

# Random geometric graphs with smooth kernels: sharp detection threshold and a spectral conjecture

Jiaming Xu

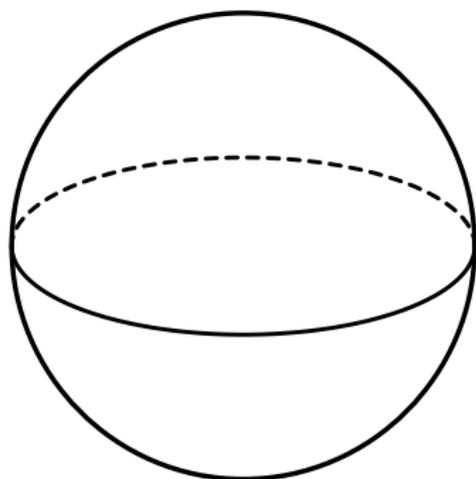
The Fuqua School of Business  
Duke University

Joint work with  
Cheng Mao (GTech) and Yihong Wu (Yale)

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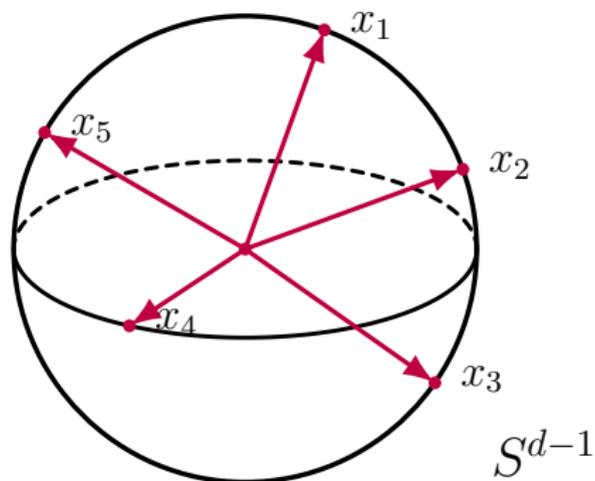
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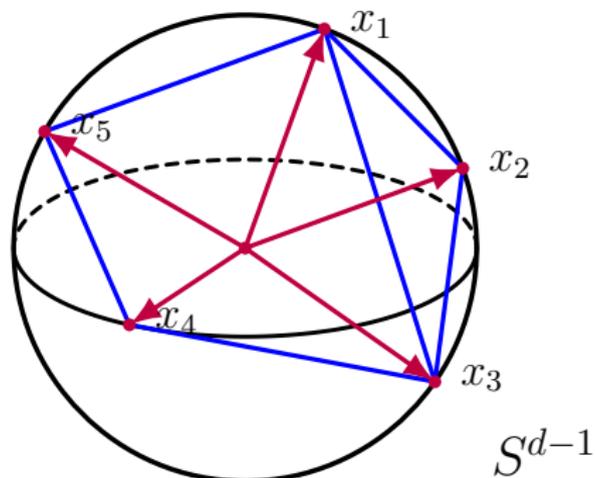
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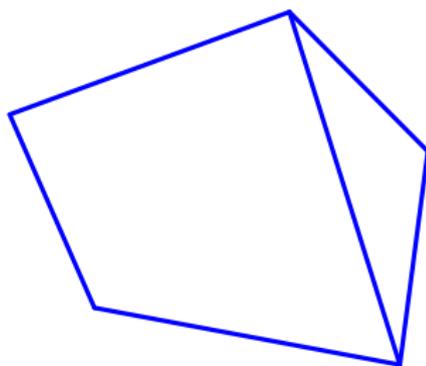
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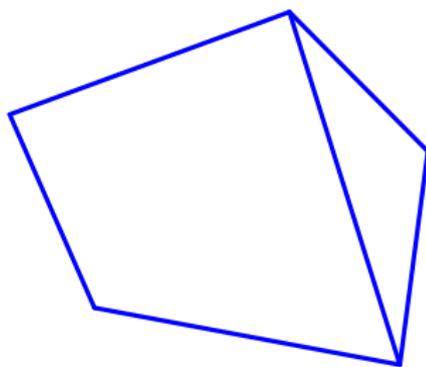
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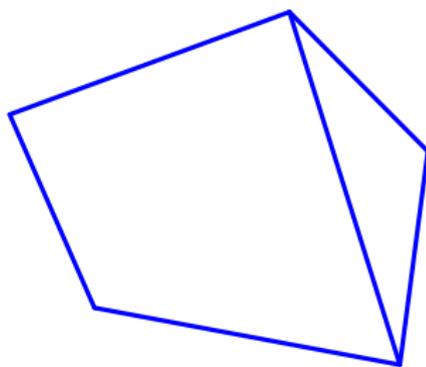
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- Special case of the latent space model and graphon model [Hoff-Raftery-Handcock'02, Lovász-Szegedy'06]

