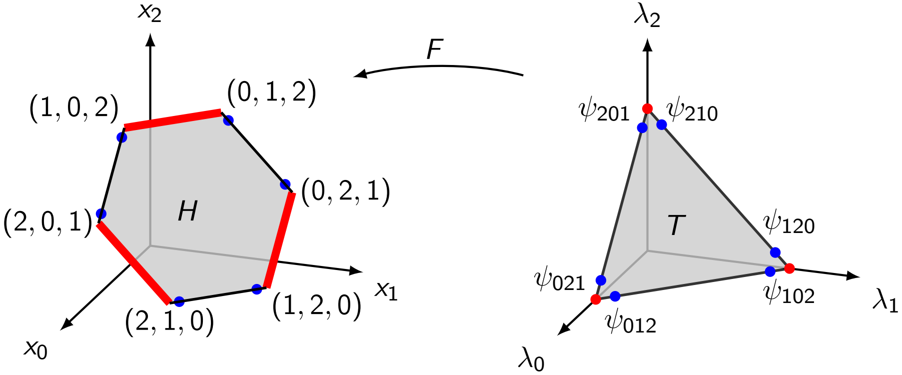


Mapping a triangle to a hexagon

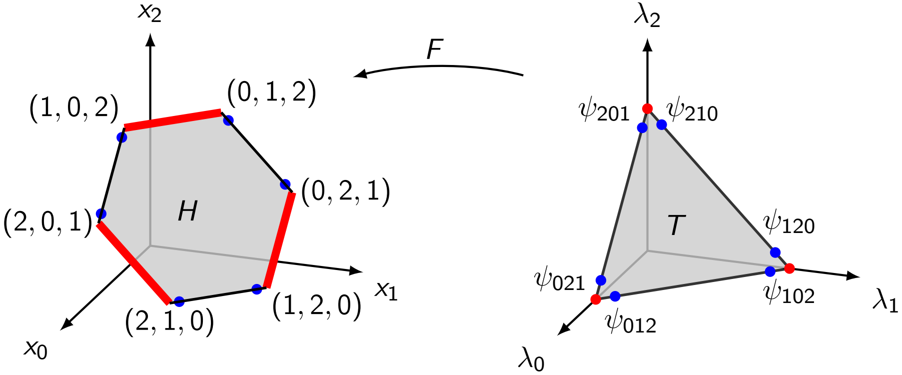


Mapping a triangle to a hexagon



$$\begin{aligned} F(\lambda_0, \lambda_1, \lambda_2) = \quad & (2,1,0)\psi_{012} + (1,2,0)\psi_{102} + (0,2,1)\psi_{120} \\ & + (0,1,2)\psi_{210} + (1,0,2)\psi_{201} + (2,0,1)\psi_{021} \end{aligned}$$

Mapping a triangle to a hexagon



$$\begin{aligned}F(\lambda_0, \lambda_1, \lambda_2) = \quad & (2,1,0)\psi_{012} + (1,2,0)\psi_{102} + (0,2,1)\psi_{120} \\ & + (0,1,2)\psi_{210} + (1,0,2)\psi_{201} + (2,0,1)\psi_{021} \\ & = \sum_{\sigma} (\sigma^{-1}(0), \sigma^{-1}(1), \sigma^{-1}(2))\psi_{\sigma(2)\sigma(1)\sigma(0)}.\end{aligned}$$

Mapping a triangle to a hexagon

$$F(\lambda_0, \lambda_1, \lambda_2) = \sum_{\sigma} (\sigma^{-1}(0), \sigma^{-1}(1), \sigma^{-1}(2)) \psi_{\sigma(2)\sigma(1)\sigma(0)}.$$

Lemma

$$\sum_{\sigma} \sigma^{-1}(i) \psi_{\sigma(2)\sigma(1)\sigma(0)} = \lambda_i \sum_{j \neq i} \frac{1}{\lambda_i + \lambda_j}$$

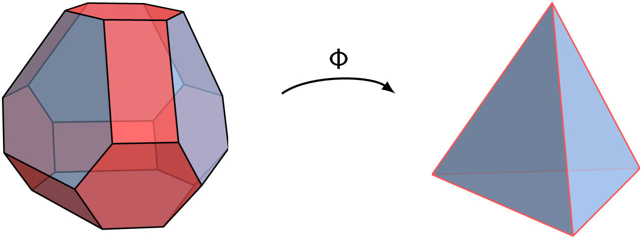
Theorem [Matúš 2003]

The map $F : (\lambda_0, \lambda_1, \lambda_2) \mapsto (x_0, x_1, x_2)$ given by

$$x_i = \lambda_i \sum_{j \neq i} \frac{1}{\lambda_i + \lambda_j}$$

is a diffeomorphism from $\mathring{T}$ to $\mathring{H}$.

