Statistically and Computationally Optimal Estimation and Inference in the Common Subspace Model

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IMSI, 2026

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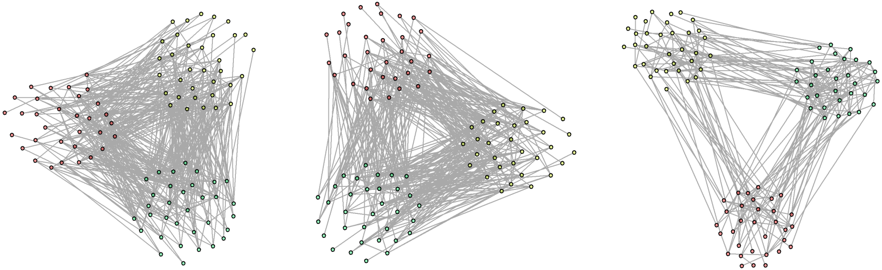
Outline

Multilayer Networks

Setting

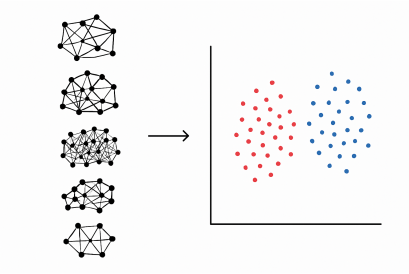
Observe L networks on same n vertices

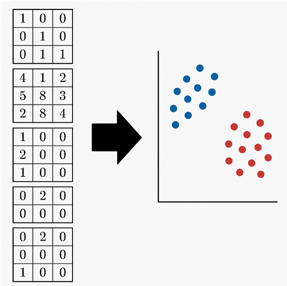


Data Integration

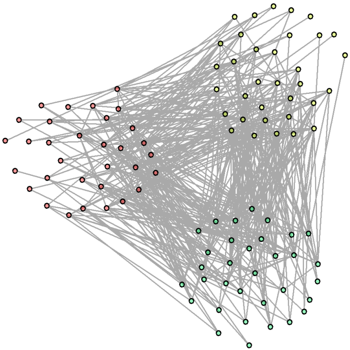
Goal

Find a common representation of all datasets

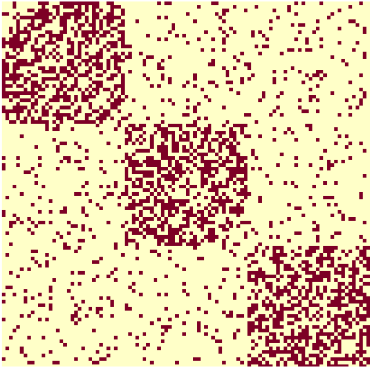




Adjacency Matrix

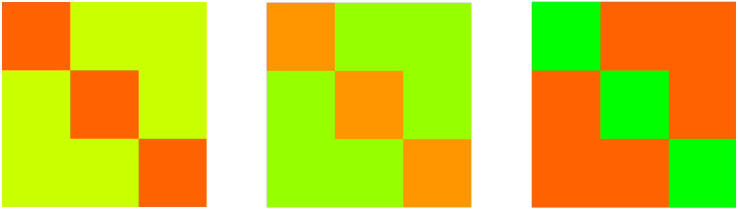


$\Longrightarrow$



Multilayer Blockmodel

Multilayer SBM (e.g. (Lei and Lin, 2022; Lei et al., 2020)) incorporates network-level idiosyncracies:



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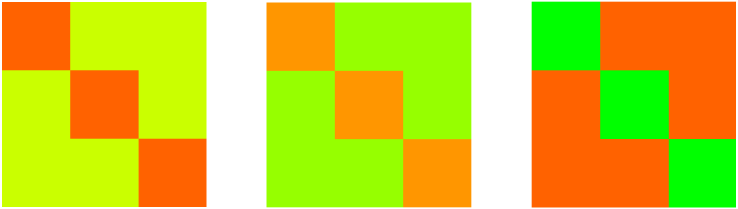

Problem

Edges determined solely by community memberships!

Many Generalizations

- Jing et al. (2021)
- Fan et al. (2021)
- Pensky and Wang (2021)
- Noroozi and Pensky (2022)
- Arroyo et al. (2021)
- Young et al. (2022)
- Agterberg and Zhang (2025)
- Agterberg et al. (2025)
- Hu and Wang (2023)
- Xie (2024)
- Pensky (2025)
- Agterberg and Cape (2025)
- Lyu et al. (2021)
- Lyu and Xia (2023)
- Lyu and Xia (2025)
- Wang et al. (2025)
- Yang et al. (2024)
- Chen et al. (2022)
- Chen and Hero (2017)
- Tang et al. (2025)
- Tian et al. (2025)
- Han et al. (2015)
- Han et al. (2021)
- Yan and Levin (2025)

(This list is incomplete...)

