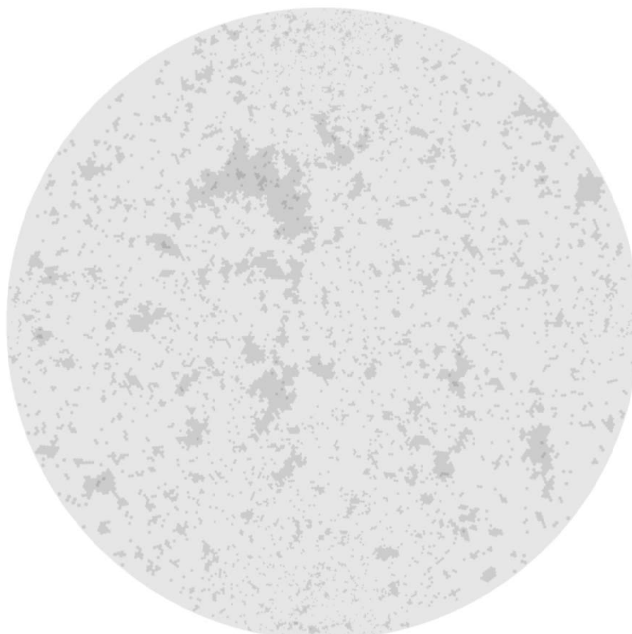
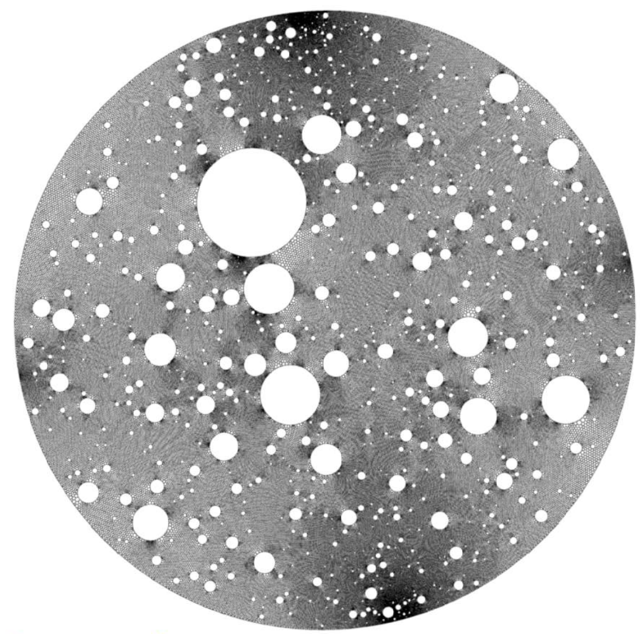
Uniformization of CLE

S. Rohde, UW

Greg Lawler fest, IMSI, Chicago, 7/9/2026



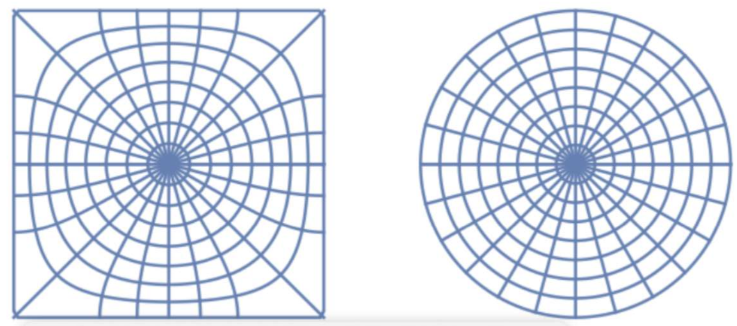


images courtesy David Wilson

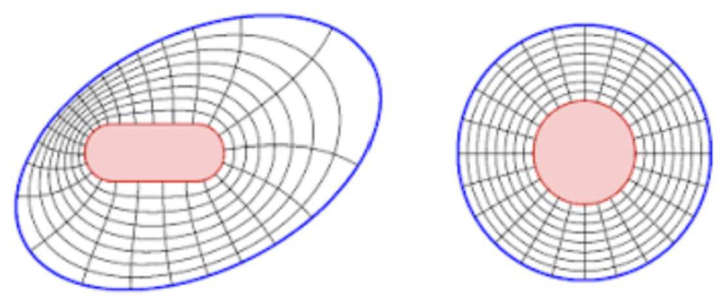
joint work with Juhan Aru, Philémon Bordereau, Brent Werness

Riemann – Koebe mapping theorem : If $D \subset \hat{\mathbb{C}}$ is n-connected, then D is conformally a circle domain.

$n = 1$



$n = 2$



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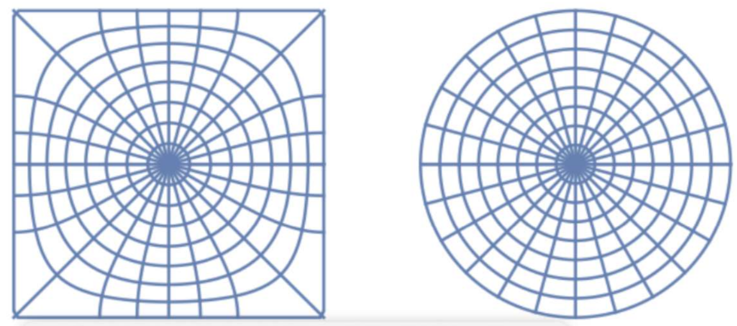

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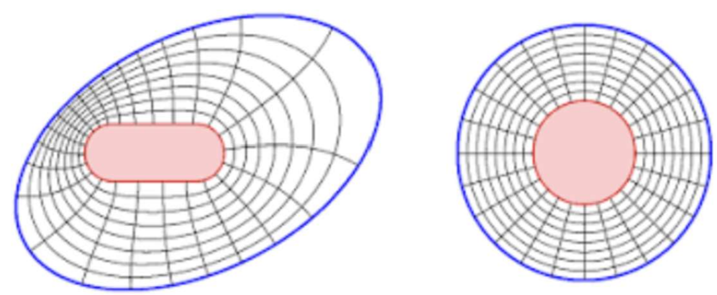

How to prove this if you are Greg Lawler?

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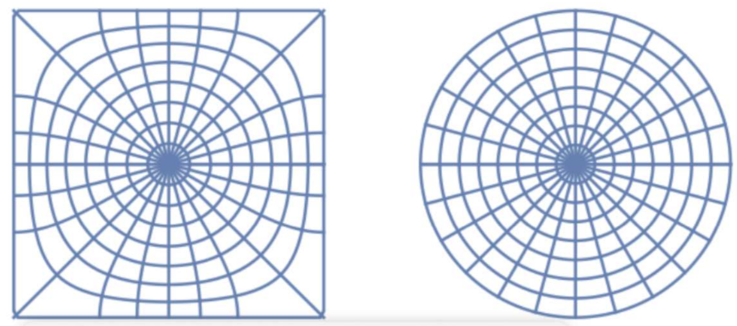

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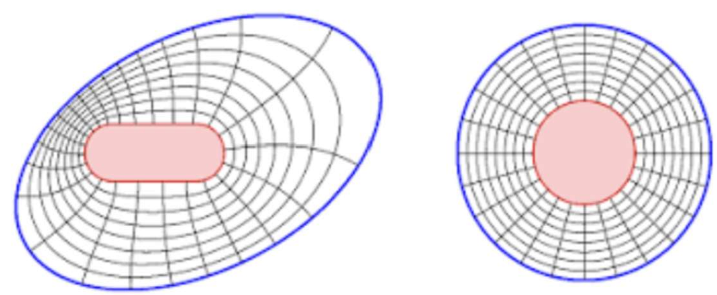

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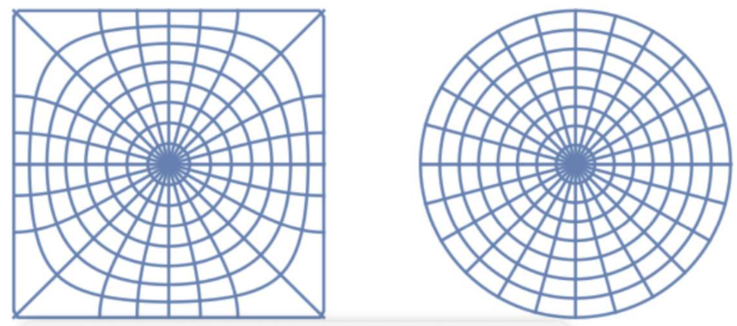

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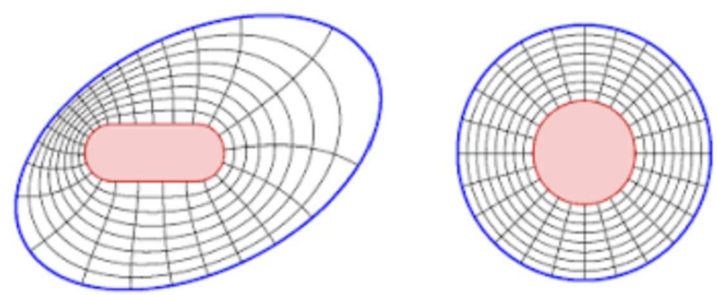

How to prove this if you are Greg Lawler ?

If $n=1$: Run BM. If $n \geq 2$: Invent ERBM, notice it does s.th. else.

