JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Retaining the Structure of Second Order Boundary Value Problems in Finite Dimensional Settings

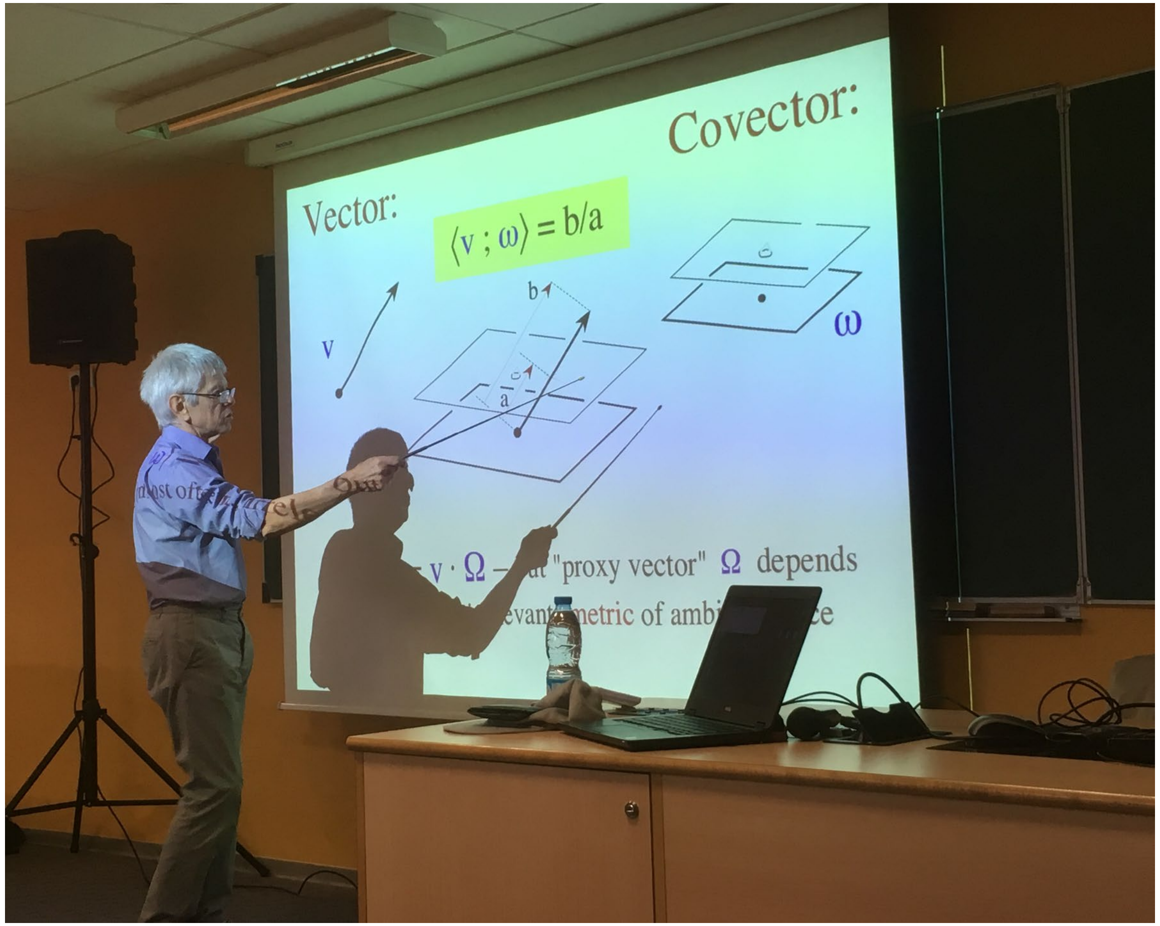
Sept. 5th, 2025

Lauri Kettunen

Background

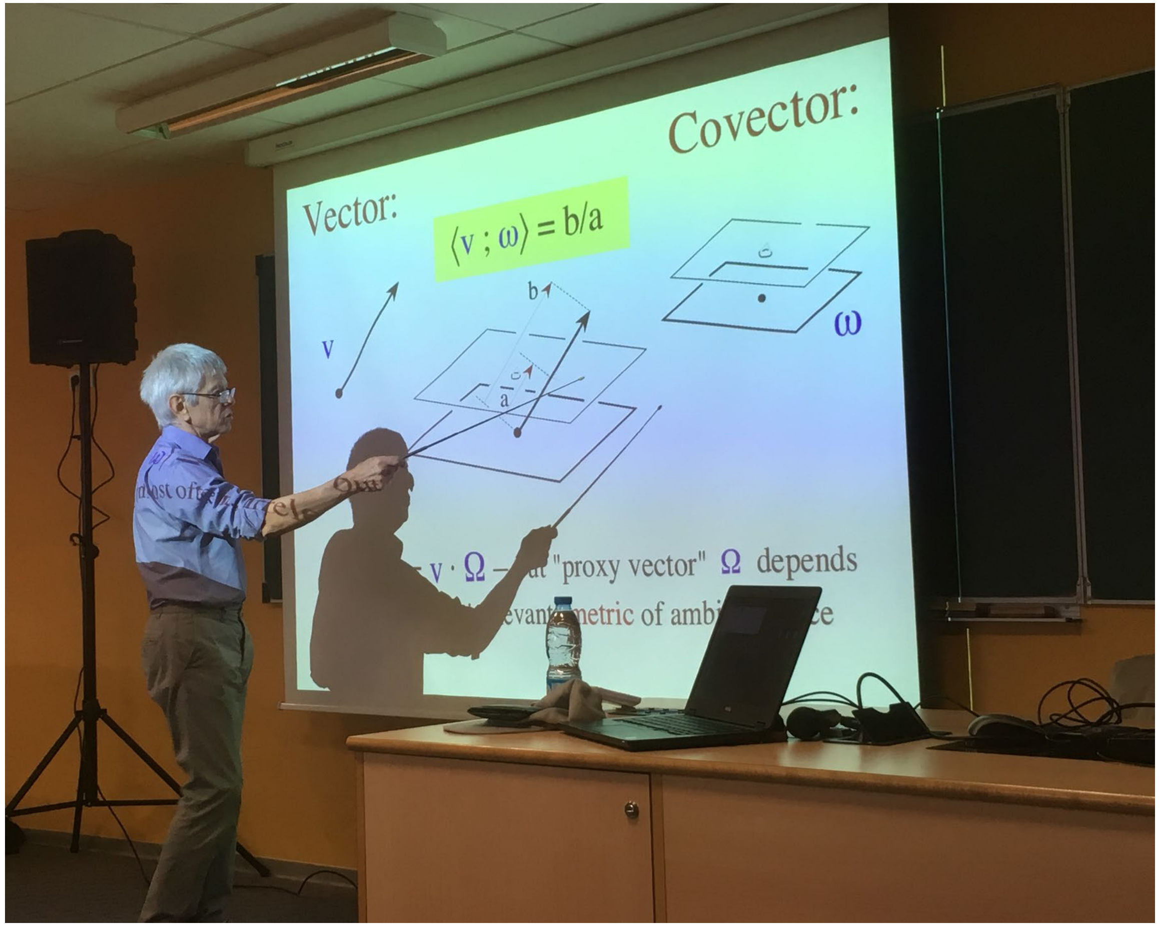
- This talk is very much motivated by the question:
  - *How should software for solving boundary value problems be designed to efficiently address diverse problem classes while remaining extensible for new needs?*

Background



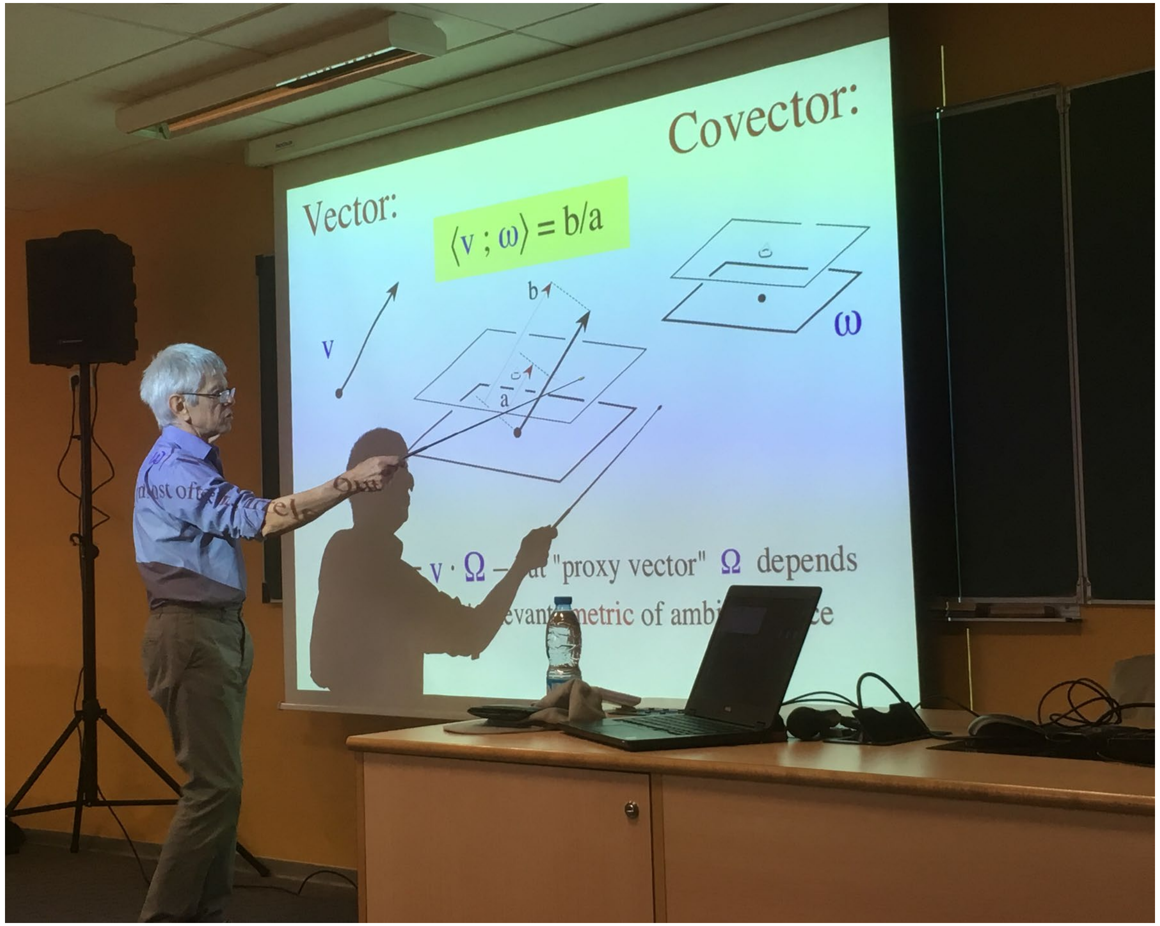
I would like to dedicate this presentation to the memory of my long-term collaborator *Alain Bossavit*, 1942-2025.