Common Subspace Model of Arroyo et al. (2021)

Assume each matrix shares a *common subspace*

$$\mathbb{E}(\boldsymbol{A}^{(l)}) = \boldsymbol{U}\boldsymbol{R}^{(l)}\boldsymbol{U}^\top \in \mathbb{R}^{n\times n}$$

$$\boldsymbol{U}^\top\boldsymbol{U} = \boldsymbol{I}_r; \qquad \boldsymbol{R}^{(l)} \in \mathbb{R}^{r\times r}$$

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Generalizes many other models!

A Simple Statistical Model

- Observe

$$\boldsymbol{A}^{(l)} = \underbrace{\boldsymbol{U}\boldsymbol{R}^{(l)}\boldsymbol{U}^\top}_{\text{Signal}} + \underbrace{\boldsymbol{N}^{(l)}}_{\text{Noise}}; \qquad l \in [L]$$

A Simple Statistical Model

- Observe

$$A^{(l)} = \underbrace{UR^{(l)}U^{\top}}_{\text{Signal}} + \underbrace{N^{(l)}}_{\text{Noise}}; \qquad l \in [L]$$

- $N^{(l)}$ is symmetric and satisfies

$$N_{ij}^{(l)} \sim \begin{cases} \mathcal{N}(0, \frac{\sigma^2}{2}) & i < j; \\ \mathcal{N}(0, \sigma^2) & i = j \\ N_{ji}^{(l)} & i > j. \end{cases}$$

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L observations of the classical *matrix denoising model*.

(Doesn't need to be Gaussian)

Key Parameters

Signal Strength Parameter

$$\lambda = \sigma_{\min}\left(\frac{1}{\sqrt{L}}\mathcal{R}\right) = \sigma_{\min}\left(\frac{1}{\sqrt{L}}\left[\boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)}, \cdots, \boldsymbol{R}^{(L)}\right]\right)$$

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Noise and Dimensions

σ = noise level;

n = dimension of each matrix;

L = number of Layers;

r = rank of each matrix

A Simple Statistical Question

What are the fundamental limits for:

- Estimation of $\boldsymbol{U}$?
- Inference for $\boldsymbol{U}$?

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$0 \qquad \frac{\sqrt{n}}{\sqrt{L}} \qquad \frac{\sqrt{n}}{L^{1/4}} \qquad \frac{n}{\sqrt{L}} \qquad \longrightarrow \text{SNR} = \lambda/\sigma$

Loss Function

Projection Loss

$$\mathrm{dist}_F(\boldsymbol{U}_1, \boldsymbol{U}_2) = \|\sin\Theta(\boldsymbol{U}_1, \boldsymbol{U}_2)\|_F = \frac{1}{\sqrt{2}}\|\boldsymbol{U}_1\boldsymbol{U}_1^\top - \boldsymbol{U}_2\boldsymbol{U}_2^\top\|_F$$

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- Focus on $r, \kappa = O(1)$ for simplicity.
- Assume that $L \lesssim n$ also for simplicity.

Minimax Lower Bound

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Theorem (Minimax Lower Bound)

It holds that

$$\inf_{\widehat{\boldsymbol{U}}} \sup_{\mathcal{P}(\lambda)} \mathbb{E}\| \sin \Theta(\widehat{\boldsymbol{U}}, \boldsymbol{U})\|_F^2 \gtrsim \frac{\sigma^2 n}{\lambda^2 L},$$

where the infimum is over all estimators $\widehat{\boldsymbol{U}}$ of $\boldsymbol{U}$.

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Projected Gradient Descent

Under Gaussian noise, the MLE is the minimizer of

$$L\left(\widehat{\boldsymbol{U}}, \{\widehat{\boldsymbol{R}}^{(l)}\}_{l=1}^{L}\right) = \frac{1}{2}\sum_{l=1}^{L} \left\|\boldsymbol{A}^{(l)} - \widehat{\boldsymbol{U}}\widehat{\boldsymbol{R}}^{(l)}\widehat{\boldsymbol{U}}^{\top}\right\|_F^2$$

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Can try minimizing this loss function directly!