Mathematical
Surveys
and
Monographs
Volume 114

Conformally Invariant Processes in the Plane

Gregory F. Lawler

American Mathematical Society

Difficulties : No known extremal problem ; uniqueness

Koebe conjecture : Every open connected set conf. equivalent to circle domain

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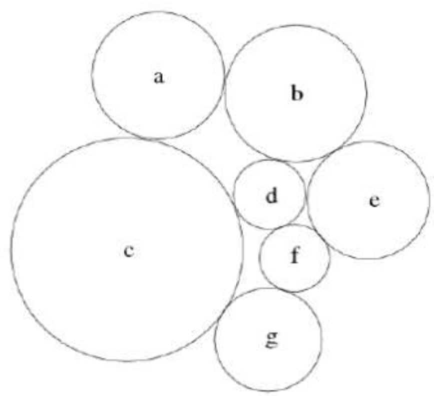
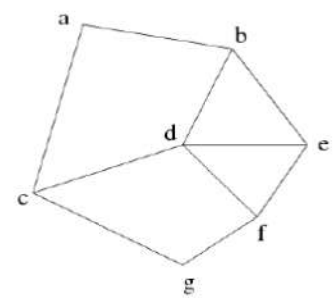
Circle packing





tangency graph

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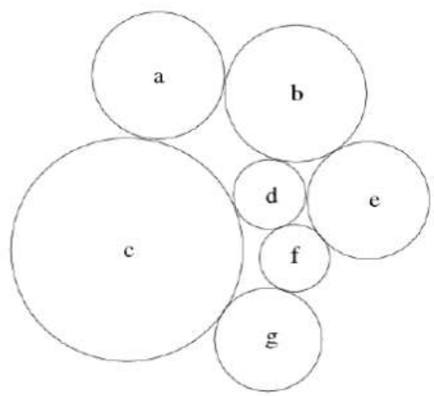

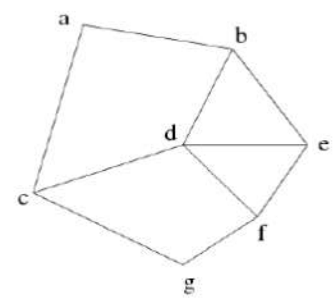

tangency graph

Thm. For every finite planar graph G, there is a circle packing in the plane with tangency graph G.

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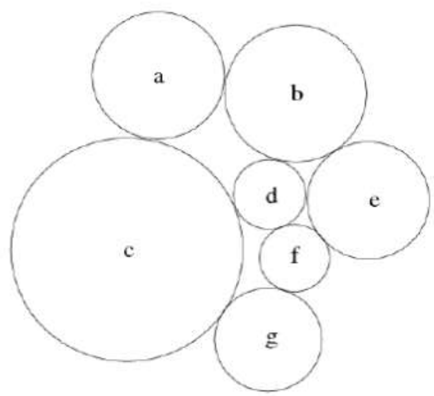

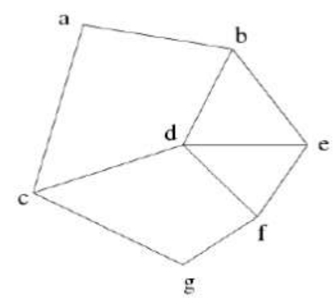

tangency graph

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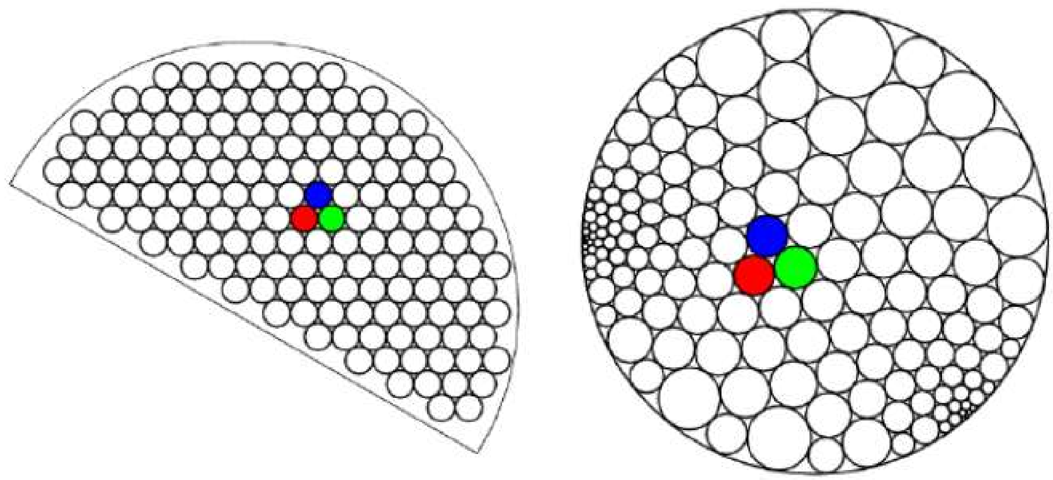
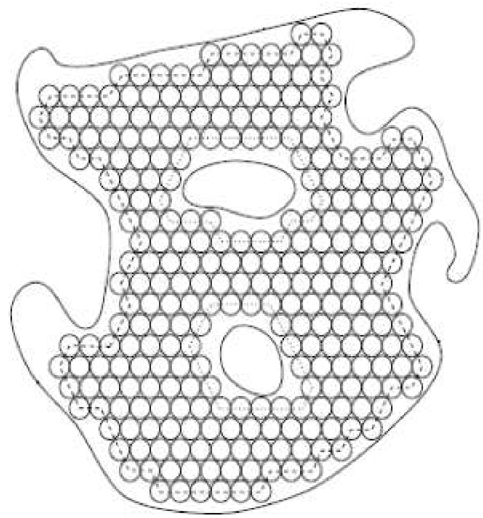
If G is triangulation, unique (up to Möbius transform).

Rodin - Sullivan Thm: CP converge to Riemann map ...



... even in multiply connected domains:



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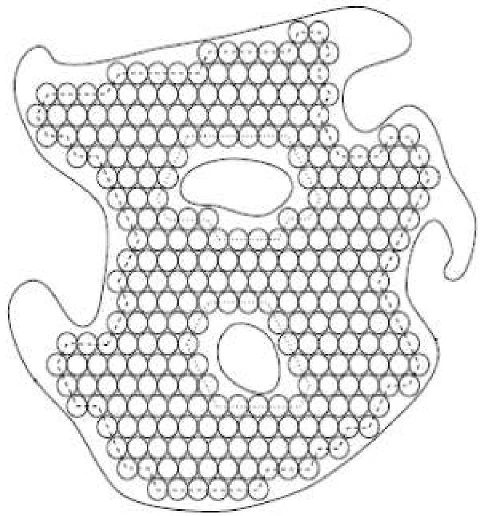

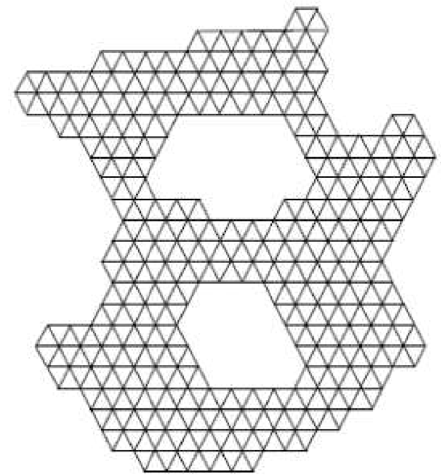

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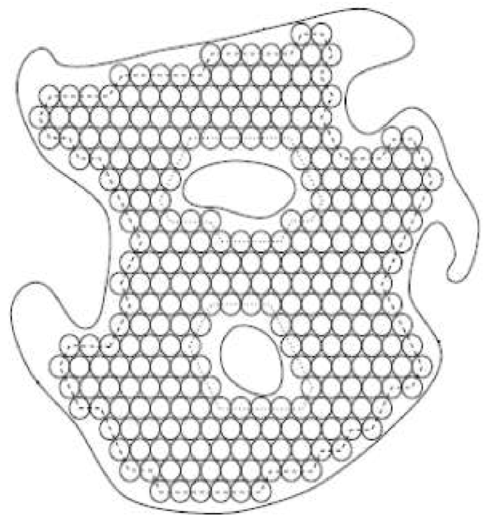