I Structure



"*It's because the methods themselves are just* superstructures *above the real infrastructure*."

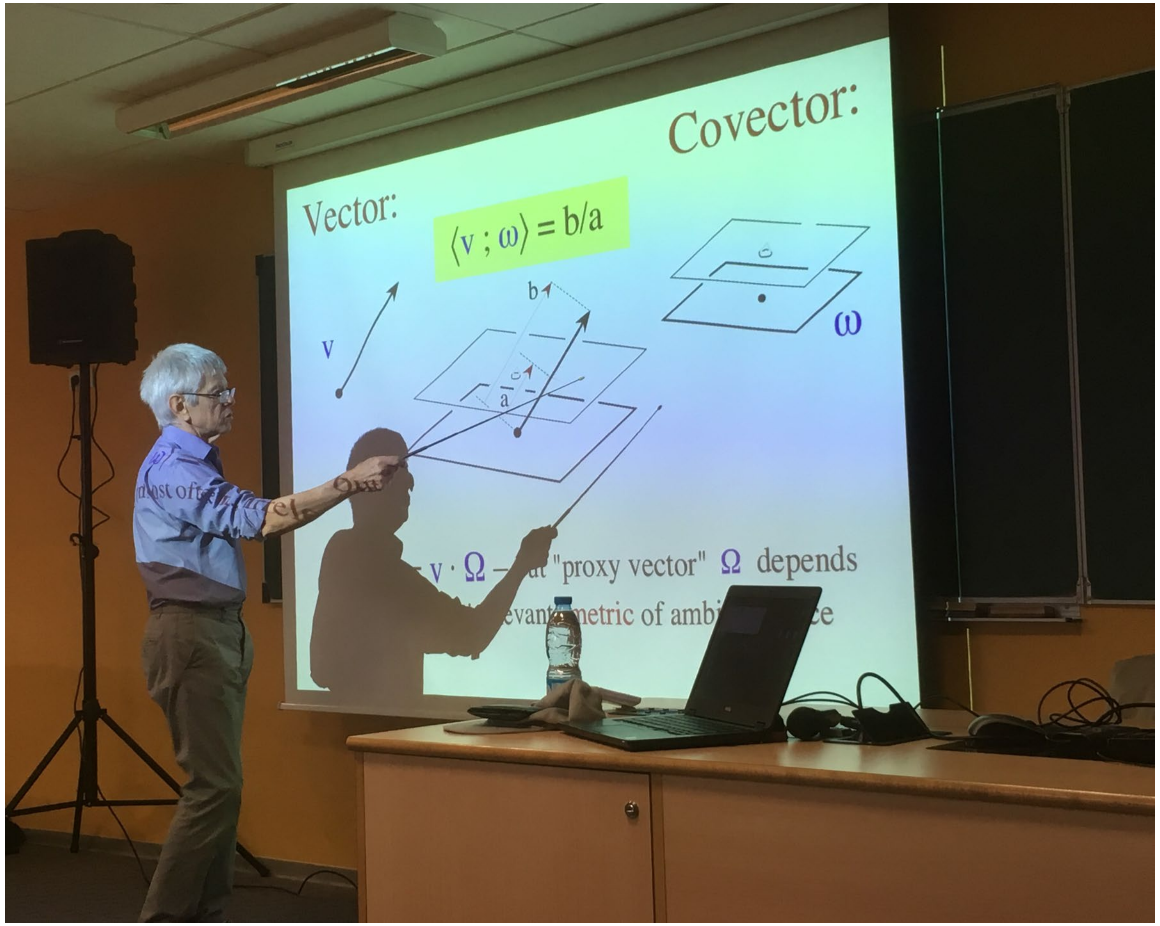
I Structure



Fredholm's equation of the second kind:

I is the identity operator, and
K is the integral operator

I Structure



"All the complexity to understand wave propagation is already in magnetostatics. The two share the same structure."

I Structure

What is meant by *structure*?

- Naïve view:
  - *The structure tells what you can do with the elements of a set.*
  - Example:
  - *A **type** is a structured set.*

I Structure

- More profound view:
  - *The question, what is meant by structure, was a major philosophical and mathematical motivation behind the birth of category theory in the mid-20th century.*
  - The essence of structure:
    - *not by what elements an object has, but by how it relates to other objects through (structure-preserving maps between objects called) morphisms*.

I Structure: Objects and morphism

- In set theory function f is not defined without specifying the domain X and codomain Y. (To know X and Y, one has take sides on whether *z* is an element of *X* and *Y* or not.)

- In category theory morphism f is defined between objects, but one does not need to care what the objects are internally.

I Structure

- Wonderful example of the power of structures:

arXiv > quant-ph > arXiv:0903.0340

Quantum Physics

[Submitted on 2 Mar 2009 (v1), last revised 6 Jun 2009 (this version, v3)]

Physics, Topology, Logic and Computation: A Rosetta Stone

John C. Baez, Mike Stay

I Structure: Boundary value problems

- Second order boundary value problems:
  - a homogeneous and
  - non-homogenous 1st order differntial equation,
  - and a constitutive law

These should hold in some domain

I Structure: Boundary value problems

- Magnetostatics:

Euclidean manifold
(spatial space)

- Electromagnetic waves

Minkowski manifold
(space-time)

I Structure: Boundary value problems

- If we start from a Minkowski manifold and introduce a formal sum of (sufficiently smooth) differential form spaces of degree $p = 0$ to $p = n$:

$$F(\Omega) = \bigoplus_{p=0}^{n} F^p(\Omega)$$

- then in dimension n = 4 an element of this space is of the form

$$f = f^0 + f^1 + f^2 + f^3 + f^4 \in F(\Omega)$$

I Structure: Boundary value problems

- Next we may write

$$\Leftrightarrow \qquad\qquad \Leftrightarrow \begin{bmatrix} & -\delta_g & & & \\ \mathrm{d}_f & & \delta_g & & \\ & \mathrm{d}_f & & -\delta_g & \\ & & \mathrm{d}_f & & \delta_g \\ & & & \mathrm{d}_f & \end{bmatrix} \begin{bmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \\ f^4 \end{bmatrix} = \begin{bmatrix} h^0 \\ h^1 \\ h^2 \\ h^3 \\ h^4 \end{bmatrix}$$

I Structure: Boundary value problems

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On each line this pair of diff. operators has to do with so called Hodge-Kodaira decompositions that generalize the idea of classical Helmholtz decompositons

I Structure: Boundary value problems

- Decomposition into space and time:

$$
\begin{aligned}
\mathrm{d} f^0 &= \mathrm{d}t \wedge \partial_t f^0 &&+ \mathrm{d}^s f^0 \,, \\
\mathrm{d} f^p &= \mathrm{d}t \wedge \partial_t f^p_s &&+ \mathrm{d}^s f^p_t &&+ \mathrm{d}^s f^p_s , && \forall p > 0 \,, \\
\delta f^p = *\mathrm{d}* f^p &= \star\mathrm{d}t \wedge \partial_t \star f^p_t &&+ \star\mathrm{d}^s \star f^p_t &&+ \star\mathrm{d}^s \star f^p_s \,, && \forall p < n \,, \\
\delta f^n = *\mathrm{d}* f^n &= \star\mathrm{d}t \wedge \partial_t \star f^n_t &&+ \star\mathrm{d}^s \star f^n_t \,.
\end{aligned}
$$