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Initialize with leading r left singular vectors of $\sum_l (\boldsymbol{A}^{(l)})^2$
(no bias because of homoskedasticity)

Projected Gradient Descent

Algorithm Projected Gradient Descent Initialized via $\widehat{\boldsymbol{U}}_0$

Input: Collection of matrices $\{\boldsymbol{A}^{(l)}\}_{l=1}^{L}$; rank r, step-size η, number of iterations T, initialization $\widehat{\boldsymbol{U}}_0$.

1. While $t \leq T$:
    1. Set $\widehat{\boldsymbol{R}}_t^{(l)} := \widehat{\boldsymbol{U}}_t^{\top} \boldsymbol{A}^{(l)} \widehat{\boldsymbol{U}}_t$.
    2. Gradient step:
    $$\widehat{\boldsymbol{U}}_{t+.5} := \widehat{\boldsymbol{U}}_t - \frac{\eta}{L} \sum_{l=1}^{L} \left(\widehat{\boldsymbol{U}}_t \widehat{\boldsymbol{R}}_t^{(l)} \widehat{\boldsymbol{U}}_t^{\top} - \boldsymbol{A}^{(l)} \right) \widehat{\boldsymbol{U}}_t \widehat{\boldsymbol{R}}_t^{(l)}. \quad (1)$$
    3. Projection step:
    $$\widehat{\boldsymbol{U}}_{t+1} := \mathrm{SVD}_r(\widehat{\boldsymbol{U}}_{t+.5}) \quad (2)$$

Output: $\widehat{\boldsymbol{U}}_T$.

Theoretical Guarantee

Theorem (Upper Bounds for Iterates)

Suppose that

$$\mathrm{SNR} = \lambda/\sigma \geq C_0 \frac{\sqrt{n}}{L^{1/4}}. \tag{3}$$

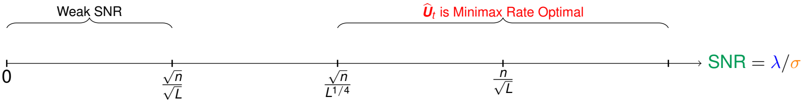
Then with high probability,

$$\|\sin\Theta(\widehat{\boldsymbol{U}}_t, \boldsymbol{U})\|_F \leq \frac{C_1 \sigma \sqrt{n}}{\lambda \sqrt{L}} + \frac{1}{2^{t+1}}.$$

Theoretical Guarantee

When SNR $= \lambda/\sigma \gtrsim \frac{\sqrt{n}}{L^{1/4}}$, after logarithmically many iterations,

$$\|\sin\Theta(\widehat{\boldsymbol{U}}_t, \boldsymbol{U})\|_F \lesssim \frac{\sigma\sqrt{n}}{\lambda\sqrt{L}}.$$



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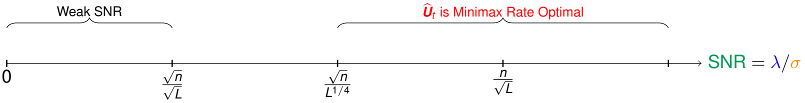

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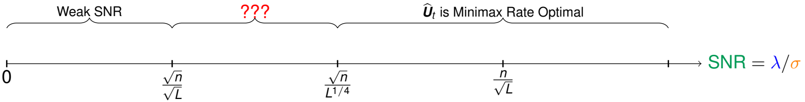

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Theorem

If $\mathrm{SNR} = \lambda/\sigma \ll \frac{\sqrt{n}}{L^{1/4}}$*, then* $\mathbb{E}_{H_a}(LR^{\leq D})^2 = 1 + o(1)$.

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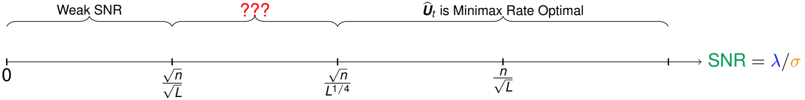

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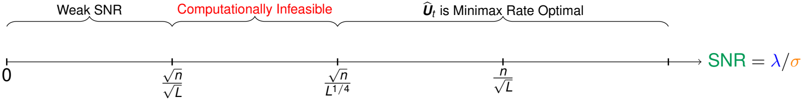

Confidence Intervals for the Loss

Goal

Find *rate-optimal* confidence intervals for $\| \sin \Theta(\widehat{\boldsymbol{U}}_t, \boldsymbol{U}) \|_F^2$.