Higher dimensions

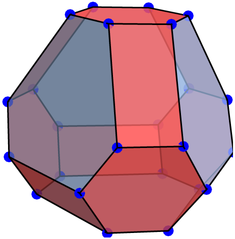


$P^n = n$-permutohedron $T^n = n$-simplex

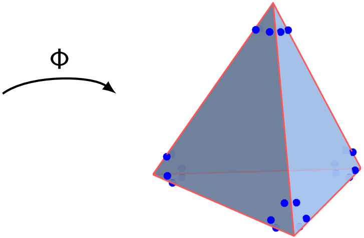
Problem

Find a continuous map $\Phi : P^n \to T^n$ that collapses the red faces and restricts to a diffeomorphism $\Phi|_{\mathring{P}^n} : \mathring{P}^n \to \mathring{T}^n$.

Higher dimensions



$P^n = n$-permutohedron



$T^n = n$-simplex

Problem

Find a continuous map $\Phi : P^n \to T^n$ that collapses the red faces and restricts to a diffeomorphism $\Phi|_{\mathring{P}^n} : \mathring{P}^n \to \mathring{T}^n$.

Scalar-valued blow-up finite elements in n dimensions

$$\psi_{i_0 i_1 \dots i_n} = \frac{\lambda_{i_0}\lambda_{i_1}\cdots\lambda_{i_{n-1}}}{\prod_{j=0}^{n-1}(\lambda_{i_j} + \lambda_{i_{j+1}} + \cdots + \lambda_{i_n})}$$

Key property

$$\lim_{\lambda_{i'_1}\to 0}\lim_{\lambda_{i'_2}\to 0}\cdots\lim_{\lambda_{i'_n}\to 0}\psi_{i_0 i_1 \dots i_n} = \begin{cases} 1 & \text{if } (i_1, i_2, \dots, i_n) = (i'_1, i'_2, \dots, i'_n), \\ 0 & \text{otherwise.} \end{cases}$$

Example (Dimension $n = 3$)

$$\psi_{0123} = \frac{\lambda_0\lambda_1\lambda_2}{(\lambda_0+\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)(\lambda_2+\lambda_3)}.$$

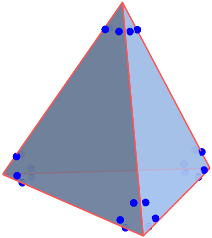
We have

$$\lim_{\lambda_1\to 0}\lim_{\lambda_2\to 0}\lim_{\lambda_3\to 0}\psi_{0123} = 1,$$

whereas, e.g.,

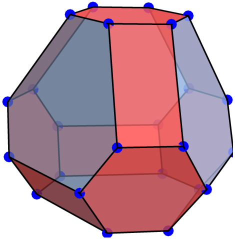
$$\lim_{\lambda_2\to 0}\lim_{\lambda_1\to 0}\lim_{\lambda_3\to 0}\psi_{0123} = 0.$$

Scalar-valued blow-up finite elements in $n = 3$ dimensions

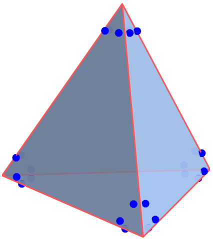


Mnemonic: ψ_{ijkl} "equals" 1 at one blue point and 0 at the others.