I Structure: Boundary value problems

- Consequently

$$
\begin{bmatrix}
\partial_t & & & & & -\mathrm{d} & & \\
& \star\partial_t\star & & & \mathrm{d} & & & \\
& & \partial_t & & & \star\mathrm{d}\star & & -\mathrm{d} \\
& & & \star\partial_t\star & \star\mathrm{d}\star & & \mathrm{d} & \\
& \star\mathrm{d}\star & & -\mathrm{d} & \partial_t & & & \\
\star\mathrm{d}\star & & \mathrm{d} & & & -\star\partial_t\star & & \\
& & & \star\mathrm{d}\star & & & \partial_t & \\
& & \star\mathrm{d}\star & & & & & -\star\partial_t\star
\end{bmatrix}
\begin{bmatrix} f^3 \\ F^3 \\ f^1 \\ F^1 \\ f^2 \\ F^2 \\ f^0 \\ F^0 \end{bmatrix}
=
\begin{bmatrix} H^3 \\ h^3 \\ H^1 \\ h^1 \\ H^2 \\ h^2 \\ H^0 \\ h^0 \end{bmatrix}
\quad \Leftrightarrow \quad
\begin{bmatrix}
 & -\delta_g & & & \\
\mathrm{d}_f & & \delta_g & & \\
 & \mathrm{d}_f & & -\delta_g & \\
 & & \mathrm{d}_f & & \delta_g \\
 & & & \mathrm{d}_f &
\end{bmatrix}
\begin{bmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \\ f^4 \end{bmatrix}
=
\begin{bmatrix} h^0 \\ h^1 \\ h^2 \\ h^3 \\ h^4 \end{bmatrix}
$$

- f^p and h^p are space-like p-forms, and
- F^p and H^p are the space-like components of $(p+1)$-forms $dt \wedge F^p$ and $dt \wedge H^p$

I Structure: Boundary value problems

- and now

$$
\begin{bmatrix}
\partial_t & & & & & -\mathrm{d} & & \\
& \star\partial_t\star & & & \mathrm{d} & & & \\
& & \partial_t & & & \star\mathrm{d}\star & & -\mathrm{d} \\
& & & \star\partial_t\star & \star\mathrm{d}\star & & \mathrm{d} & \\
& \star\mathrm{d}\star & & -\mathrm{d} & \partial_t & & & \\
\star\mathrm{d}\star & & \mathrm{d} & & & -\star\partial_t\star & & \\
& & & \star\mathrm{d}\star & & & \partial_t & \\
& & \star\mathrm{d}\star & & & & & -\star\partial_t\star
\end{bmatrix}
\begin{bmatrix} f^3 \\ F^3 \\ f^1 \\ F^1 \\ f^2 \\ F^2 \\ f^0 \\ F^0 \end{bmatrix}
=
\begin{bmatrix} H^3 \\ h^3 \\ H^1 \\ h^1 \\ H^2 \\ h^2 \\ H^0 \\ h^0 \end{bmatrix}
$$

$$
\text{the choice} \quad \begin{cases} F^1 &= -e \\ f^2 &= b \\ h^1 &= \star j \\ H^0 &= -\star q \end{cases} \quad \text{results in} \quad \begin{cases} \mathrm{d}b &= 0 \\ \mathrm{d}e + \partial_t b &= 0 \\ -\star\partial_t\star_\varepsilon e + \star\mathrm{d}\star_\nu b &= \star j \\ -\star\mathrm{d}\star_\varepsilon e &= -\star q \end{cases} \iff \begin{cases} \mathrm{d}b &= 0 \\ \mathrm{d}e + \partial_t b &= 0 \\ -\partial_t\star_\varepsilon e + \mathrm{d}\star_\nu b &= j \\ \mathrm{d}\star_\varepsilon e &= q \end{cases}
$$

Maxwell's equations

I Structure: Boundary value problems

- Or, if f is about E-valued forms and d is about the exterior covariant derivative

$$
\begin{bmatrix}
\partial_t & & & & & -\mathrm{d} & & \\
& \star\partial_t\star & & & \mathrm{d} & & & \\
& & \partial_t & & & \star\mathrm{d}\star & & -\mathrm{d} \\
& & & \star\partial_t\star & \star\mathrm{d}\star & & \mathrm{d} & \\
& \star\mathrm{d}\star & & -\mathrm{d} & \partial_t & & & \\
\star\mathrm{d}\star & & \mathrm{d} & & & -\star\partial_t\star & & \\
& & & \star\mathrm{d}\star & & & \partial_t & \\
& & \star\mathrm{d}\star & & & & & -\star\partial_t\star
\end{bmatrix}
\begin{bmatrix} f^3 \\ F^3 \\ f^1 \\ F^1 \\ f^2 \\ F^2 \\ f^0 \\ F^0 \end{bmatrix}
=
\begin{bmatrix} H^3 \\ h^3 \\ H^1 \\ h^1 \\ H^2 \\ h^2 \\ H^0 \\ h^0 \end{bmatrix}
$$

$$
\text{the choice} \quad \begin{cases} F^0 &= u \\ f^1 &= \varepsilon \\ g^0 &= -\star f_V \end{cases} \quad \text{results in} \quad \begin{cases} -\partial_t\varepsilon + \mathrm{d}_\nabla u &= 0 \\ \mathrm{d}_\nabla \varepsilon &= 0 \\ \star\mathrm{d}_\nabla \star^C \varepsilon - \star\partial_t \star_\rho u &= -\star f_V \end{cases} \iff \begin{cases} -\partial_t\varepsilon + \mathrm{d}_\nabla u &= 0 \\ \star_\rho\, \partial_t u - \mathrm{d}_\nabla \sigma &= f_V \\ \sigma = \star^C \varepsilon,\ u &= \partial_t \end{cases}
$$

small strain elasticity

I Structure: Boundary value problems

- Small strain elasticity

I Structure: Boundary value problems

I Structure: Boundary value problems

- My point is,
  - *if we do not focus on what elements an object has, but instead, on how it relates to other objects through morphisms, we start to recognize analogies.*

I Structure: Structures and functors

- *John Baez:*
  - *"Every good analogy is yearning to become a functor."*
- A *functor* is a
  - mapping between categories –objects to objects and morphisms to morphism– that
  - translates structures from one category to another, and
  - preserves the relationships between objects and morphisms.

I Structure: Conclusion

- Recall Alain's words:
  - *"All the complexity to understand wave propagation is already in magnetostatics. The two share the same structure."*

In *Lawrerence*'s functorial semantics: Abstract category Model (a concrete category)

II Finite dimensional problems: de Rham complex

- the cohomology groups

—— the kernel of d

The very idea of *Whitney forms* is a family of finite dimensional spaces of differential forms that lends itself to the de Rham complex.

Graphical representation of the de Rham complex in dimension 3

I Finite dimensional problems: de Rham complex

- The de Rham complex raises a question:
  - *What is the* complement of $\mathrm{cod}(d)$ *with respect to the* L^2*- norm?*

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This is known as the *Hodge-Kodaira decomposition* that generalizes classical Helmholtx decompositions.

I Finite dimensional problems: de Rham complex

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This is known as the *Hodge-Kodaira decomposition* that generalizes classical Helmholtx decompositions.

It provides answers to the question, is a boundary value problem well-established.

I Finite dimensional problems: Formulation

- Usage of L^2-decompositions in writing boundary value problems in the weak form
- As

the orthogonal components cancel out

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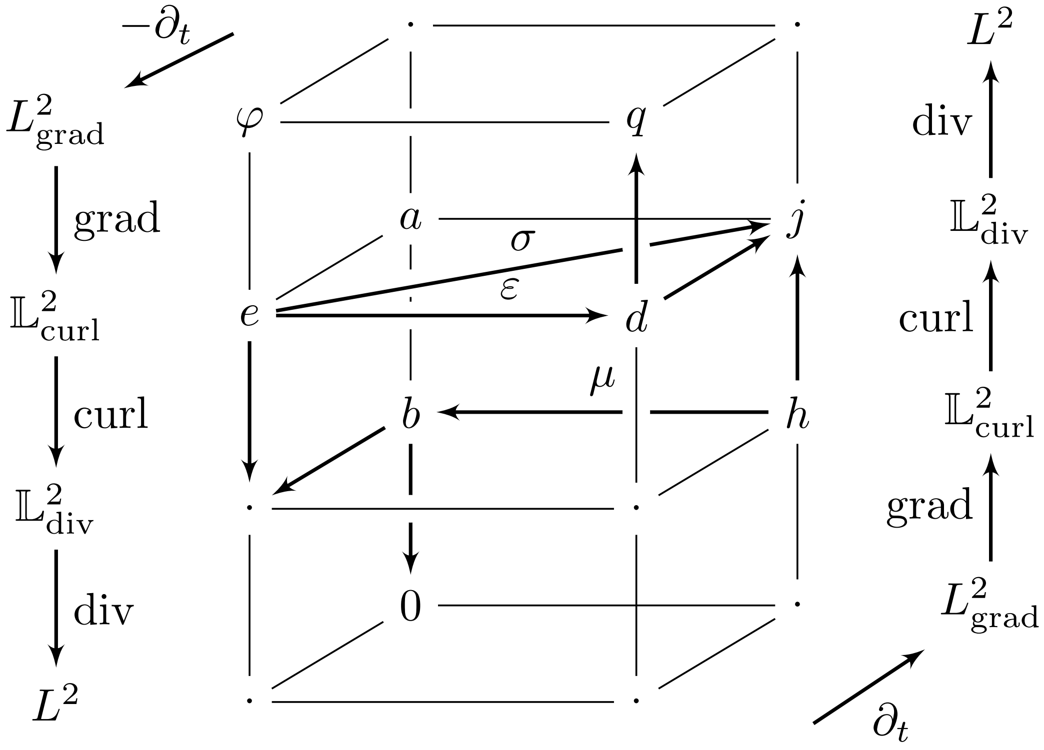
This is how topologically non-trivial domains can be tackled reliably