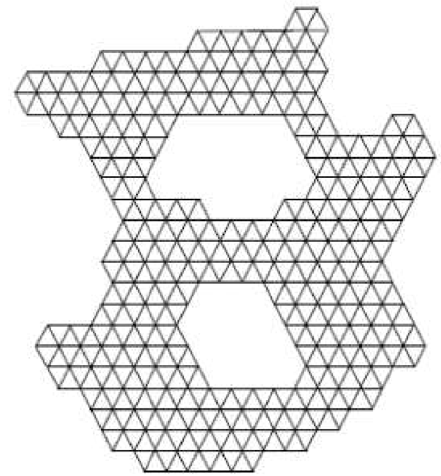

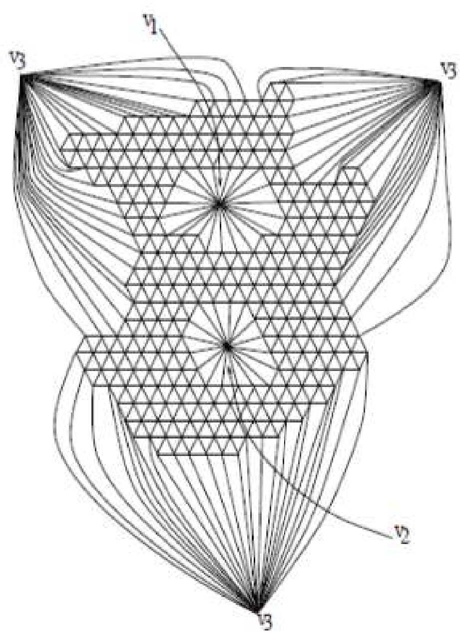

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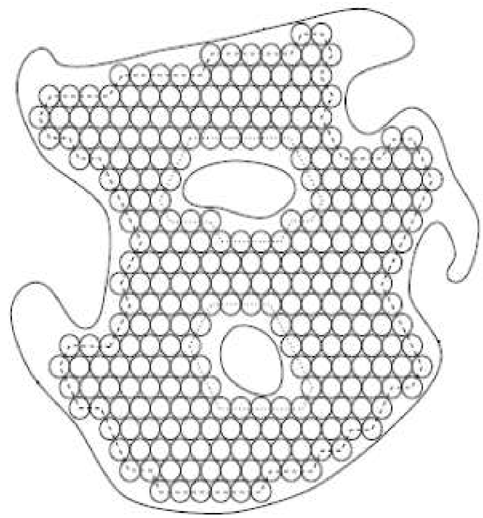

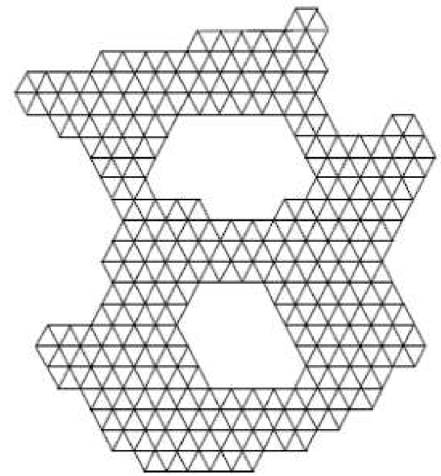

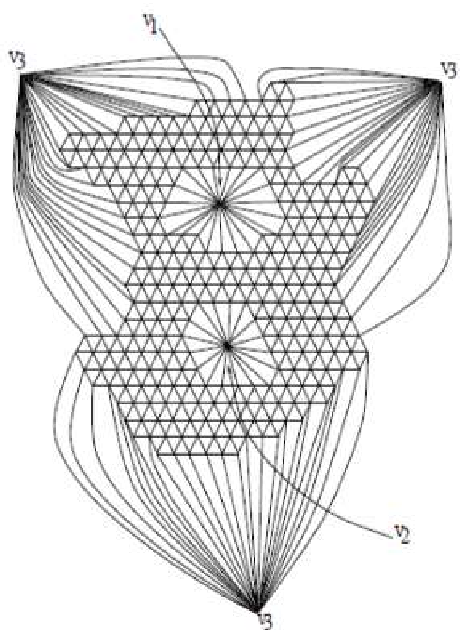

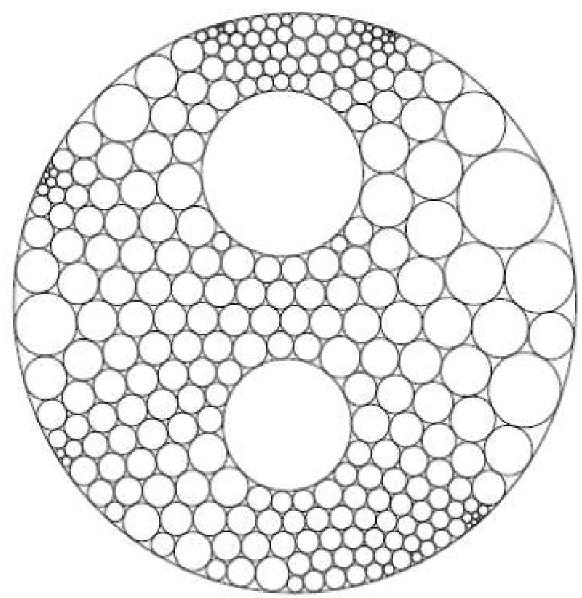

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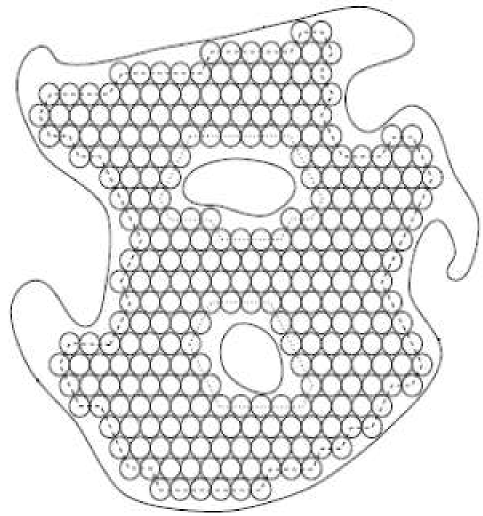
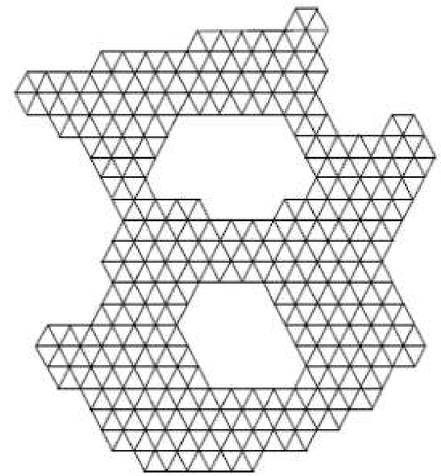
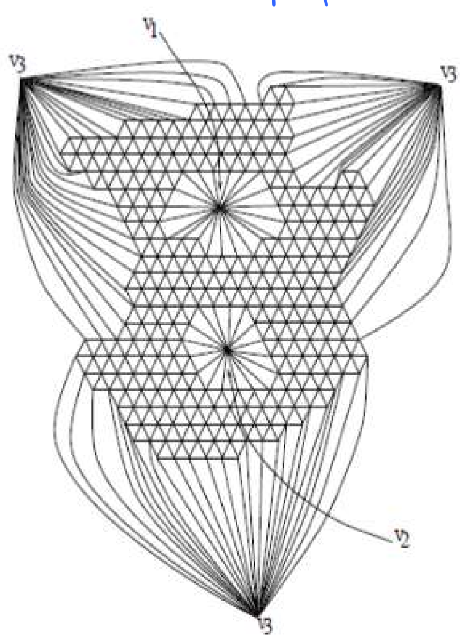
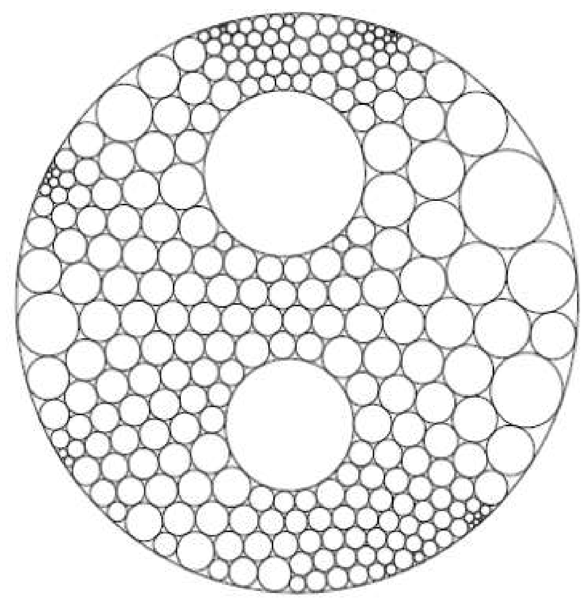