# The high-dimensional setting

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  - ▶ Numerous applications: wireless networks, manifold learning, resource allocation, etc
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- The high-dimensional regime where  $d \rightarrow \infty$  as  $n \rightarrow \infty$  is much less explored
  - ▶ High-dimensional feature space
  - ▶ Novel theoretical challenges

- 1 When can we detect the latent geometry from the observation of  $G$ ?

Testing  $\mathcal{H}_0 : G \sim \mathcal{G}(n, p)$  v.s.  $\mathcal{H}_1 : G \sim \mathcal{G}(n, d, K)$

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## Intuition

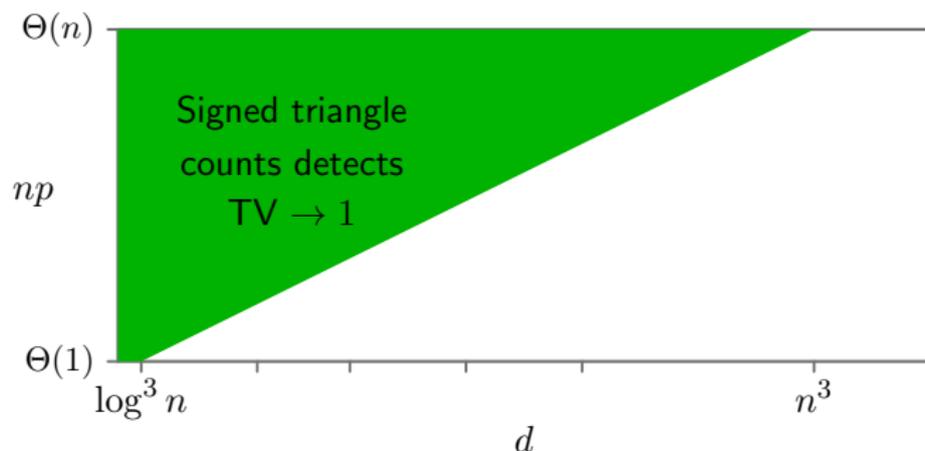
As dimension  $d$  increases, latent geometry becomes less evident, making detection and estimation harder.

## Exciting progress on hard RGG

$K(t) = \mathbf{1}_{\{t \geq \tau\}}$  for some threshold  $\tau$ , i.e.,  $i \sim j$  iff  $\langle x_i, x_j \rangle \geq \tau$