Mapping a simplex to a permutohedron



$P^n = n$-permutohedron



$T^n = n$-simplex

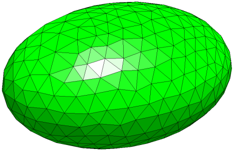
Solution

$$F(\lambda_0, \lambda_1, \ldots, \lambda_n) = \sum_{\sigma} (\sigma^{-1}(0), \sigma^{-1}(1), \ldots, \sigma^{-1}(n)) \psi_{\sigma(n)\sigma(n-1)\ldots\sigma(0)}.$$

Equivalently: $F_i(\lambda_0, \lambda_1, \ldots, \lambda_n) = \lambda_i \sum_{j \neq i} \frac{1}{\lambda_i + \lambda_j}.$

Outline

Finite elements for the Bochner Laplacian



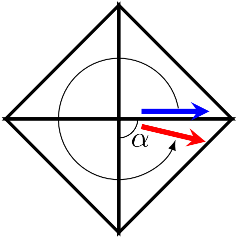
Goal: Vector-valued finite elements for the ***Bochner Laplacian*** (a.k.a. rough Laplacian or connection Laplacian)

$$\Delta = -\nabla^*\nabla = \mathrm{Tr}\,\nabla\nabla.$$

- When applied to vector fields on a surface $\Gamma \subset \mathbb{R}^3$, this is the usual surface vector Laplacian Δ_Γ.
- Note: Bochner Laplacian $\neq$ Hodge Laplacian.

Finite elements for the Bochner Laplacian

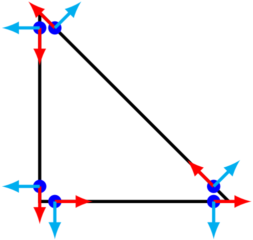
- Well-known difficulty: H^1-conforming vector fields on triangulated surfaces are difficult to construct.
- Reason: Any piecewise smooth, tangentially and normally continuous vector field must vanish at (generic) vertices.
- The ***angle defect*** is the source of the problem.



$\alpha > \frac{\pi}{2}$

- Idea: Use finite elements that incorporate "rapid rotations" near the vertices.

Vector-valued blow-up finite elements



- Recall: $\psi_{ijk} = \dfrac{\lambda_i \lambda_j}{\lambda_j + \lambda_k} =: \psi_{ij}$
- Local finite element space on a triangle T:

$$\mathbf{b}\mathcal{P}_1(T) = \left\{ \sum_{\substack{i,j=0 \\ i \neq j}}^{2} \psi_{ij}(a_{ij}\boldsymbol{\tau}_{ij} + b_{ij}\mathbf{n}_{ij}) \mid a_{ij}, b_{ij} \in \mathbb{R} \right\}$$

- Global finite element space on a triangulation $\mathcal{T}$:

$$\mathbf{b}\mathcal{P}_1(\mathcal{T}) = \{\mathbf{u} \mid \mathbf{u}|_T \in \mathbf{b}\mathcal{P}_1(T) \,\forall T,\ [\![\mathbf{u}\cdot\mathbf{n}]\!] = [\![\mathbf{u}\cdot\boldsymbol{\tau}]\!] = 0 \,\forall e\}$$

- 4 DOFs per edge: 2 a_{ij}'s (1 per endpoint) and 2 b_{ij}'s (1 per endpoint).
- $W^{1,p}$-conforming for all $p < 2$.

Numerical experiment

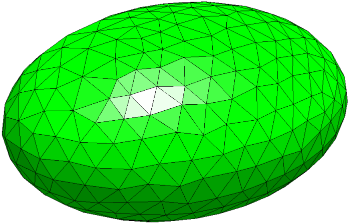
- Eigenvalue problem for the surface vector Laplacian:

$$-\Delta_\Gamma \mathbf{u} = \lambda \mathbf{u}$$

- Discretization: Find $\mathbf{u}_h \in \mathbf{b}\mathcal{P}_1(\mathcal{T})$ and $\lambda_h \in \mathbb{R}$ such that

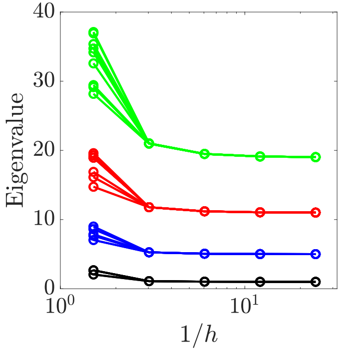
$$\int_{\Gamma_h} \langle \nabla_{\Gamma_h} \mathbf{u}_h, \nabla_{\Gamma_h} \mathbf{v}_h \rangle \omega_\varepsilon = \lambda_h \int_{\Gamma_h} \langle \mathbf{u}_h, \mathbf{v}_h \rangle \omega_\varepsilon, \quad \forall \mathbf{v}_h \in \mathbf{b}\mathcal{P}_1(\mathcal{T}),$$

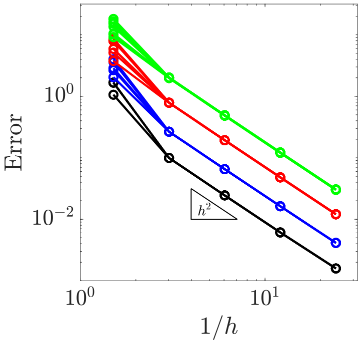
where ω_ε = modified volume form on Γ_h.