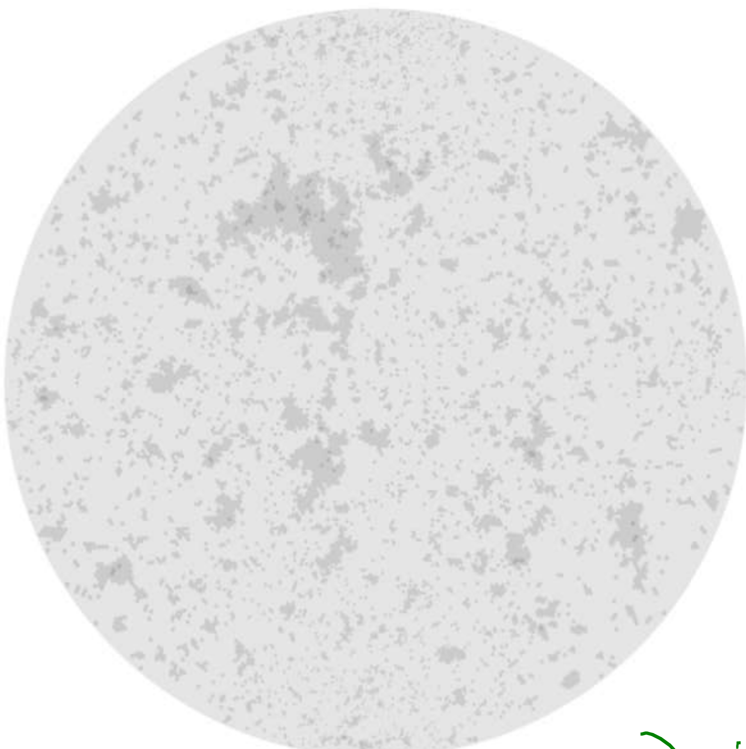


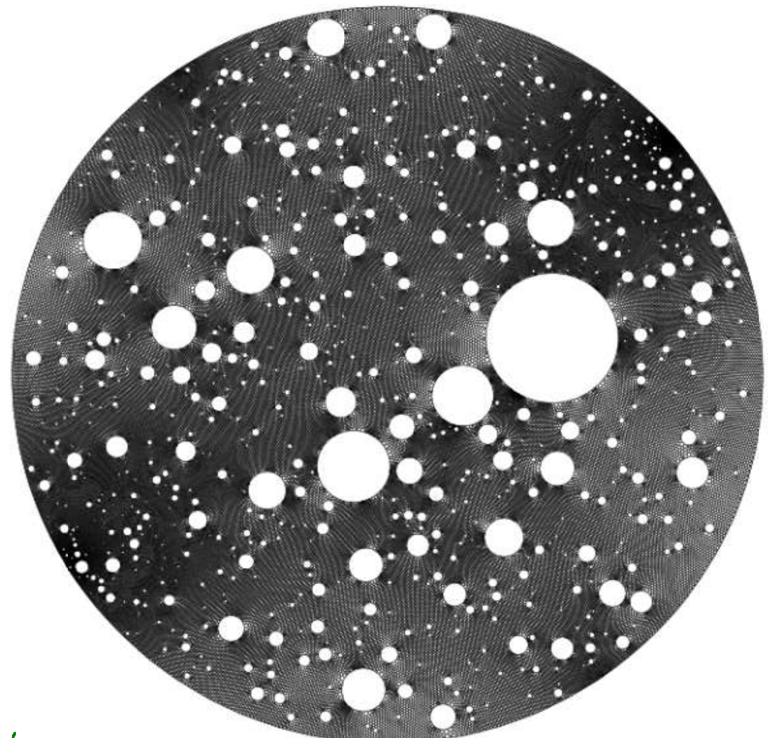
top pictures: Oded Schramm's thesis



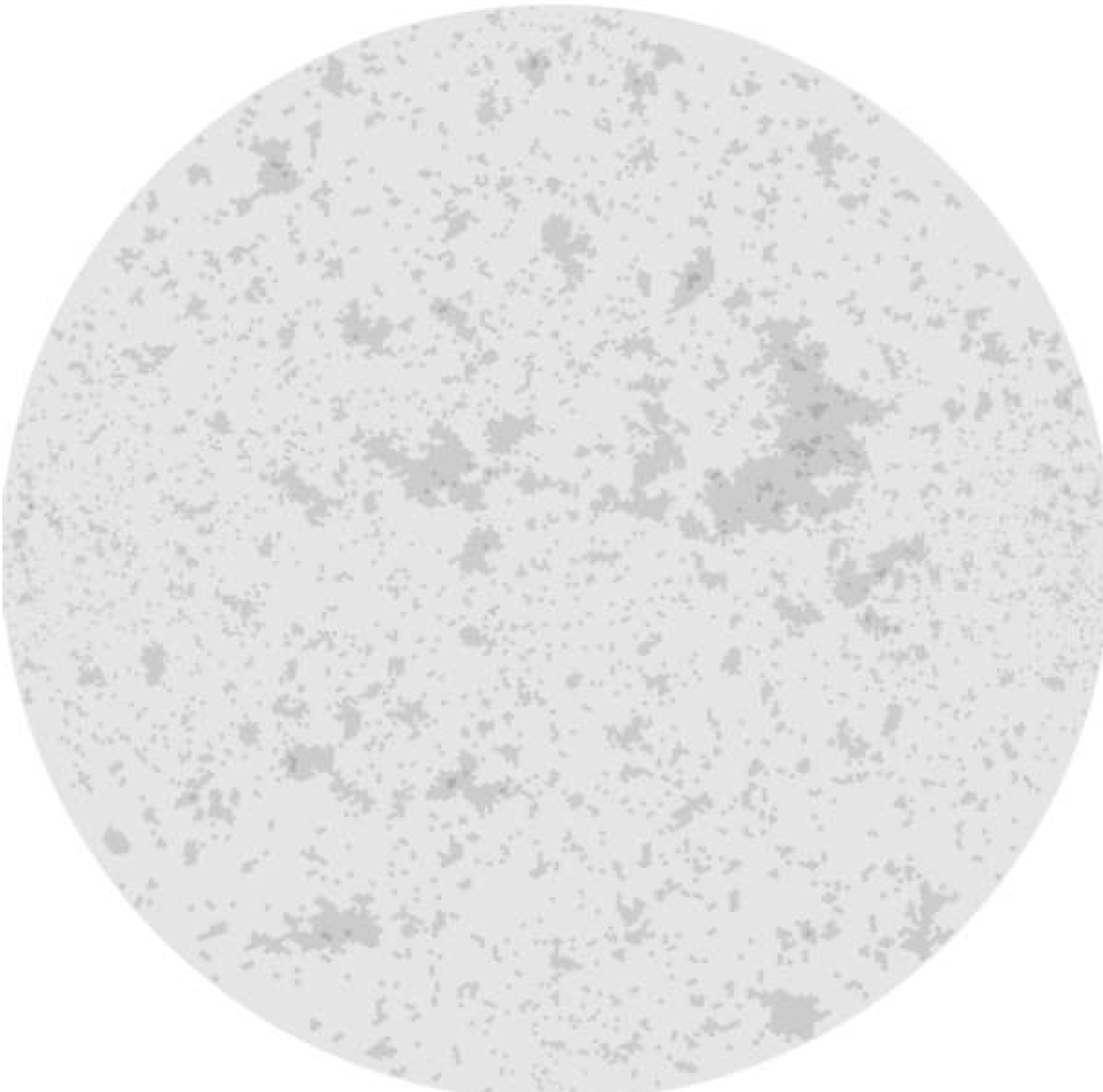


Circle packing of critical Ising loops





David Wilson



Ising model at criticality on triangular grid

Ising Model

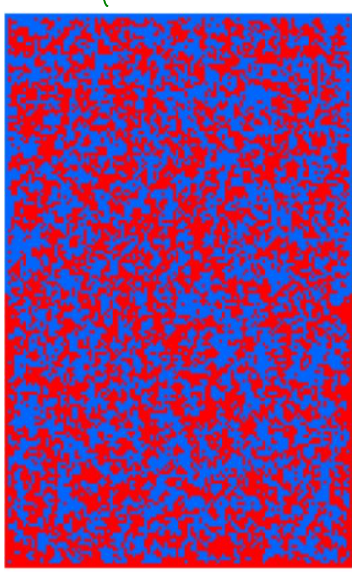
$G = (V, E)$ graph, e.g. $D \cap \varepsilon \mathbb{Z}^2$

States $\sigma : V \to \{-1, +1\}$

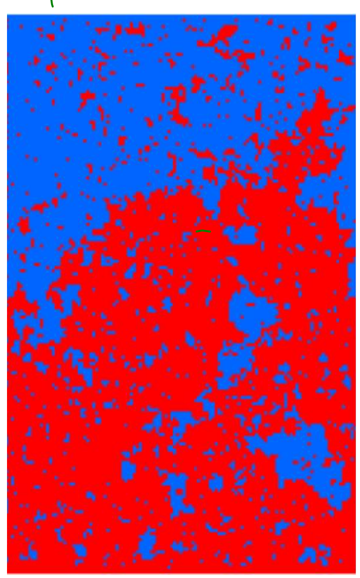
$$H(\sigma) = \sum_{(x,y) \in E} 1_{\{\sigma(x) \neq \sigma(y)\}}$$ Hamiltonian

$P_\beta(\sigma) \propto e^{-\beta \cdot H(\sigma)}$ β "inverse temperature"; $Z = \sum_\sigma e^{-\beta H(\sigma)}$ partition function