Minimax Lower Bound on Length

Theorem

Suppose $\lambda/\sigma \geq C_0\sqrt{\frac{n}{L}}$. *Then minimax length of any confidence interval* CI_α *that is valid over* $\mathcal{P}(\lambda)$ *satisfies*

$$\inf_{\mathrm{CI}_\alpha} \sup_{\theta \in \mathcal{P}(\lambda)} \mathbb{E}_\theta \mathrm{Length}(\mathrm{CI}_\alpha) \gtrsim \frac{\sigma^2 n}{\lambda^2 L}\left(\frac{\sigma\sqrt{n}}{\lambda\sqrt{L}} + \frac{1}{\sqrt{n}}\right)$$

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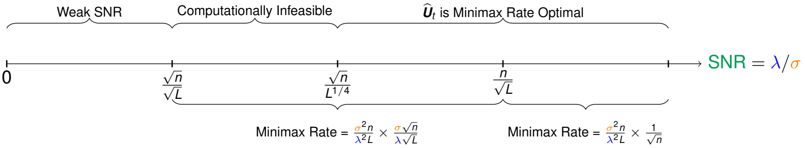

A Limit Theorem

Theorem (Asymptotic Normality)

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Suppose that $\lambda/\sigma \geq C_0 \frac{n}{\sqrt{L}}$. *Recall*

$$\mathcal{R} = \left[\boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)}, \cdots, \boldsymbol{R}^{(L)}\right].$$

Then after logarithmically many iterations,

$$\frac{\|\sin\Theta(\widehat{\boldsymbol{U}}_t, \boldsymbol{U})\|_F^2 - \frac{\sigma^2 n}{2}\mathrm{Tr}\left((\mathcal{R}\mathcal{R}^\top)^{-1}\right)}{\sigma^2\sqrt{n/2}\|(\mathcal{R}\mathcal{R}^\top)^{-1}\|_F} \xrightarrow{d} \mathcal{N}(0,1).$$

Proposed Confidence Interval

$$\widehat{\boldsymbol{R}}^{(l)} = \widehat{\boldsymbol{U}}_t^\top \boldsymbol{A}^{(l)} \widehat{\boldsymbol{U}}_t; \qquad \widehat{\mathcal{R}} := \left[\widehat{\boldsymbol{R}}^{(1)}, \widehat{\boldsymbol{R}}^{(2)}, \cdots, \widehat{\boldsymbol{R}}^{(L)}\right];$$

$$\widehat{\mathrm{CI}} := \frac{\sigma^2 n}{2}\mathrm{Tr}\left(\left(\widehat{\mathcal{R}}\widehat{\mathcal{R}}^\top\right)^{-1}\right) \pm z_{\alpha/2}\sigma^2\sqrt{\frac{n}{2}}\left\|\left(\widehat{\mathcal{R}}\widehat{\mathcal{R}}^\top\right)^{-1}\right\|_F.$$

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Theorem

Suppose that $\lambda/\sigma \geq C_0 \frac{n}{\sqrt{L}}$. *Then*

$$\mathbb{P}\Big(\| \sin \Theta(\widehat{\boldsymbol{U}}_t, \boldsymbol{U}) \|_F^2 \in \widehat{\mathrm{CI}} \Big) \geq 1 - \alpha;$$

$$\mathrm{L}(\widehat{\mathrm{CI}}) \lesssim \frac{\sigma^2 n}{\lambda^2 L} \frac{1}{\sqrt{n}} = \textit{ minimax optimal length.}$$

Confidence Intervals

When SNR $\gtrsim \frac{n}{\sqrt{L}}$, $\widehat{\text{CI}}$ attains the minimax-optimal length!

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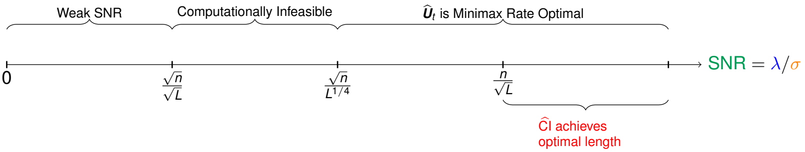

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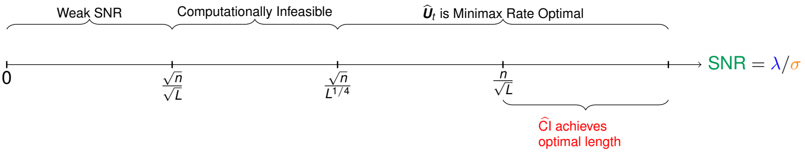