II Finite dimensional problems: Maxwell's house



Functional framework for Maxwell's equations

… base further study of models derived from Maxwell's equations on the systematic exploitation of these structural properties (an ambitious working program, to which the present book can only begin to contribute)

A. Bossavit:
Computational Electromagnetism,
Academic Press, 1998

II Remark i

- *How should the Maxwell house be approximated in finite dimensional spaces?*
  - **Be aware:**
    - *As soon as one restricts oneself to finite dimensional spaces, all the properties of the infinite dimensional model will no longer hold.*

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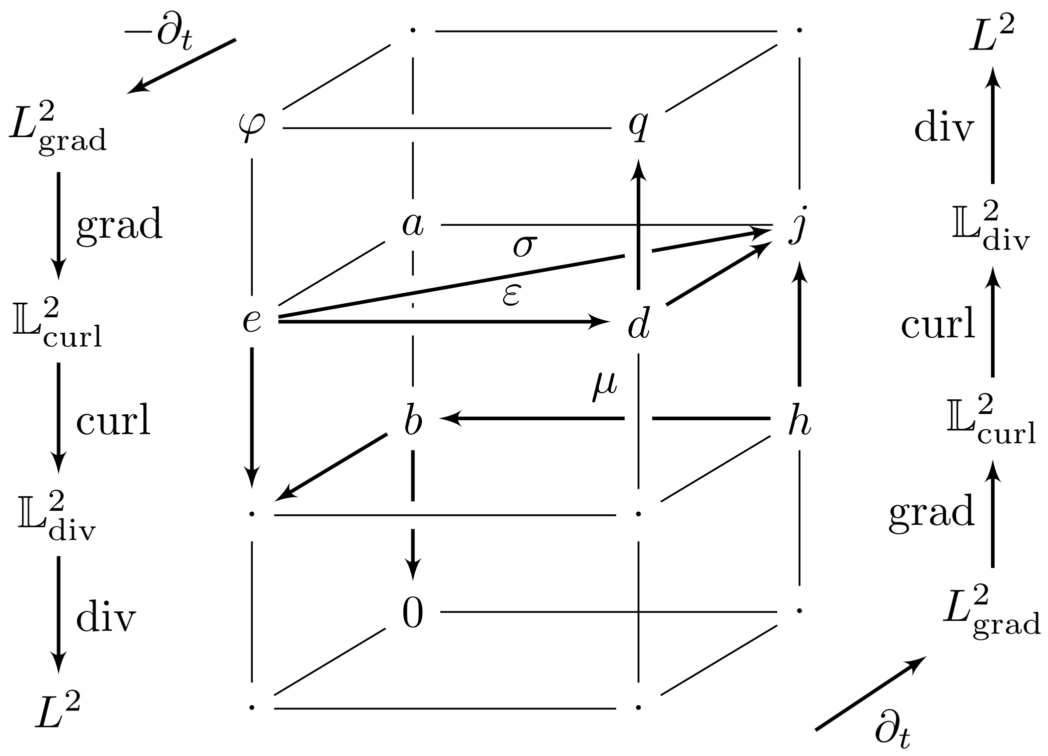
  - **Be aware:**
    - *As soon as one restricts oneself to finite dimensional spaces, all the properties of the infinite dimensional model will no longer hold.*

      It's up to the modelling decision which properties are retained and which not. (Alain: "It is better to make a conscious decision")

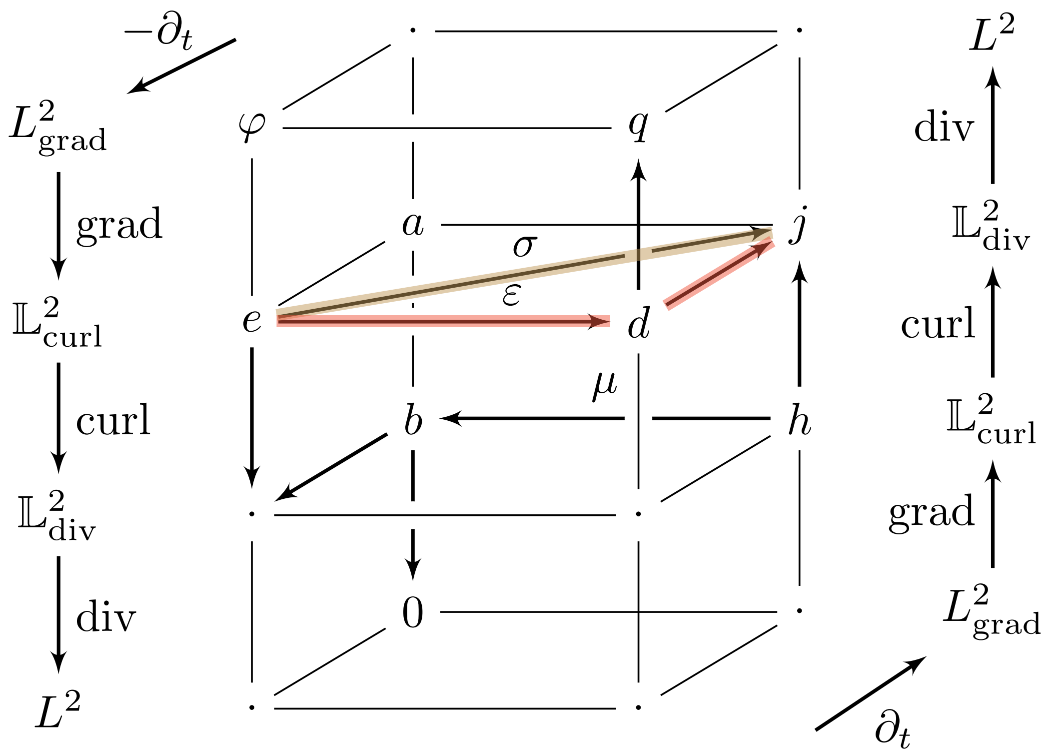
II Remark ii



- The Maxwell's house and the underlying de Rham complex

  ***is not a category*.**

II Remark ii



- The Maxwell's house and the underlying de Rham complex

  is not a category.

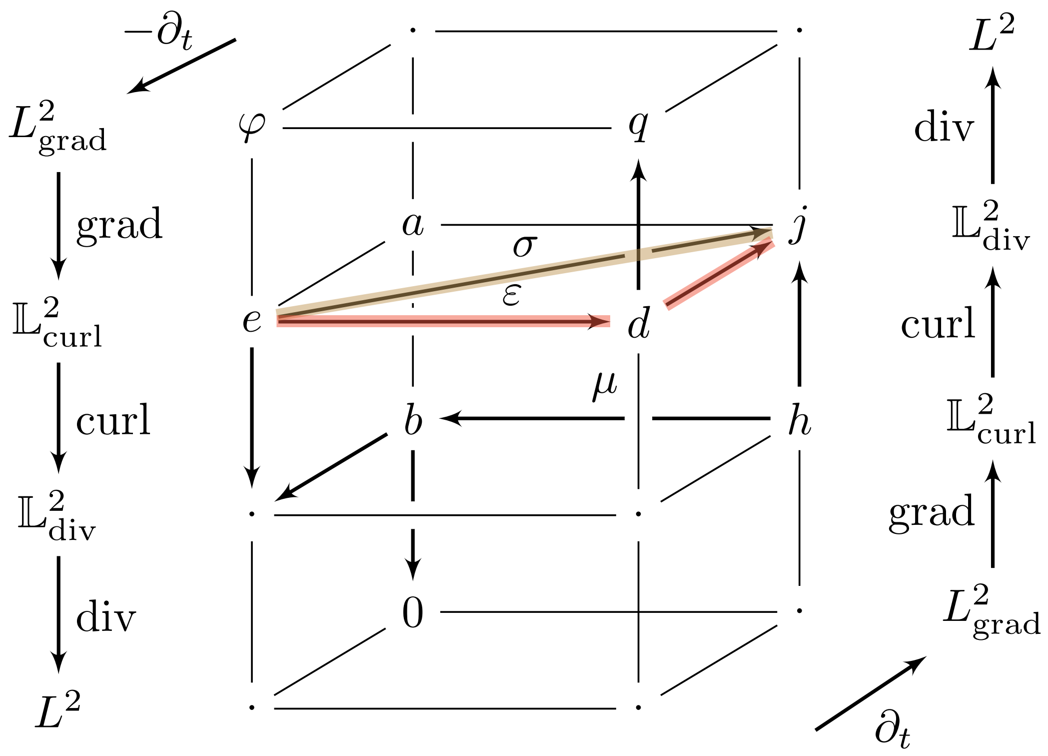
- For, the constitutive laws do not form a commutative diagram. That is, we have

$$e \xrightarrow{\varepsilon} d, \quad d \xrightarrow{\partial_t} j \quad \text{and} \quad e \xrightarrow{\sigma} j$$

but still,

$$e \xrightarrow{\varepsilon} d \xrightarrow{\partial_t} j \quad \text{is not} \quad e \xrightarrow{\sigma} j$$

II Remark ii



- The axioms of categories insist on compositions of morphisms:

II Finite dimensional problems: DEC

- There is no canonical approach to express the notion of a field.
  - *The choice between quaternions, vector fields, differential forms, or cochains is a modeling decision*

II Finite dimensional problems: DEC

- Equivalence between p-forms and p-chains:

II Finite dimensional problems: DEC

- Equivalence between *p*-forms and *p*-chains:

- Modelling decision behind DEC:
  - Instead of all chains, the differential equations are replaced with cochain equations over finite sets of chains.