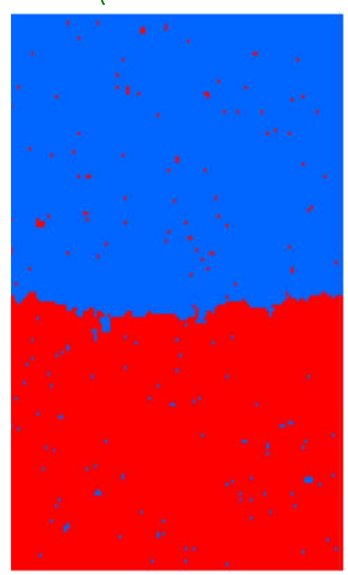
$\beta \approx 0$

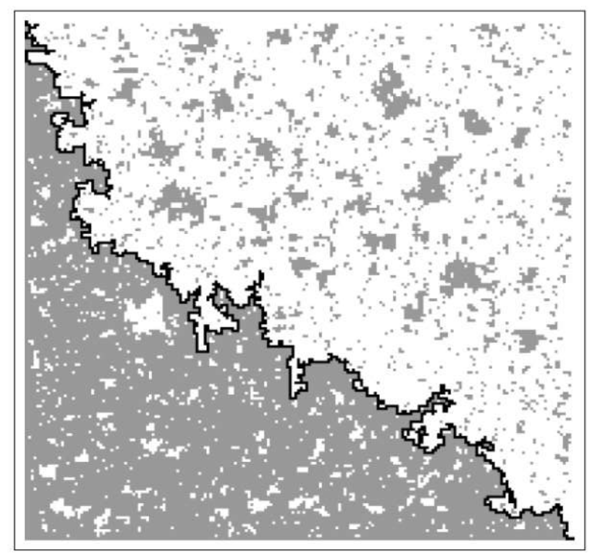


β critical



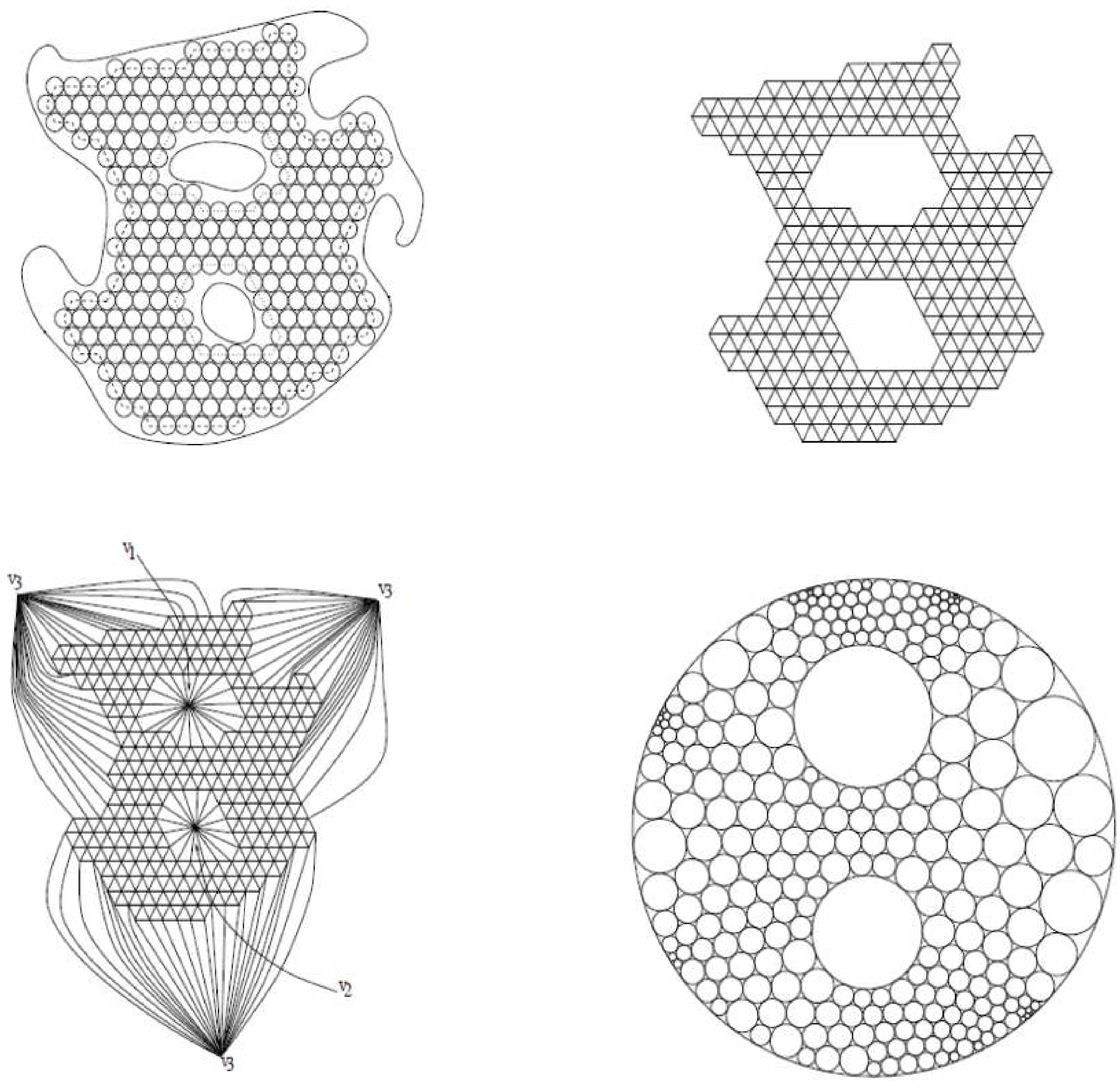
$\beta >> 1$



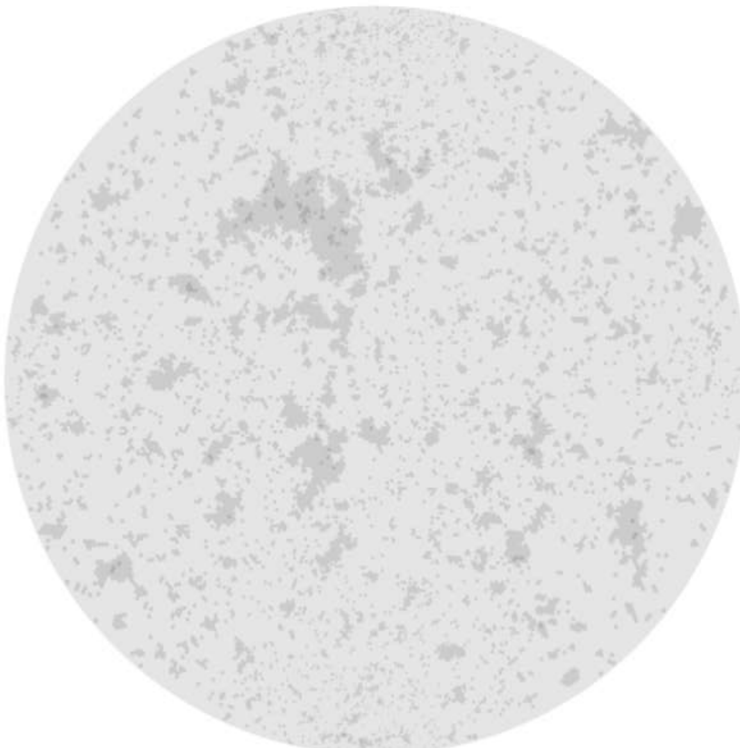


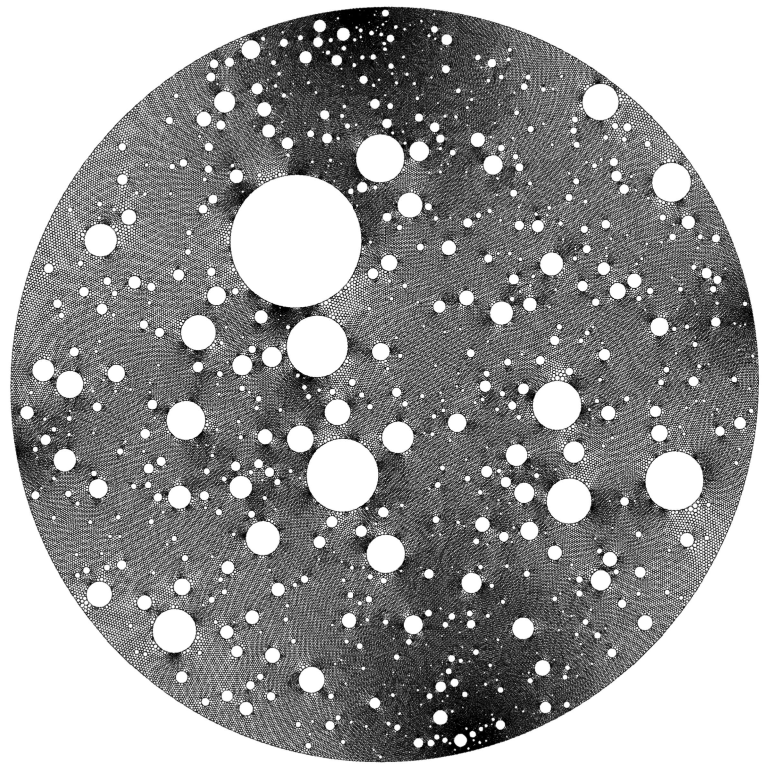
build triangulation with one vertex per outer loop

... as we did here:



and use Ken Stephenson's program to Circle Pack :