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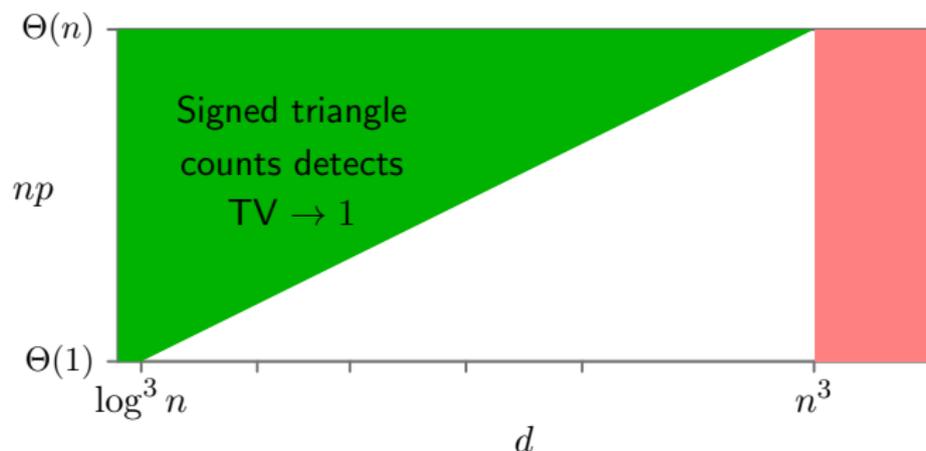
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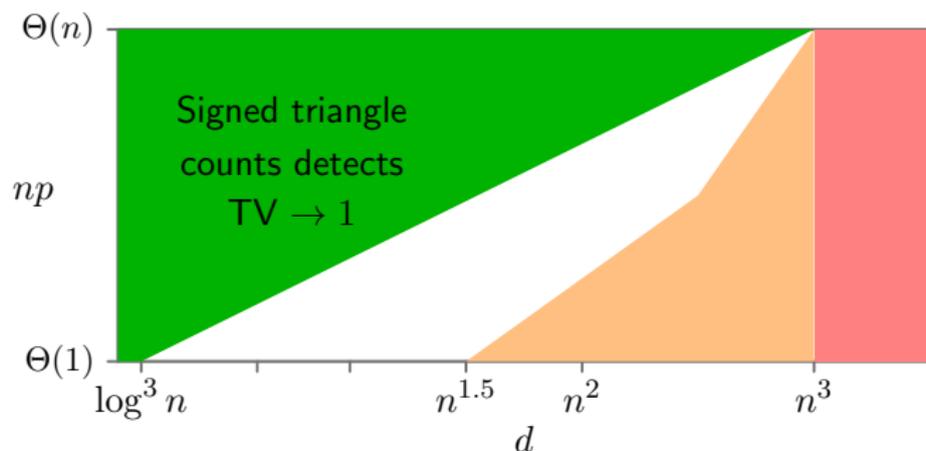
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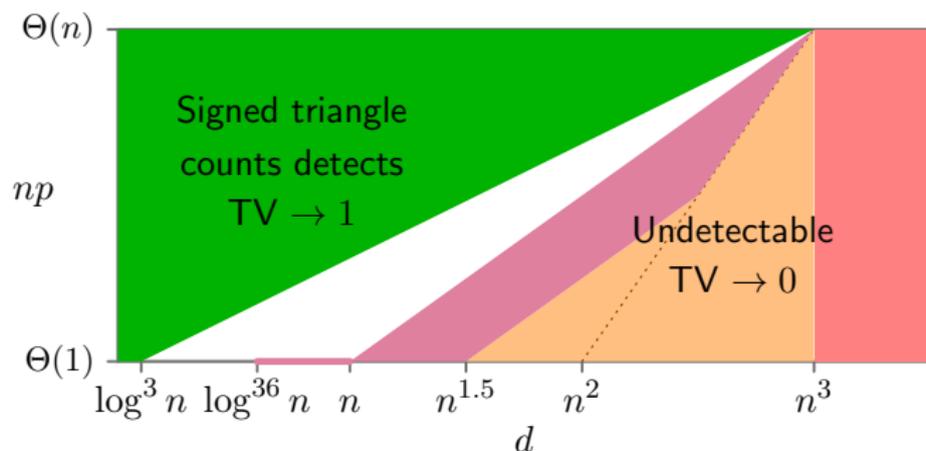
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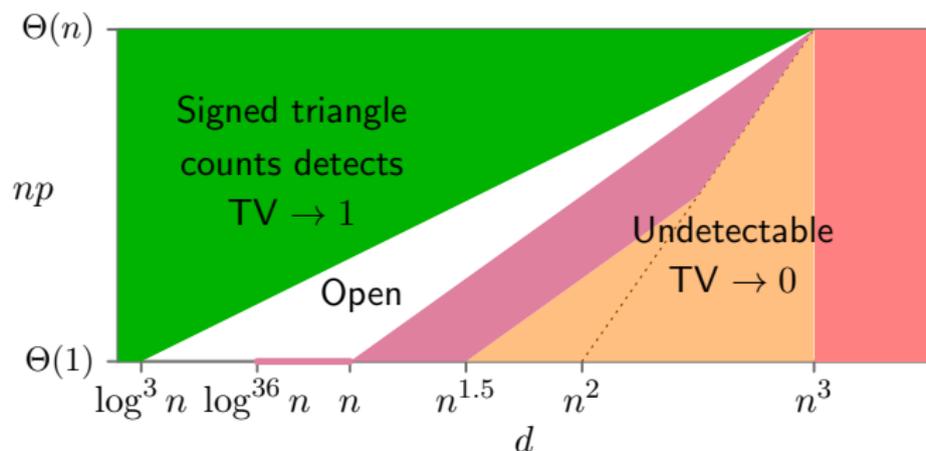
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# Focus of this talk: Soft RGG