II Finite dimensional problems: DEC

- In the same spirit, the constitutive law is imposed only on a *finite set of points* of the domain.

II Finite dimensional problems: DEC

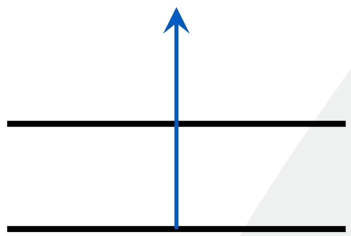
- The orthogonality condition (between the primal and dual grids) follows from the definition of the Hodge:

Let ω_n be the unit n-vector.

The Hodge operator is the map $\star : V_p \to V_{n-p}, v \mapsto \star v$ such that for all $u \in V_p$ condition

$$u \wedge \star v = \omega_n \langle u, v \rangle$$

holds.



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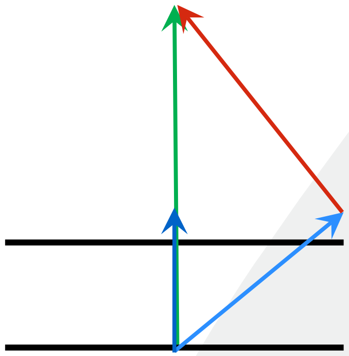


II Finite dimensional problems: DEC

- Once the values of the cochains on the primal and dual chains are known
  - Whitney forms is a just machinery to construct a covector field from the primal side cochain.

II Finite dimensional problems: DEC

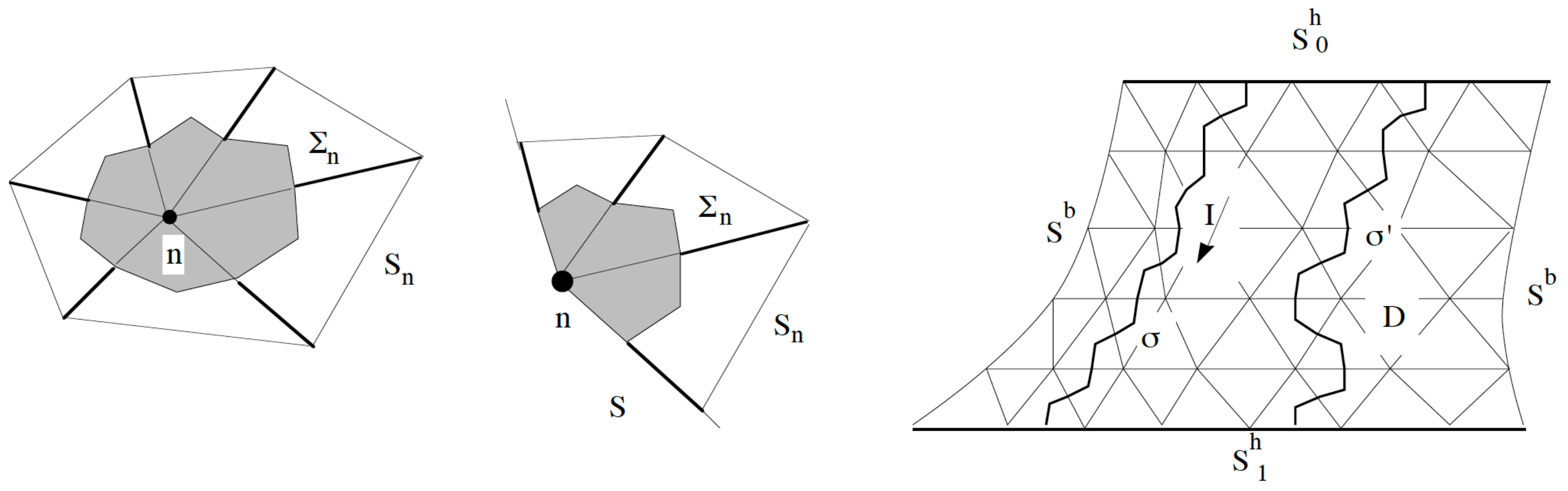
- The remaining issue is,
  - *there are neither Whitney forms on the dual side*
  - *nor a   for Whitney forms*
- Let f be the primal side differential form, the usual workaround is to employ the constitutive equation locally within each primal element

II Finite dimensional problems

- Perhaps surprisingly,
    - *The solutions of finite elements and DEC solutions share the same properties*

II Finite dimensional problems

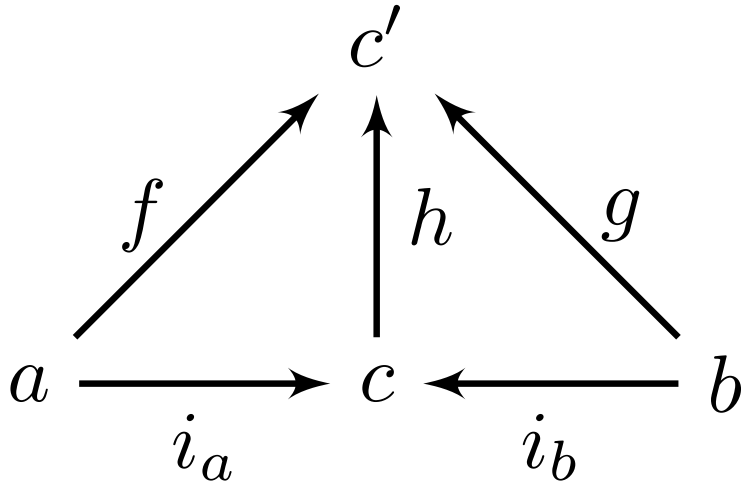
- Assuming barycentric subdivision, the integrals of finite element solutions over the dual (relative bounding) cycles are exact



A. Bossavit: How weak is the Weak Solution in finite elements methods? IEEE TMAG 1998

III Formalization of structure

- Coproduct (intuitively “or”)



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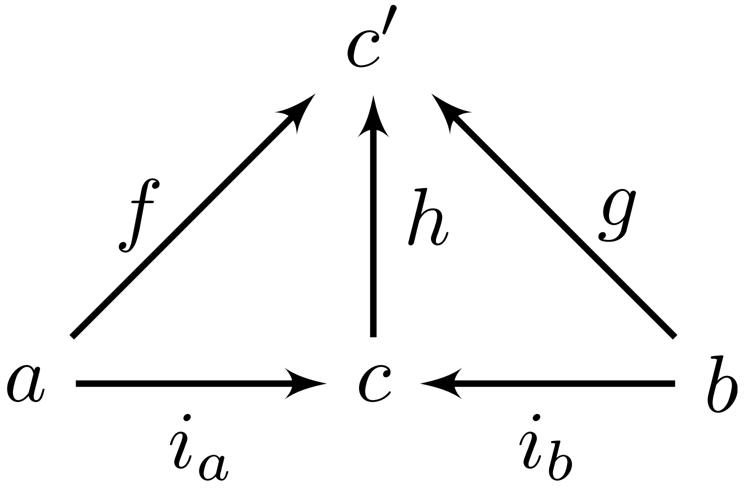


$$\begin{cases} f = h \circ i_a \\ g = h \circ i_b \end{cases}$$

Notice, maps f and g are factorized by map h.