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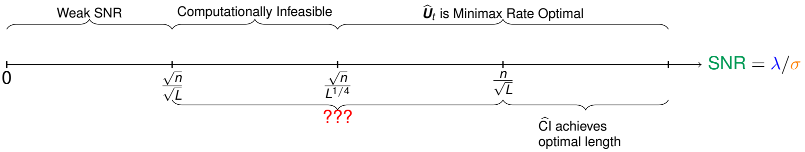

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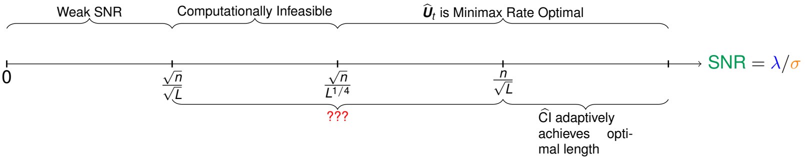

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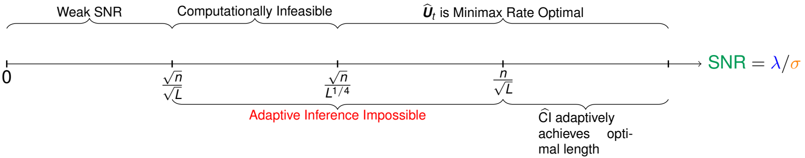

Conclusion

Main Question

What are the fundamental limits for:

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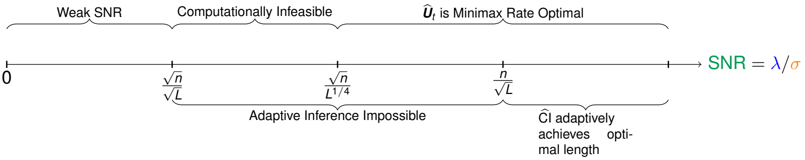

Questions?

(ArXiv paper to appear soon...)

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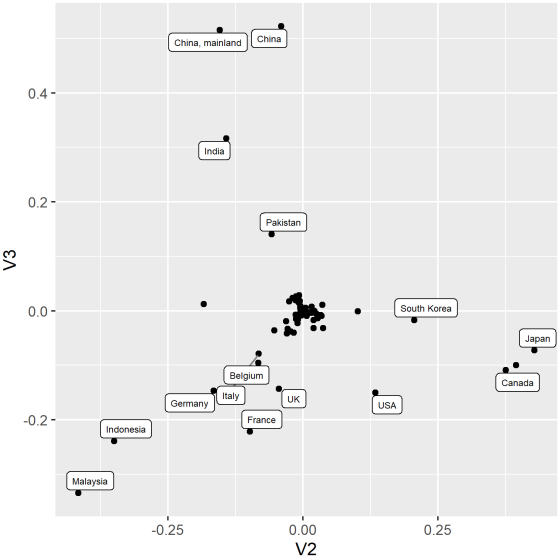
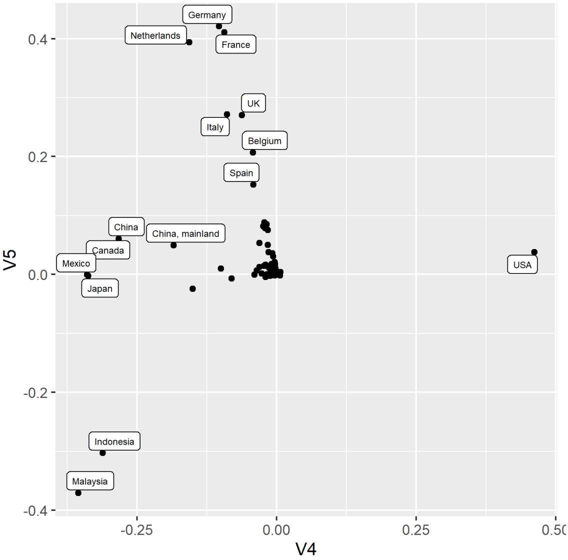
Trade Data: Dimensions 2 and 3



Trade Data: Dimensions 4 and 5



Dimension 1

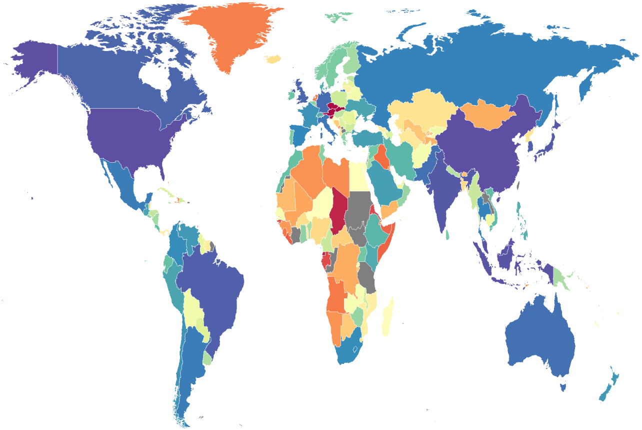


Figure: Percentile of latent dimension 1.