- Statistics literature commonly assumes a *soft* RGG with a smooth kernel  $K$  [Hoff-Raftery-Handcock'02, Ma-Ma-Yuan'20, Duchemin-Castro'23]
  - ▶  $K(t)$  is a polynomial, e.g.,  $K(t) = p + rt$
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## Theorem (Liu-Rácz '23)

Suppose  $K$  is a fixed smooth kernel with  $K'(0) \neq 0$ .

- If  $d \ll n^{3/4}$ , then  $\text{TV} \rightarrow 1$  and the signed triangle count succeeds;
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- $d_{\text{test}}^* \asymp n^{3/4}$  is conjectured to be the right threshold

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Detection ( $d_{\text{test}}^* \asymp n^{3/4}$ )   v.s.   Estimation ( $d_{\text{est}}^* \asymp n^{1/2}$ )

# A unifying spectral conjecture

Consider the standardized kernel

$$\kappa(t) \triangleq \frac{K(t) - p}{\sqrt{p(1-p)}}$$

Identify  $\kappa$  as an integral operator on the sphere:

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- If  $\text{tr}(\kappa^3)$  dominated by contribution from  $\lambda_1$  ( $\text{tr}(\kappa^3) \asymp \lambda_1^3 d$ ), then threshold simplifies to  $d \asymp n^{3/4} b_1^{3/2}$ , where  $b_1 = d\lambda_1$ 
  - ▶ Match the threshold in smooth kernels and the (conjectured) threshold in hard RGG

## Proof ideas

(focus on linear kernel  $\kappa(t) = t$  for clarity)

# Failure of naïve second moment method

Define the likelihood ratio  $L(G) \triangleq \frac{\mathbb{P}(G)}{\mathbb{Q}(G)}$ . Then

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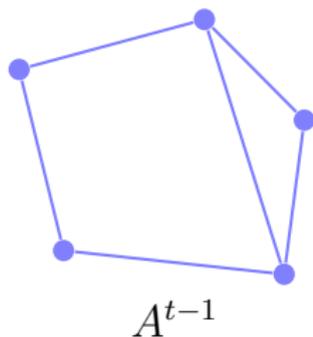
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- In fact, the second-moment blows up to  $\infty$ , when  $d \ll n$