III Formalization of structure

- Constitutive law

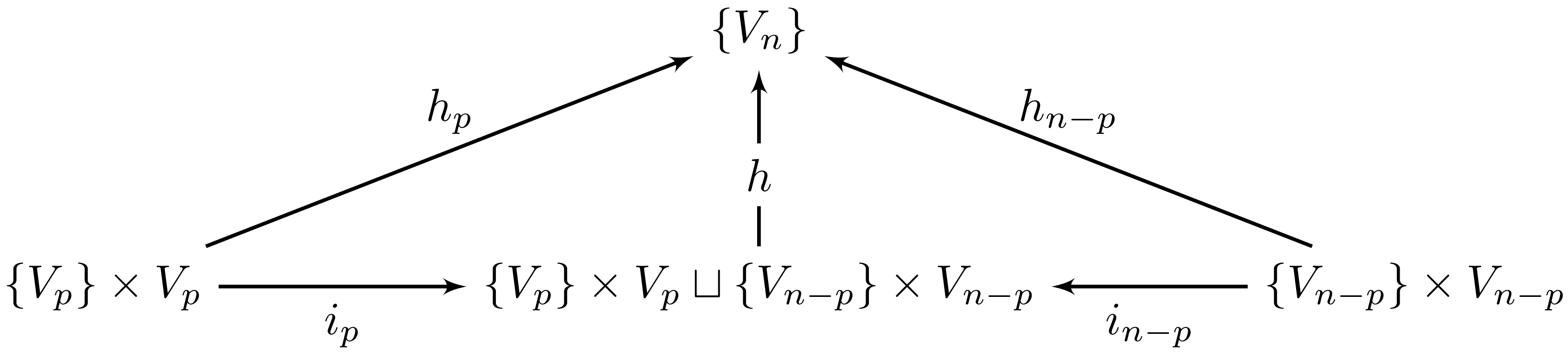
$$u \wedge \star v = \omega_n \langle u, v \rangle \quad \forall u \in V_p$$

$$h_p : \{V_p\} \times V_p \to \{V_n\}, \quad (v', v)_{v' \in V_p} \mapsto \omega_n \langle v', v \rangle$$

$$h_{n-p} : \{V_{n-p}\} \times V_{n-p} \to \{V_n\}, \quad (w', w)_{w' \in V_{n-p}} \mapsto \omega_n \langle w', w \rangle$$