Dimension 2

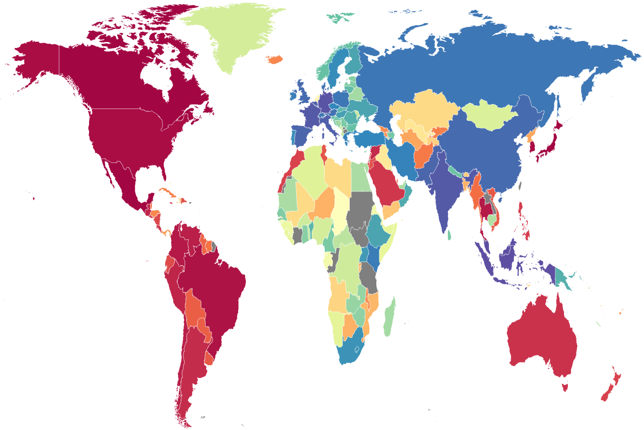


Figure: Percentile of latent dimension 2.

Dimension 3

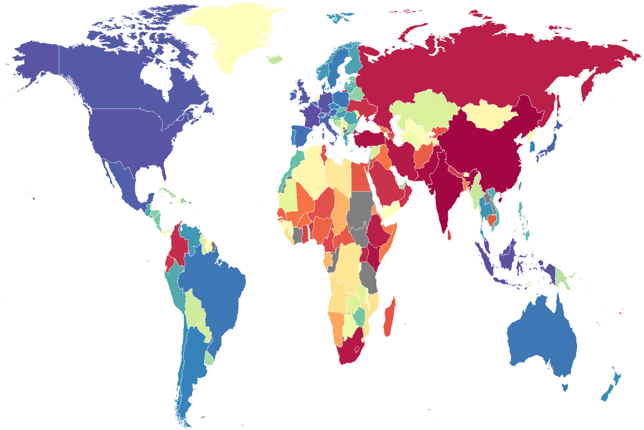


Figure: Percentile of latent dimension 3.

Dimension 4

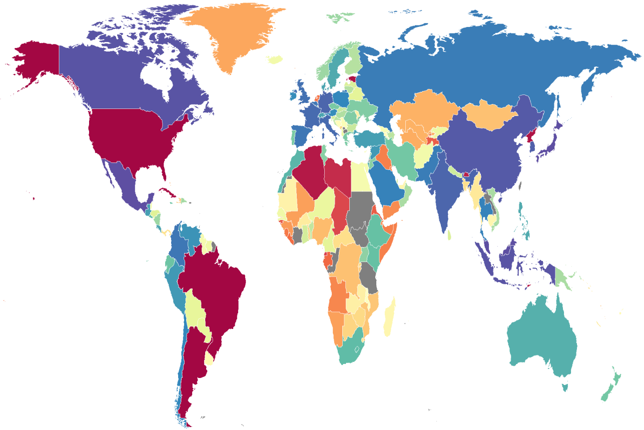


Figure: Percentile of latent dimension 4.

Dimension 5

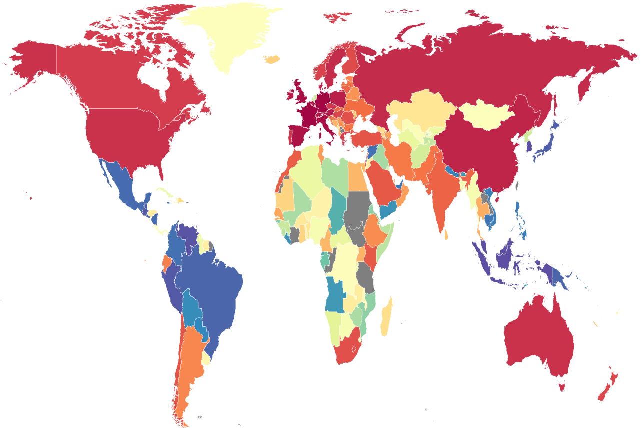


Figure: Percentile of latent dimension 5.