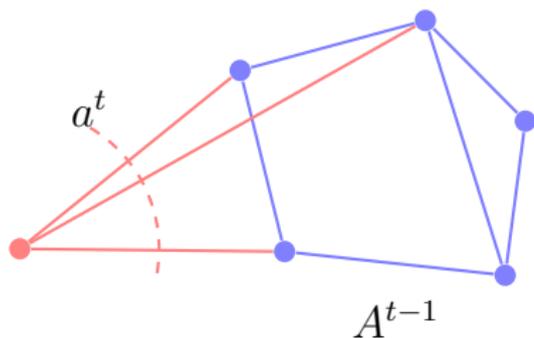
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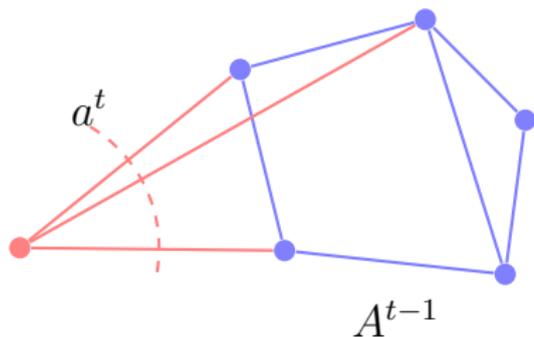
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- Apply the chain rule for KL divergence to  $A = (a^2, \dots, a^n)$ :

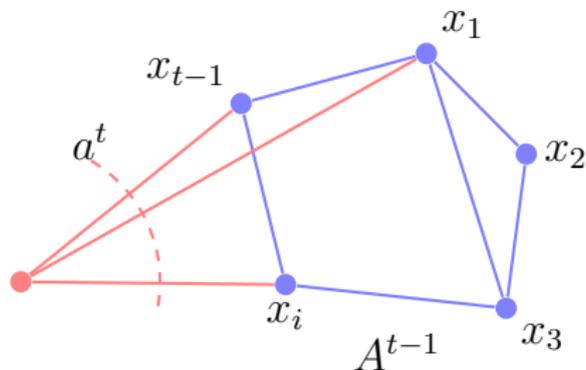
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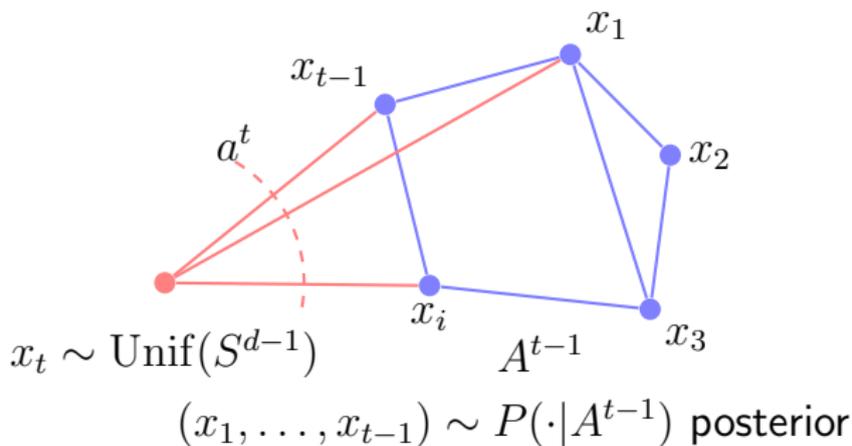