Numerical experiment

Eigenvalues of the vector Laplacian on the unit sphere:

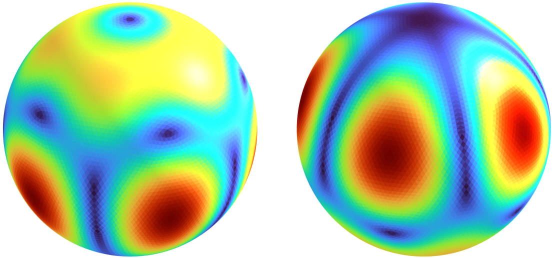




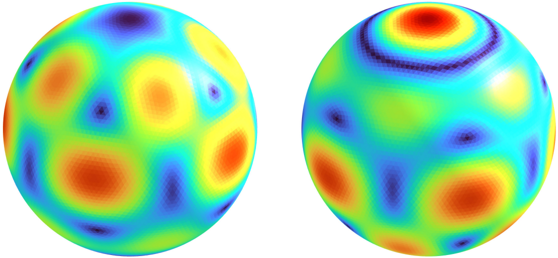
The exact eigenvalues are $\lambda = 1, 5, 11, 19, \ldots$ with multiplicities $6, 10, 14, 18, \ldots$.

Numerical experiment

$\lambda = 11$:



$\lambda = 19$:



Numerical experiment

Remark: This discretization is ***intrinsic***. Vertex coordinates play no role in the calculation; only edge lengths do.

Analysis of convergence

Analysis still in progress, but one encouraging result:

Proposition

Consider the following discretization of $-\Delta u = \lambda u$ on a flat domain $\Omega \subset \mathbb{R}^n$: Find $u_h \in V_h = \mathcal{P}_1(\mathcal{T})$ and $\lambda_h \in \mathbb{R}$ such that

$$a_\varepsilon(u_h, v_h) = \lambda_h m_\varepsilon(u_h, v_h), \quad \forall v_h \in V_h,$$

where

$$m_\varepsilon(u, v) = \sum_{T \in \mathcal{T}_h} \int_{T_\varepsilon} uv\, dx, \quad a_\varepsilon(u, v) = \sum_{T \in \mathcal{T}_h} \int_{T_\varepsilon} \nabla u \cdot \nabla v\, dx,$$

$$T_\varepsilon = \{\lambda \in T \mid \lambda_i + \lambda_j \geq \varepsilon \text{ for each } i \neq j\},$$

and $\varepsilon > 0$ is fixed. Then for each $i = 1, 2, \ldots, \dim V_h$, we have

$$\min_j |\lambda_h^{(j)} - \lambda^{(i)}| \leq Ch.$$

Outline

A differential complex

- The space $b\mathcal{P}_1$ can be generalized to differential forms on n-simplices, leading to a differential complex

$$b\mathcal{P}_1^-\Lambda^0(T^n) \xrightarrow{d} b\mathcal{P}_1^-\Lambda^1(T^n) \xrightarrow{d} b\mathcal{P}_1^-\Lambda^2(T^n) \xrightarrow{d} \cdots \xrightarrow{d} b\mathcal{P}_1^-\Lambda^n(T^n)$$

that was studied by [Brasselet, Goresky, & MacPherson 1991].

- These differential forms are called ***shadow forms*** / ***blow-up Whitney forms***.

- In our paper [Berchenko-Kogan & G. 2025], we construct:
  - Degrees of freedom (DOFs) for these spaces.
  - A combinatorial procedure for computing bases dual to these DOFs (via a surprising link with Poisson processes).