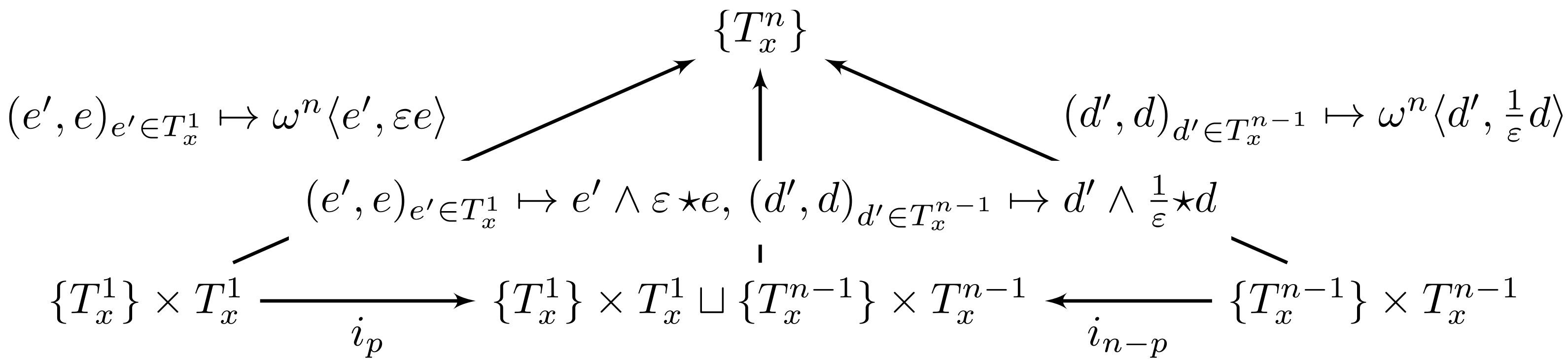
III Formalization of structure

- Constitutive law



III Formalization of structure

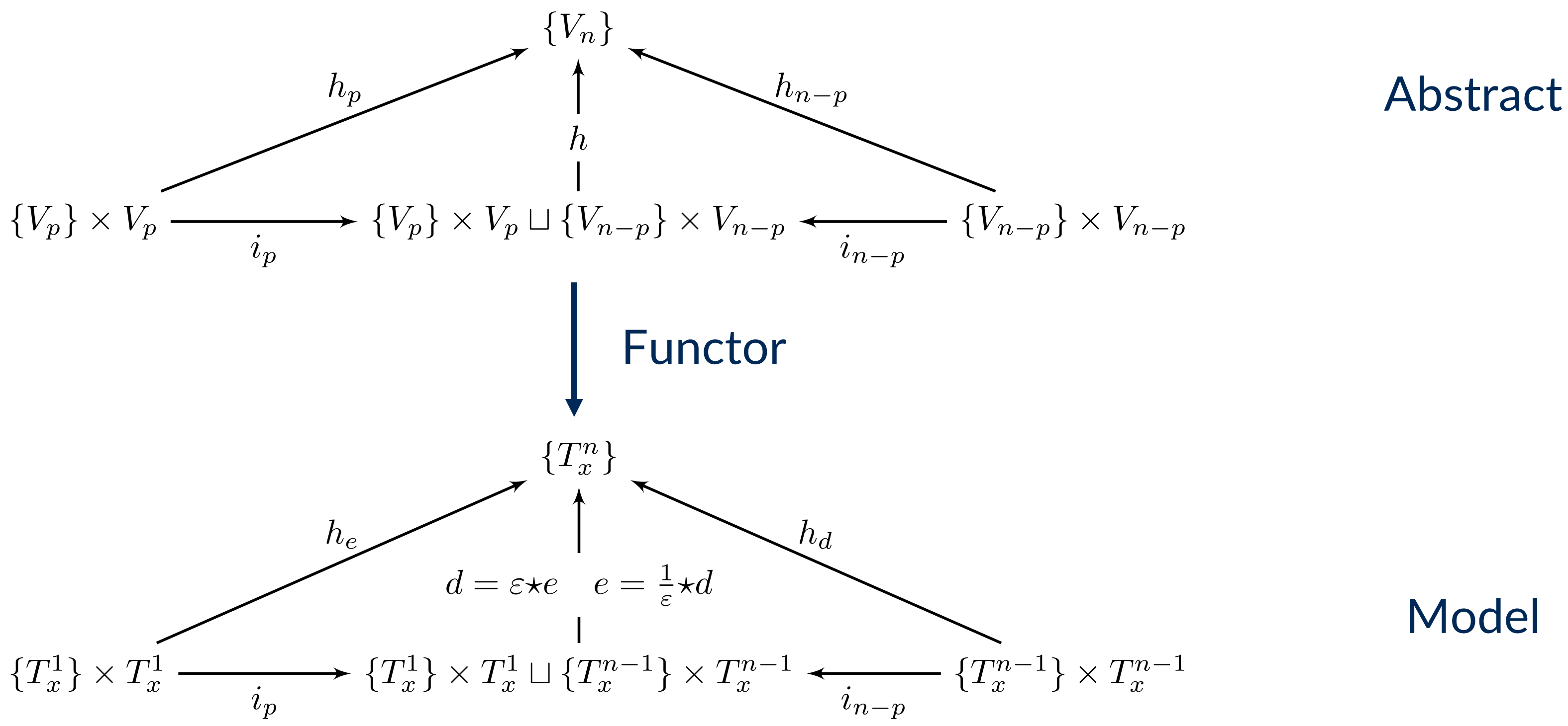
- Constitutive law in electricity



$$d = \varepsilon \star e \qquad\qquad e = \frac{1}{\varepsilon} \star d$$

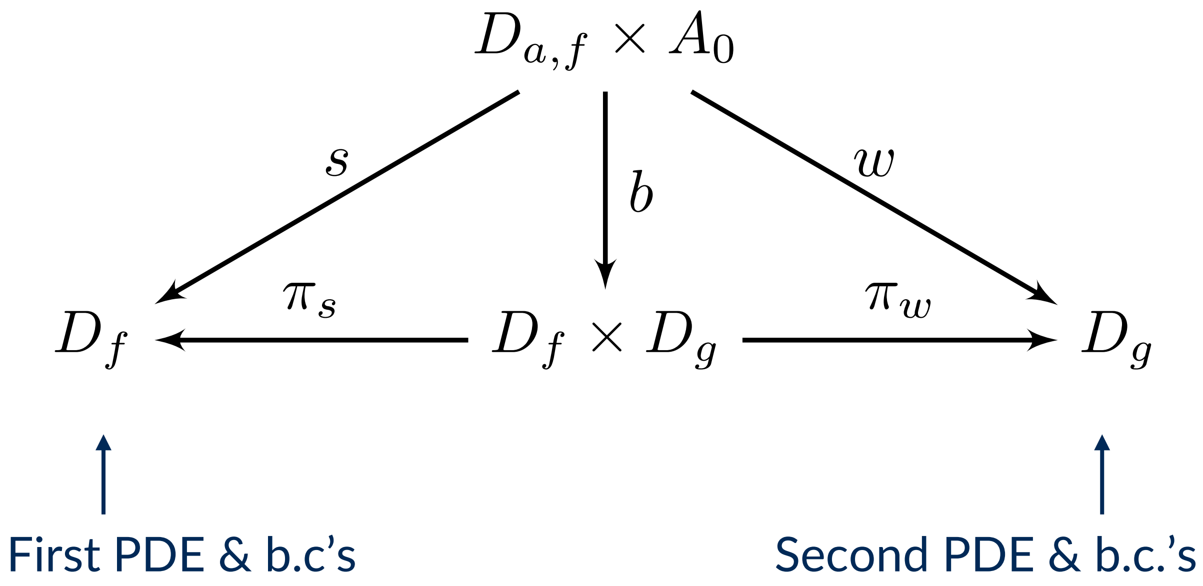
III Formalization of structure

- Constitutive law in electricity



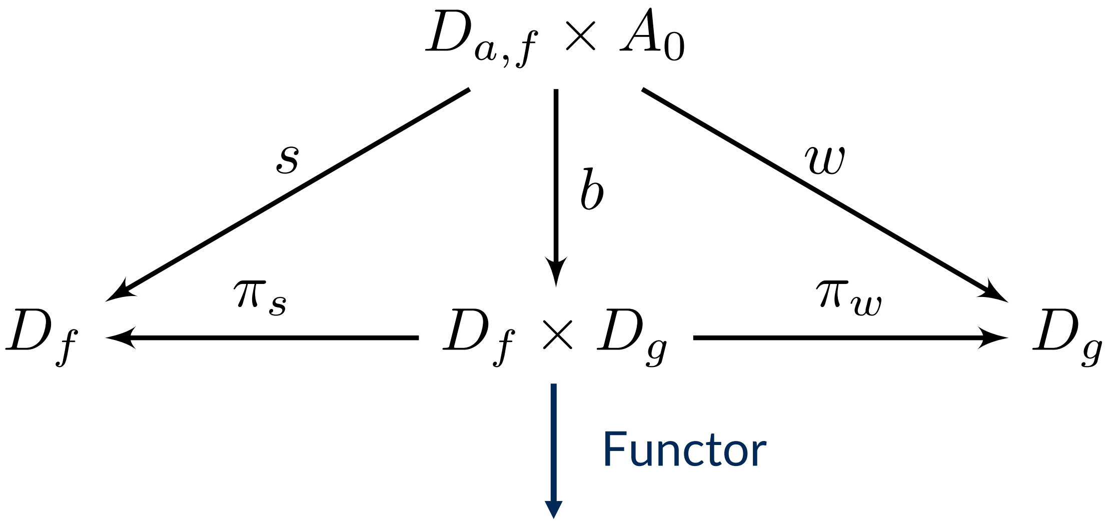
III Formalization of structure

- Product: *Action principle* (intuitively “and”)



III Formalization of structure

- Product: *Hodge-Kodaira decomposition*

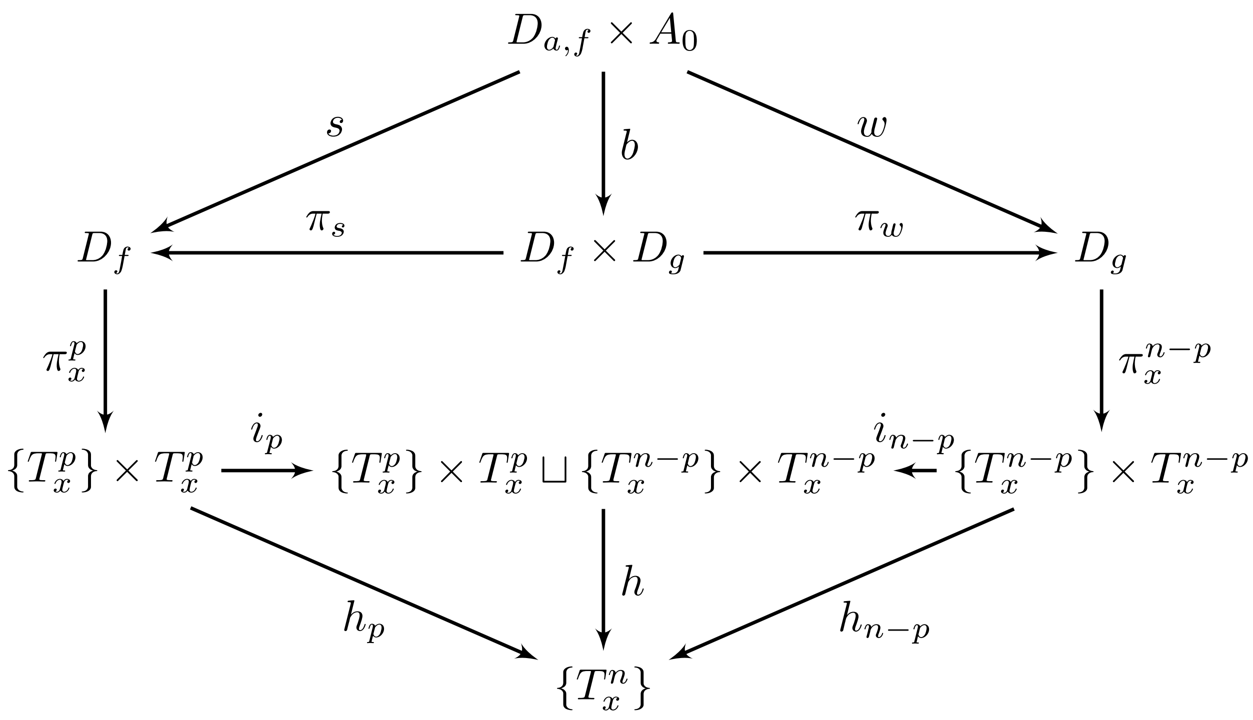


Abstract

Model

III Formalization of structure

- Second order boundary value problem



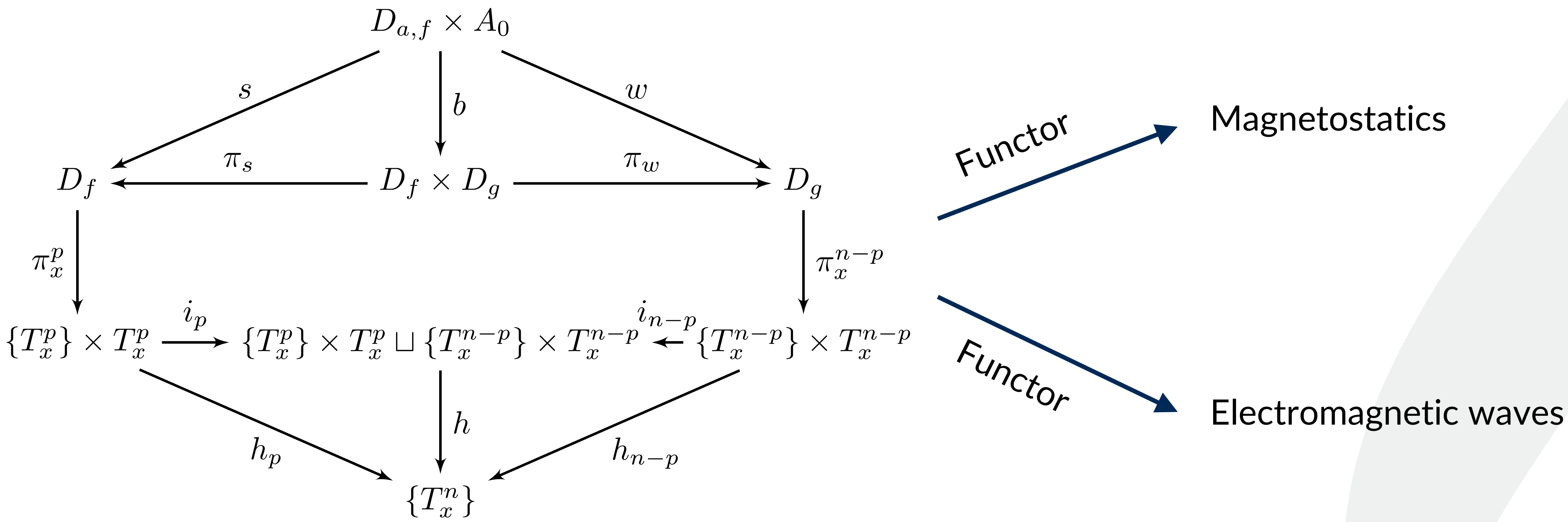
Conditions on differential forms on a manifold