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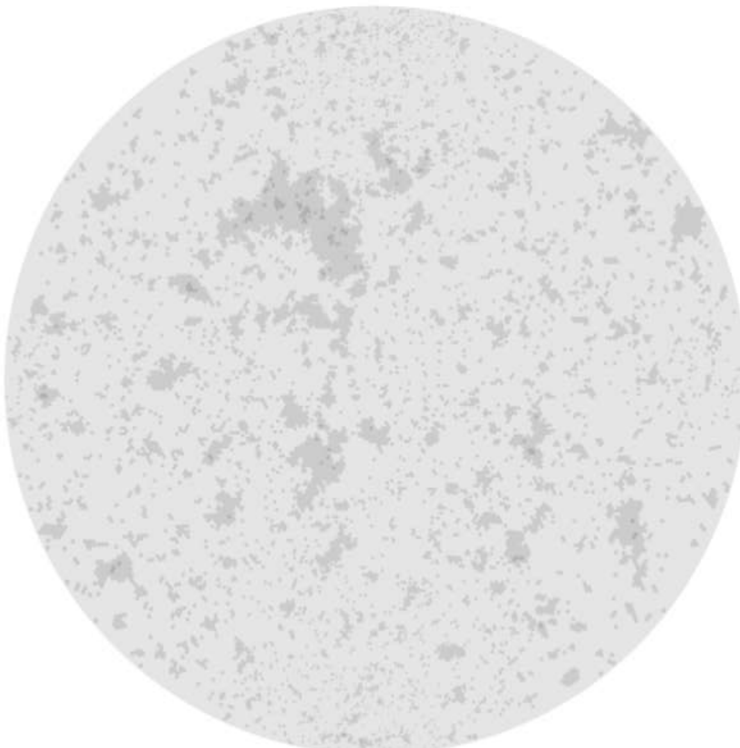
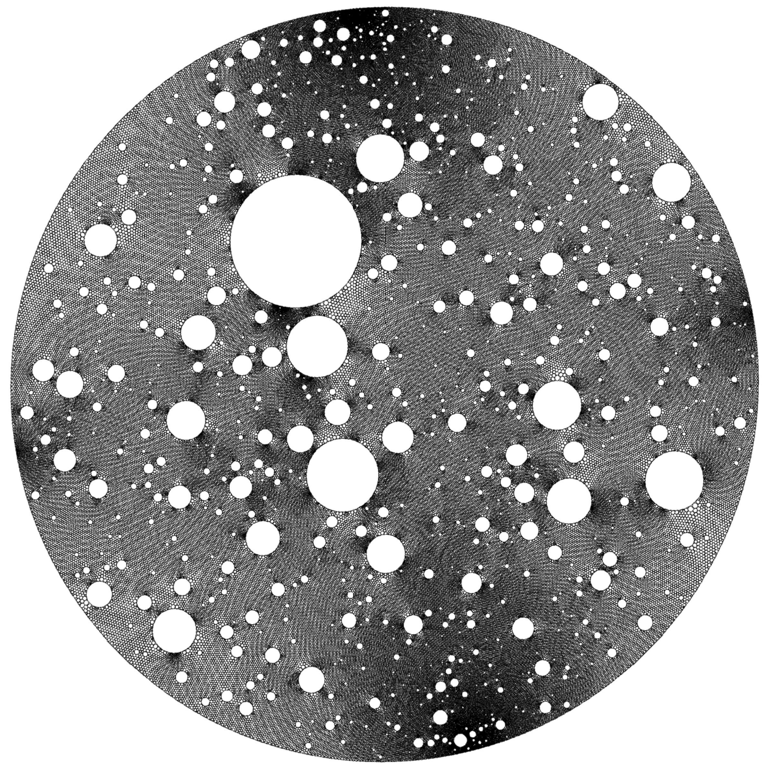
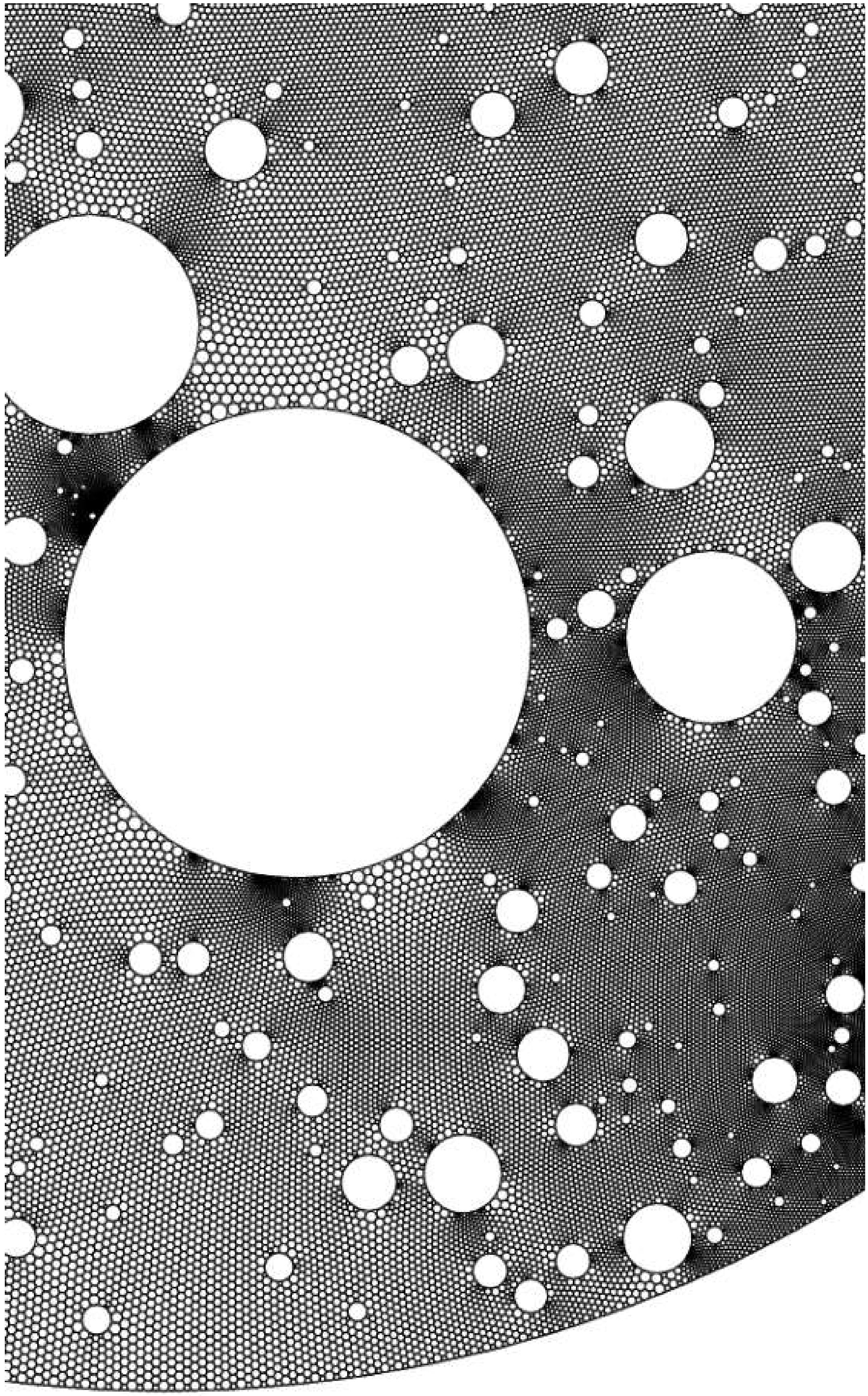


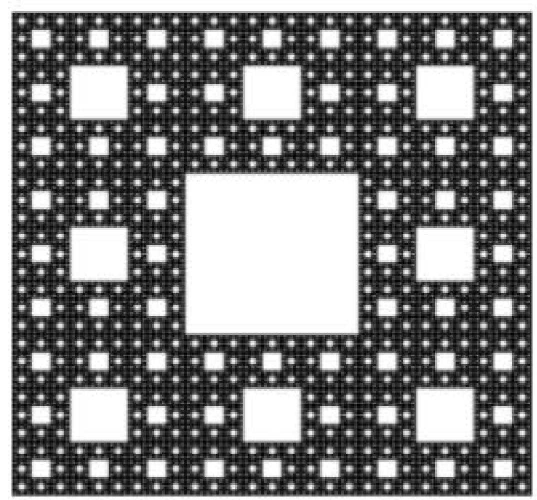


CP Theorem suggests this approximates a conformal map $CLE_3 \to$ round carpet



Sierpinski carpets $\left(X = \hat{\mathbb{C}} \setminus \bigcup_j D_j \,,\ \operatorname{diam}(D_j) \to 0,\ \operatorname{int} X = \emptyset \right)$

disjoint Jordan domains



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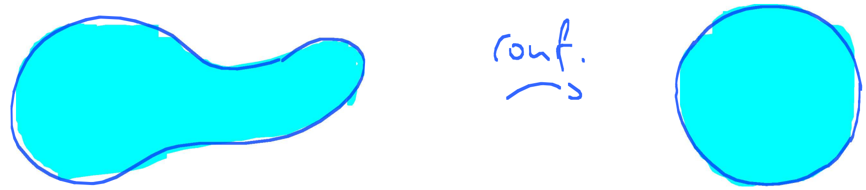
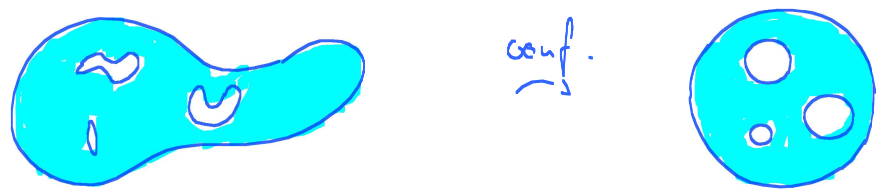
Riemann :



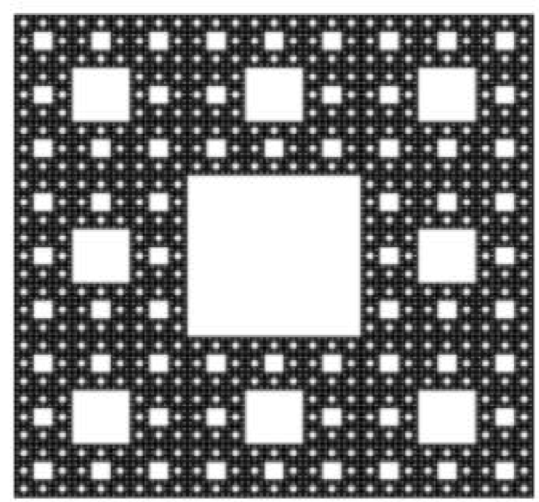
Koebe :



Koebe conjecture: Every <u>open</u> connected set conf. equivalent to circle domain

Are Sierpinski carpets conf. equivalent to "round carpets"?