Blow-up Whitney forms in 2D

Spaces:

0-forms: $\left\{\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}, \frac{\lambda_0\lambda_2}{\lambda_2+\lambda_1}, \frac{\lambda_1\lambda_0}{\lambda_0+\lambda_2}, \frac{\lambda_1\lambda_2}{\lambda_2+\lambda_0}, \frac{\lambda_2\lambda_0}{\lambda_0+\lambda_1}, \frac{\lambda_2\lambda_1}{\lambda_1+\lambda_0}\right\}$

1-forms: $\left\{\varphi_{01}, \varphi_{12}, \varphi_{20}, \frac{p_{2\{01\}}\varphi_{01}}{(\lambda_0+\lambda_1)^2}, \frac{p_{0\{12\}}\varphi_{12}}{(\lambda_1+\lambda_2)^2}, \frac{p_{1\{20\}}\varphi_{20}}{(\lambda_2+\lambda_0)^2}\right\}$

2-forms: $\{\varphi_{012}\}$

where

$$\begin{aligned}
\varphi_{ij} &= \lambda_i d\lambda_j - \lambda_j d\lambda_i, \\
\varphi_{ijk} &= \lambda_i d\lambda_j \wedge d\lambda_k + \lambda_j d\lambda_k \wedge d\lambda_i + \lambda_k d\lambda_i \wedge d\lambda_j, \\
p_{i\{jk\}} &= \lambda_i(1+\lambda_j+\lambda_k).
\end{aligned}$$

Degrees of freedom:

0-forms: "Evaluation" at the 6 edge endpoints.

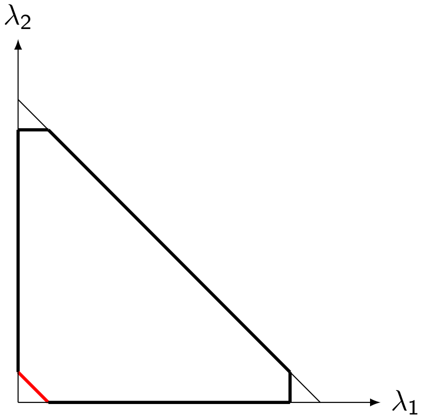
1-forms: Integration over the 3 original edges and 3 "tiny edges".

2-forms: Integration over T.

Degrees of freedom

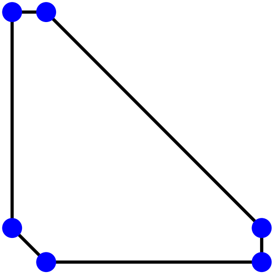
Integration over a "tiny edge" means:

- Change from $(\lambda_0, \lambda_1, \lambda_2)$ to $(r, \theta) := \left(\lambda_1 + \lambda_2, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$.
- Take the limit as $r \to 0$.
- Integrate with respect to θ.



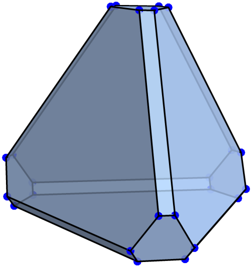
Degrees of freedom

Observe: Each degree of freedom is associated with a k-face of the hexagon.



Blow-up Whitney forms in 3D

In 3D, each degree of freedom is associated with a k-face of the permutohedron P^3.



Number of degrees of freedom:

0-forms: 24

1-forms: 36 (12 big edges, 12 small edges, 12 tiny edges)

2-forms: 14 (4 big hexagons, 6 rectangles, 4 small hexagons)

3-forms: 1

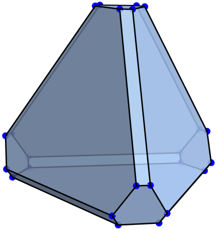
The exterior derivative

Proposition

Let F be a k-dimensional face of P^n, and let ψ_F be the corresponding blow-up Whitney k-form. Then

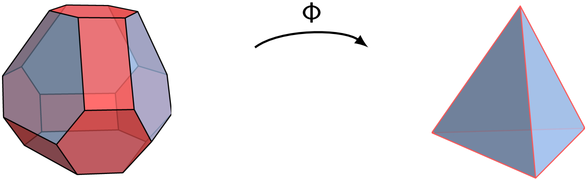
$$d\psi_F = \pm \sum_{j=1}^{n-k} \psi_{F_j},$$

where $F_1, F_2, \ldots, F_{n-k}$ are the $(k+1)$-dimensional faces incident to F.