$(x_1, \dots, x_{t-1}) \sim P(\cdot | A^{t-1})$  posterior

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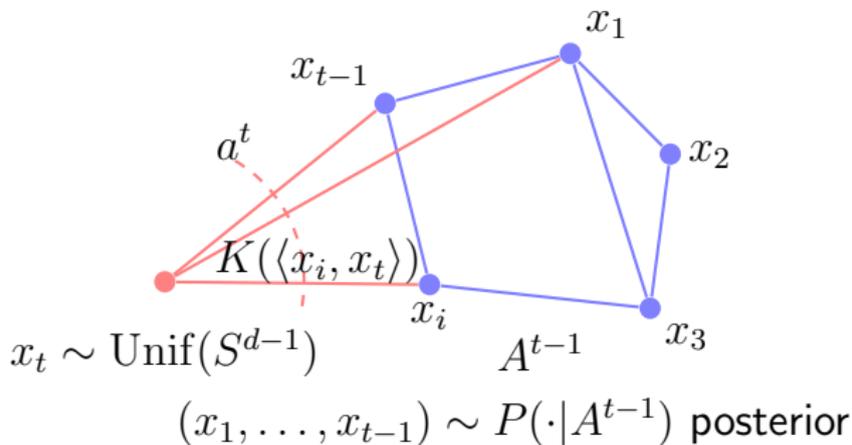
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- **Key challenge:**  $P_{a^t|A^{t-1}}$  is a mixture of products of Bernoulli:

$$P_{a^t|A^{t-1}} = \mathbb{E}_{x_1, \dots, x_t | A^{t-1}} \left[ \otimes_{i=1}^{t-1} \text{Bern}(K(\langle x_i, x_t \rangle)) \right]$$

where the mixing distribution is the *posterior* distribution of the latent points  $(x_1, \dots, x_t)$  given  $A^{t-1}$ .

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- Main source of looseness in [Liu-Mohanty-Schramm-Yang'21, Liu-Rácz'23]

# Our proof strategy to break $d \asymp n$ barrier

## ① KL expansion:

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## ③ Posterior analysis: $g(k) \lesssim d^{-2k}$ for $k \leq k_0$ and $g(k) \lesssim k^k d^{-3k/2}$ , so $\sum_t \sum_k \binom{t-1}{k} g(k) = o(1)$ , when $n^3 g(2) = o(1) \Leftrightarrow d \gg n^{3/4}$

$$g(k) \triangleq \mathbb{E}_{A^{t-1}} \left( \mathbb{E}_{x^k, x_t | A^{t-1}} \left[ \prod_{i=1}^k \langle x_i, x_t \rangle \right] \right)^2$$

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- Key:** show the lower bound is tight, so  $g(2) \asymp d^{-4}$

- Let  $\tilde{x}$  be an independent draw from the posterior  $P(\cdot|A^{t-1})$ ,

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## Posterior analysis (cont'd)

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- Let  $X_{ij} = \langle x_i, x_j \rangle$  for  $i \neq j$  and  $X_{ii} = 0$ . To show the above is  $O(d^{-2})$ , it suffices to prove, for  $\delta = \Theta(1/d)$ ,

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- By change-of-measure (“planting trick” [\[Achlioptas-Coja-Oghlan'08\]](#)),

$$\begin{aligned} \mathbb{P} \left\{ \langle \tilde{X}, X^* \rangle \gtrsim \delta \frac{n^2}{d} \right\} &\lesssim \mathbb{E}_{x \perp x^*} \left[ \exp \left( \frac{1}{2} \langle X, X^* \rangle \right) \mathbf{1}_{\{\langle X, X^* \rangle \gtrsim \delta n^2/d\}} \right] \\ &\leq \exp(-O(\delta^2 n^2)), \quad \text{when } d \gg \sqrt{n} \end{aligned}$$

# Extension to polynomial and general smooth kernels

- For polynomial kernels  $\kappa(t) = \sum_{\ell=0}^L b_{\ell} t^{\ell}$ , we can still derive an explicit expression of “posterior correlations”  $g(k)$  using Wick’s formula and bound it via similar but more sophisticated posterior analysis

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- Let  $P$  and  $\tilde{P}$  denote the distribution of RGG with  $K$  and  $\tilde{K}$ , respectively. Then

$$\text{TV}(P, \tilde{P}) \leq \binom{n}{2} \|K - \tilde{K}\|_2 = o(1), \quad \text{when } d \gg n^{3/4}$$