(No interior! Conformal?
Topologically: Yes, Whyburn)



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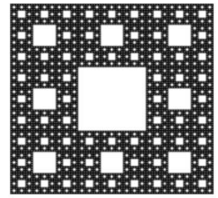

Roughly : Nice carpets have unique conformal structure

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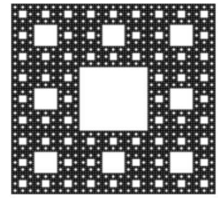

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<u>Thm</u> (Bonk; Bonk-Kleiner-Merenkov) : If X is "nice" carpet, then there is essentially unique round carpet X' and a "nice" homeomorphism $f: X \to X'$

nice : holes are uniformly separated quasicircles, full area ; f quasisymmetric
uniqueness if $\text{area}(X') = 0$.

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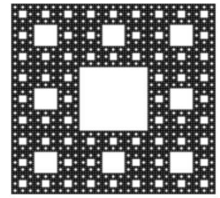

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Strategy : $f_n : X_n = D_0 \setminus \bigcup_{j=1}^{n} D_j \to \mathbb{D} \setminus \bigcup_{j=1}^{n} B_j = X_n'$ conformal (normalized)

Want to show equicontinuity

How about CLE ?

CLE (Sheffield; Lawler-Werner; S-W)

1) Brownian loop measure (Lawler-Werner)

$$\mu^{loop} = \int_{\mathbb{C}} \int_0^\infty \frac{\mu^\#(z,z,t)}{2\pi t^2}\, dt\, d^2z \,,$$ $\mu^\#$ law of Brownian bridge of duration t

image of $d^2z \otimes \frac{d\rho}{\rho} \otimes \frac{\mu^\#(0,0,1)}{\pi}(d\gamma)$ under $(z,\rho,\gamma) \mapsto \rho(z+\gamma)$

invariant under translation and dilation

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2) Brownian loop soup

$\Gamma = (\gamma_j)$ Poisson point process intensity $\alpha \cdot \mu$ $(0 < \alpha \leq \frac{1}{2})$

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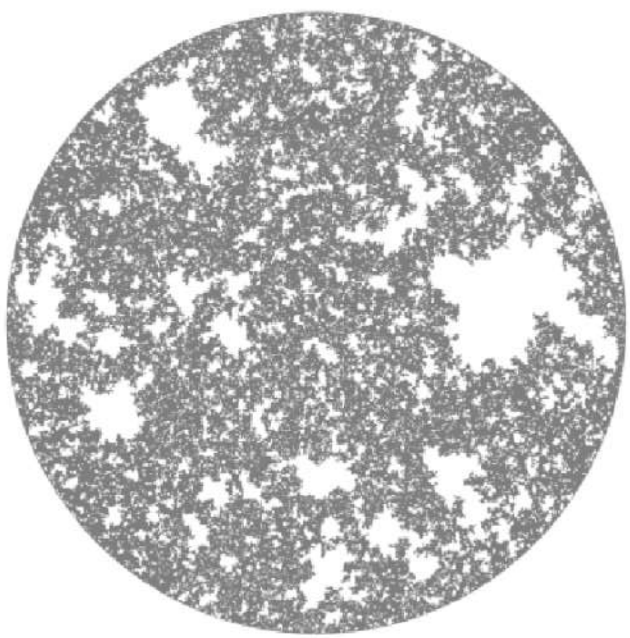
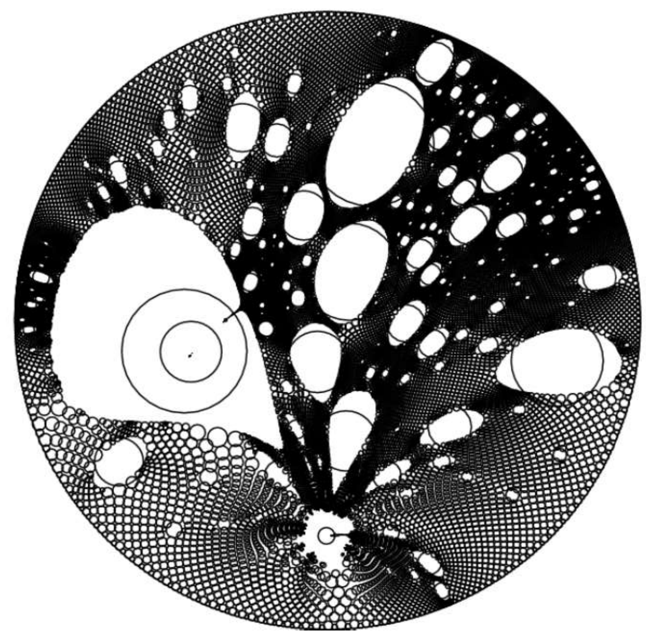
3) CLE_κ, $\alpha = \dfrac{(3\kappa-8)(6-\kappa)}{4\kappa}$

$\mathcal{L} = (\ell_j)$, ℓ_j outermost boundaries of closures of clusters of loops γ_j in D

Smirnov, Chelkak, Benoist - Hongler: Ising $\to$ CLE_3 as mesh $\to 0$

Schramm - Sheffield: If h GFF, the outermost loops of $\{h=0\}$ are CLE_4





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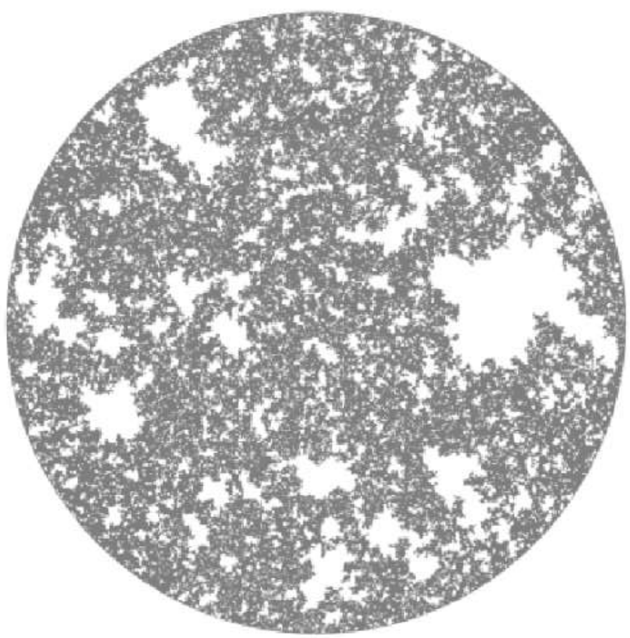
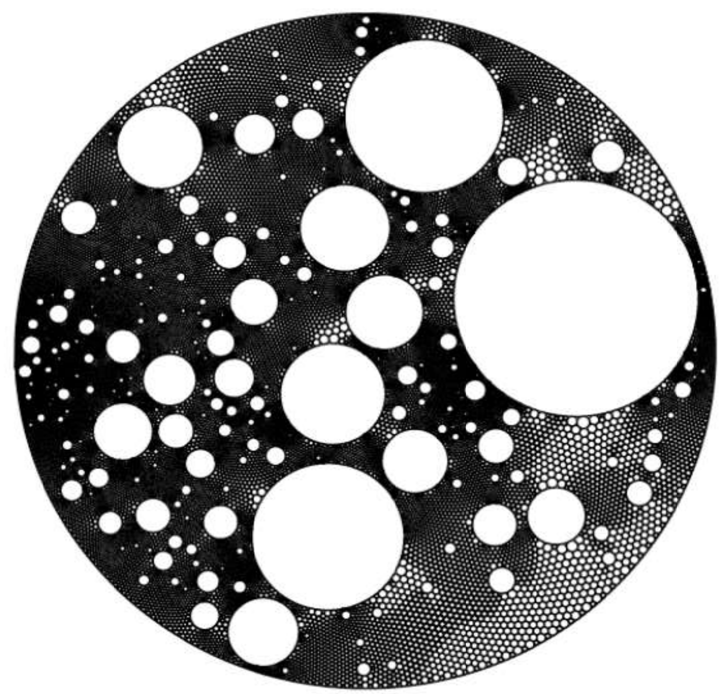
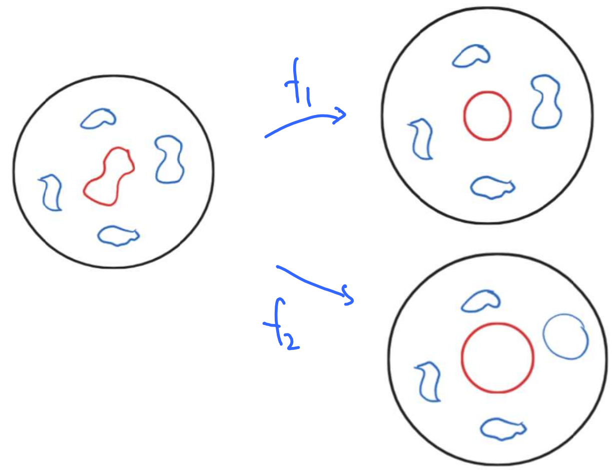