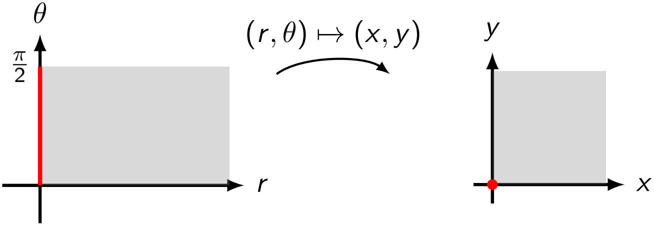


The word "blow-up"

The blow-up Whitney forms are not smooth on T^n, but their pullbacks to P^n (the ***blow-up*** of T^n) are smooth.

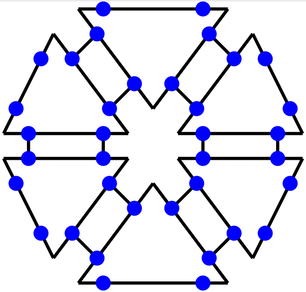


Analogy: The function $f(x, y) = \arctan(y/x)$ is not smooth on $[0, \infty) \times [0, \infty)$, but its pullback to $[0, \infty) \times [0, \pi/2]$ is smooth.

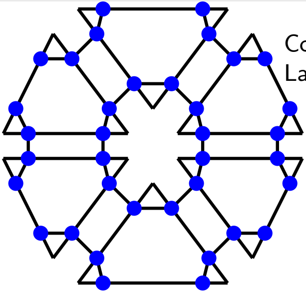


Different ways to glue degrees of freedom

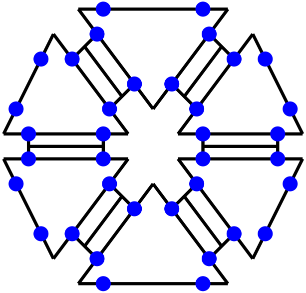
Single-valued on edges, multi-valued at vertices



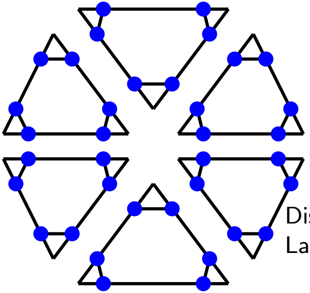
Continuous Lagrange



Single-valued and constant on edges, multi-valued at vertices



Discontinuous Lagrange



A surprising link with Poisson processes

The formulas for the bases have probabilistic interpretations.

Example 1: The 0-form $\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}$.

- Three radiation sources with rates $\lambda_0, \lambda_1, \lambda_2$ that sum to 1.
- Let t_0, t_1, t_2 be the times when the respective sources produce their first particle.
- The probability that $t_0 \leq t_1 \leq t_2$ is $\frac{\lambda_0\lambda_1}{\lambda_1+\lambda_2}$.

Example 2: The 1-form $\frac{p_{0\{12\}}\varphi_{12}}{(\lambda_1+\lambda_2)^2}$, $p_{0\{12\}} = \lambda_0(1+\lambda_1+\lambda_2)$.

- Two radiation sources A and B with rates λ_0 and $\lambda_1+\lambda_2$ that sum to 1.
- Let t_A be the time when source A produces its first particle.
- Let t_B be the time when source B produces its second particle.
- The probability that $t_A \leq t_B$ is $\lambda_0(1+\lambda_1+\lambda_2)$.

A surprising link with horse race betting

$$\text{Recall: } \psi_{ijkl} = \frac{\lambda_i\lambda_j\lambda_k}{(\lambda_i+\lambda_j+\lambda_k+\lambda_l)(\lambda_j+\lambda_k+\lambda_l)(\lambda_k+\lambda_l)} = \frac{\lambda_i\lambda_j\lambda_k}{(1-\lambda_i)(1-\lambda_i-\lambda_j)}.$$

Benter '94, "Computer based horse race handicapping & wagering systems: A report"

In exotic bets that involve specifying the finishing order of two or more horses in one race, a method is needed to estimate these probabilities. A popular approach is the Harville formula. (Harville, 1973):

For three horses (i, j, k) with win probabilities (π_i, π_j, π_k) the Harville formula specifies the probability that they will finish in order as

$$\pi_{ijk} = \frac{\pi_i\pi_j\pi_k}{(1-\pi_i)(1-\pi_i-\pi_j)} \qquad (6)$$

This formula is significantly biased, and should not be used for betting purposes, as it will lead to serious errors in probability estimations if not corrected for in some way.[3] (Henery 1981, Stern 1990,

For further reading

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Thank you!

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