Projection to the fibers over a point

Conditions on cotangent vectors at a point of a manifold

III Formalization of structure

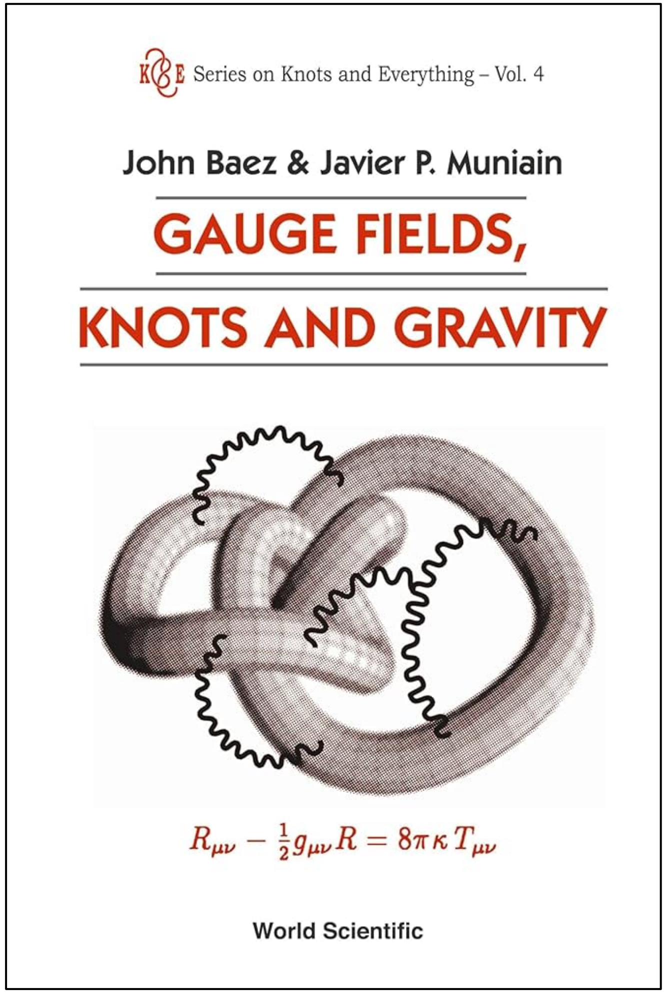
- Second order boundary value problem

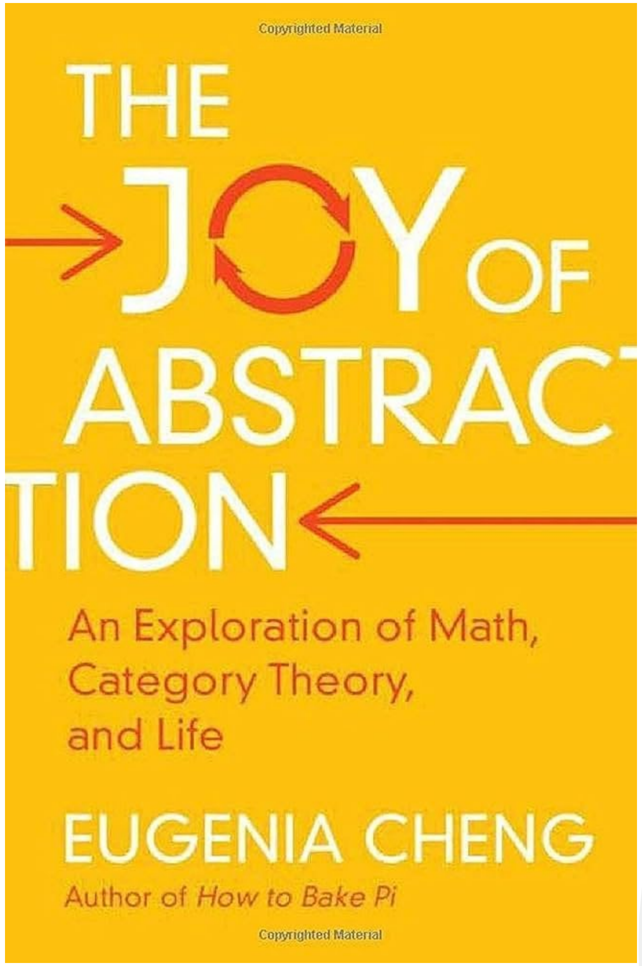


III Formalization of structure

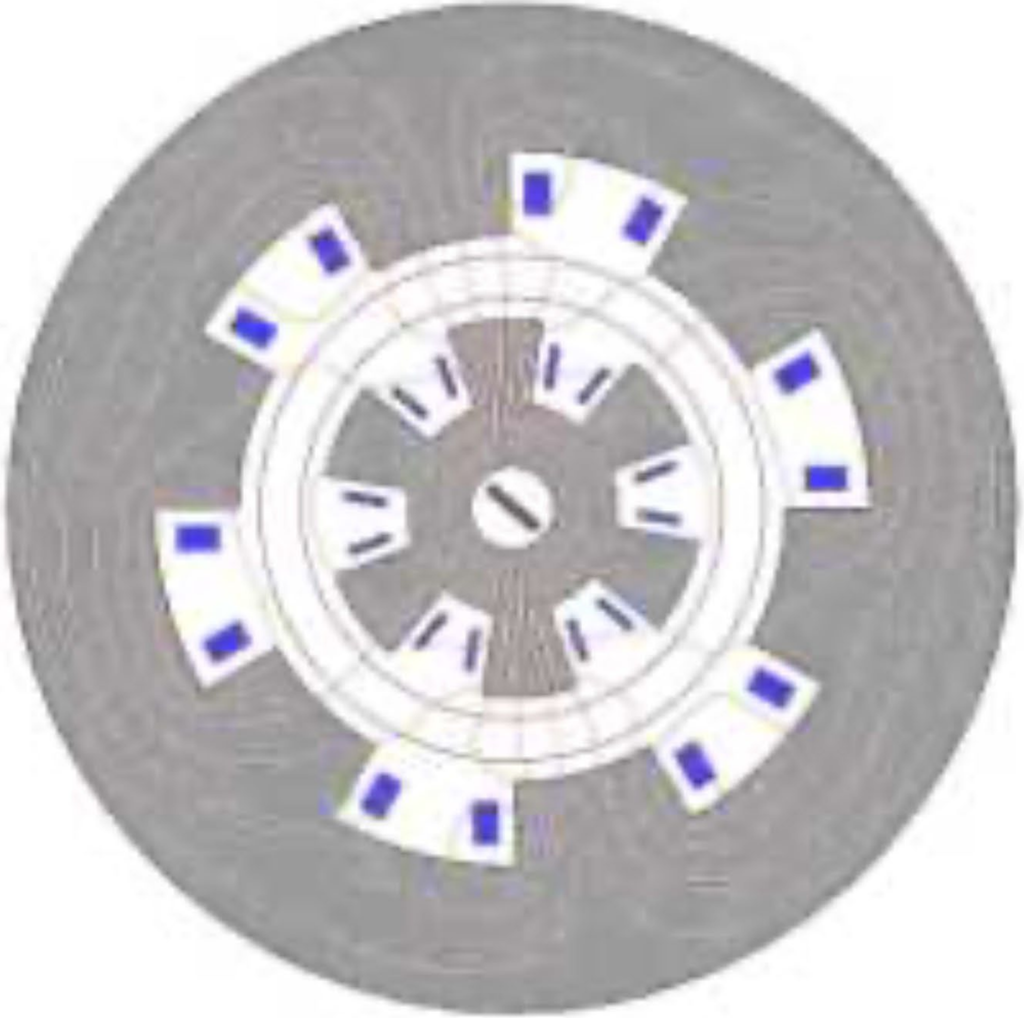
- Sketch of what led to DEC (this should not be taken too literally, need to work this our precisely)

IV Recommend reading

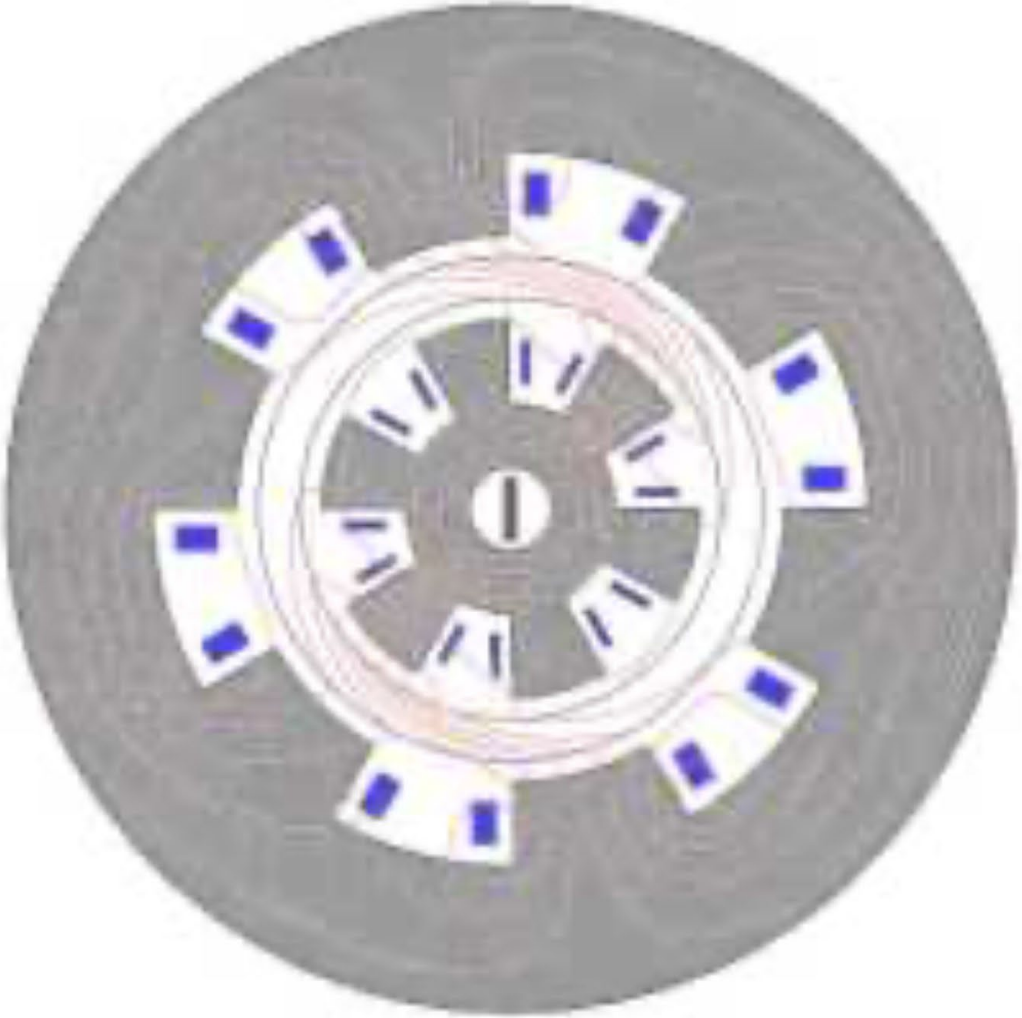




Side by side view



Standard approach



Metric tensor changes