# Concluding remarks

- We show the detection threshold for fixed smooth kernels is  $d_{\text{test}}^* \asymp n^{3/4}$ , confirming a conjecture of [Liu-Rácz '23]
- As by-product, we show the estimation threshold is  $d_{\text{est}}^* \asymp \sqrt{n}$
- We extend the analysis to sparse linear kernels and scaled kernels
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## Future directions

- Proof of hard RGG detection threshold conjecture:  $d_{\text{test}}^* \asymp (np \log(1/p))^3$  [Liu-Mohanty-Schramm-Yang'21]
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## References

- Cheng Mao, Yihong Wu, & Jiaming Xu. *Random geometric graphs with smooth kernels: sharp detection threshold and a spectral conjecture*, Preprint, 2026.

# Extension to polynomial kernels

- For polynomial kernels  $\kappa(t) = \sum_{\ell=0}^L b_{\ell} t^{\ell}$ , we can still derive explicit expression of  $g(k)$  using Wick's formula. In particular,

$$\begin{aligned} g(2) &= \mathbb{E} \left[ \left( \mathbb{E} \left[ \kappa(\langle x_t, x_1 \rangle) \kappa(\langle x_t, x_2 \rangle) \mid A^{t-1} \right] \right)^2 \right] \\ &= \mathbb{E} \left[ \left( \mathbb{E} \left[ \eta(\langle x_1, x_2 \rangle) \mid A^{t-1} \right] \right)^2 \right], \end{aligned}$$

where  $\eta(\langle x_1, x_2 \rangle) \triangleq \mathbb{E}_{x_t} [\kappa(\langle x_t, x_1 \rangle) \kappa(\langle x_t, x_2 \rangle)]$ .

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- By Jensen's inequality,

$$\underbrace{\mathbb{E} \left[ \left( \mathbb{E} \left[ \eta(\langle x_1, x_2 \rangle) \mid A_{12} \right] \right)^2 \right]}_{\text{tr}^2(\kappa^3)} \leq g(2) \leq \underbrace{\mathbb{E} \left[ \eta^2(\langle x_1, x_2 \rangle) \right]}_{\text{tr}(\kappa^4)}$$

# Extension to polynomial kernels

- For polynomial kernels  $\kappa(t) = \sum_{\ell=0}^L b_{\ell} t^{\ell}$ , we can still derive explicit expression of  $g(k)$  using Wick's formula. In particular,

$$\begin{aligned} g(2) &= \mathbb{E} \left[ \left( \mathbb{E} \left[ \kappa(\langle x_t, x_1 \rangle) \kappa(\langle x_t, x_2 \rangle) \mid A^{t-1} \right] \right)^2 \right] \\ &= \mathbb{E} \left[ \left( \mathbb{E} \left[ \eta(\langle x_1, x_2 \rangle) \mid A^{t-1} \right] \right)^2 \right], \end{aligned}$$

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- Further,  $g(k) \lesssim d^{-2k}$  for  $2 \leq k \leq k_0$  and  $g(k) \lesssim k^k d^{-3k/2}$ , so  $\sum_{t=2}^n \sum_{k=1}^t \binom{t-1}{k} g(k) = o(1)$ , when  $n^3 g(2) = o(1) \Leftrightarrow d \gg n^{3/4}$

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# Extensions to scaled kernels and sparse regimes

- Scaled kernel  $K_r(t) \equiv K(rt)$  for  $r > 0$ 
  - ▶ Low-SNR regime ( $n^{-1/2+\epsilon} \ll r \ll 1$ ): any smooth kernel  $K$  with  $K'(0) \neq 0$ :
  - ▶ High-SNR regime ( $1 \ll r \ll d^{1/3-\epsilon}$ ): Gaussian kernel  $K$ :

$$d_{\text{test}}^* \asymp n^{3/4} r^{3/2}$$

- Linear kernel in sparse regime:  $K(t) = p(1+t)$  and  $np \geq \text{polylog}(n)$  :

$$d_{\text{test}}^* \asymp n^{3/4} p^{3/2}$$