CLE Uniformization

Setup: $f_n : X_n = \mathbb{D} \setminus \bigcup_{j=1}^{n} D_j \to \mathbb{D} \setminus \bigcup_{j=1}^{n} B_j = X_n'$ conformal (normalized)

($\ell_0 = \partial\mathbb{D}$, B_1 centered at 0)



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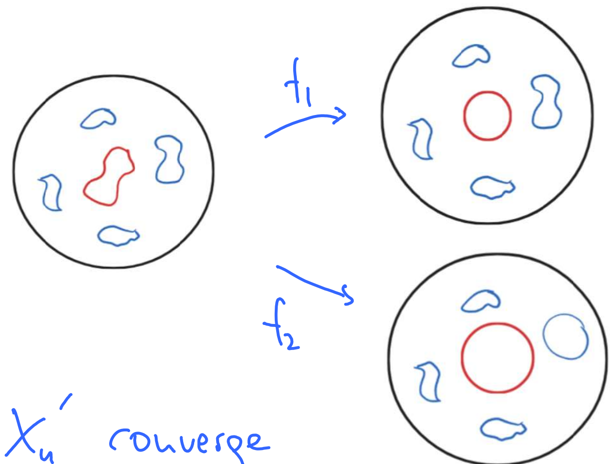

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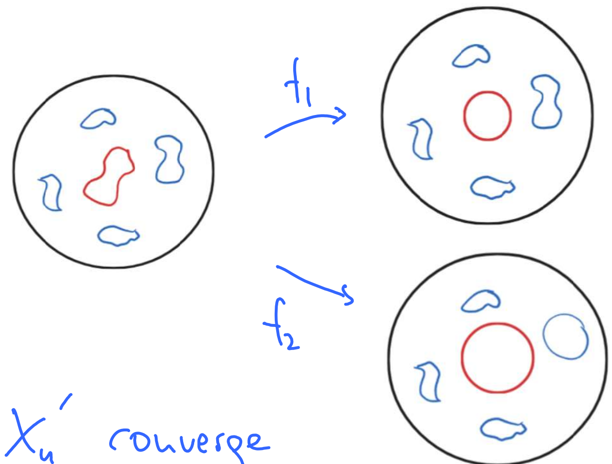

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$f : X \to X'$ homeomorphism.

Local square summability: $\exists C > 0 \; \forall x \in X \; \exists r_n \to 0$: $\displaystyle\sum_{D_j \cap A(x,r,2r) \neq \emptyset} \operatorname{dia}^2(D_j \cap A(x,r,2r)) \leq C \cdot r^2$

Ntalampekos: (GSS) $\Rightarrow$ subsequential convergence of f_n^{-1}

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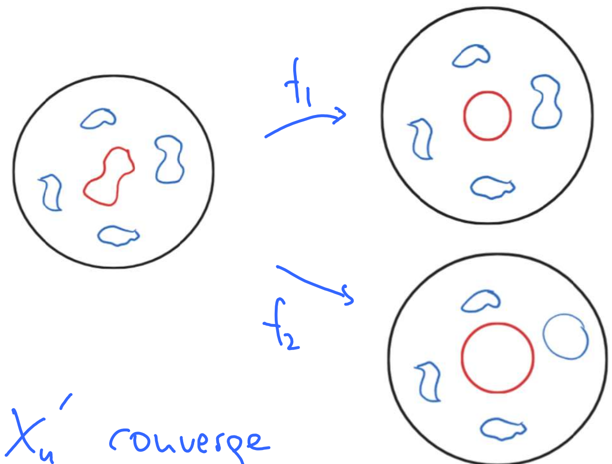

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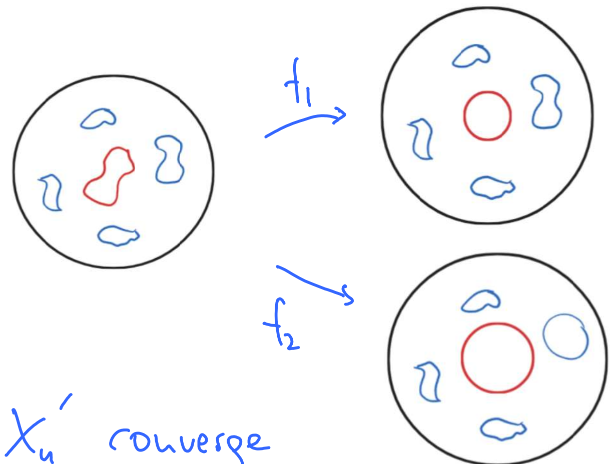

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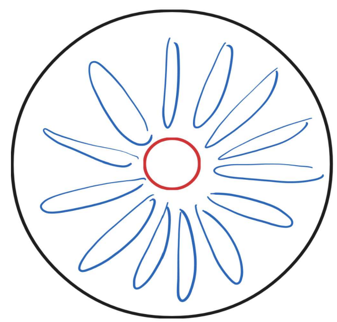
Thm (Aru, Bordereau, R, Werness): If $\frac{8}{3} < \kappa \underset{(-)}{\leq} 4$, CLE_κ has (GSS) and (LSS) a.s.

(GSS) independently by Doherty–Miller, also for $4 < \kappa < 8$

What could go wrong?



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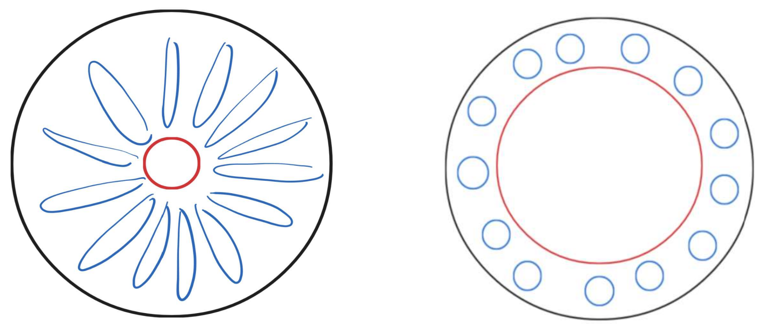

$\sum d_i a_i^2 l_i$ large

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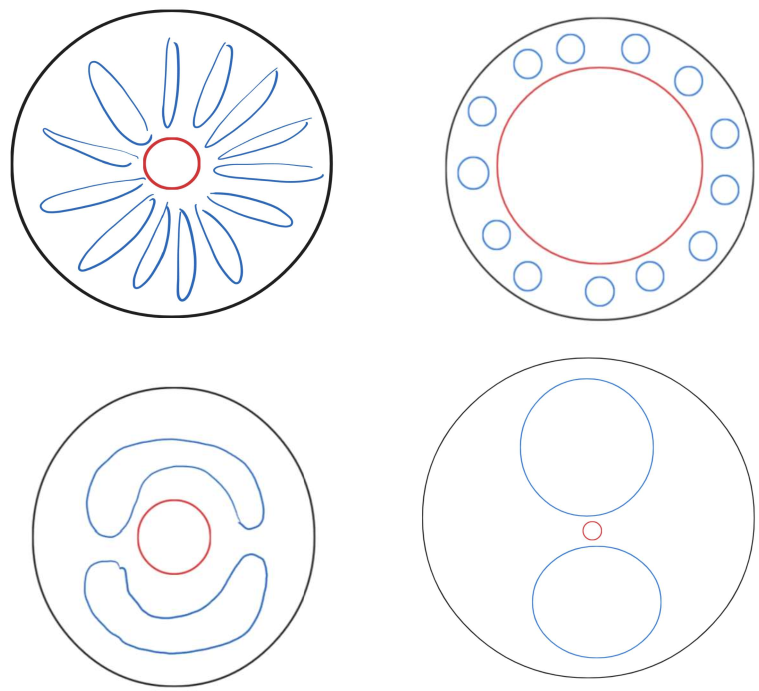

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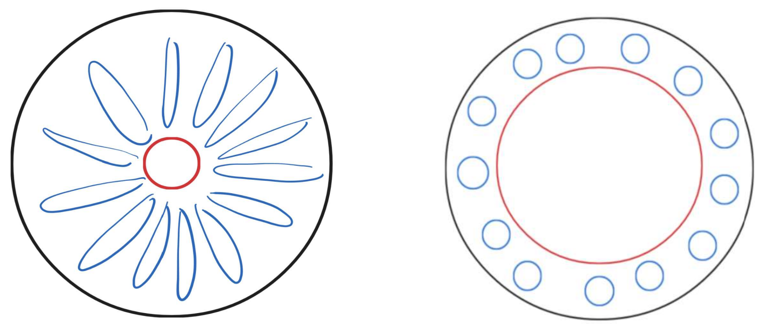

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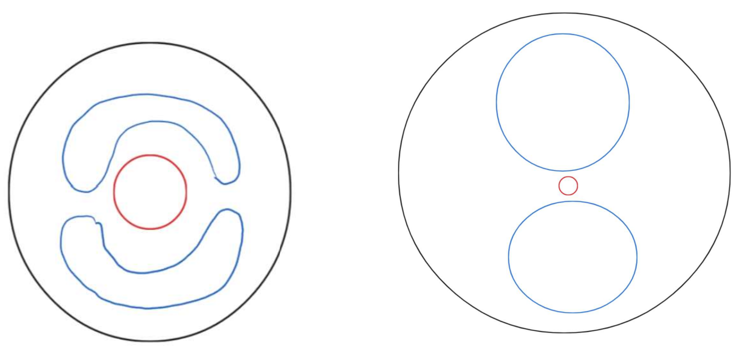
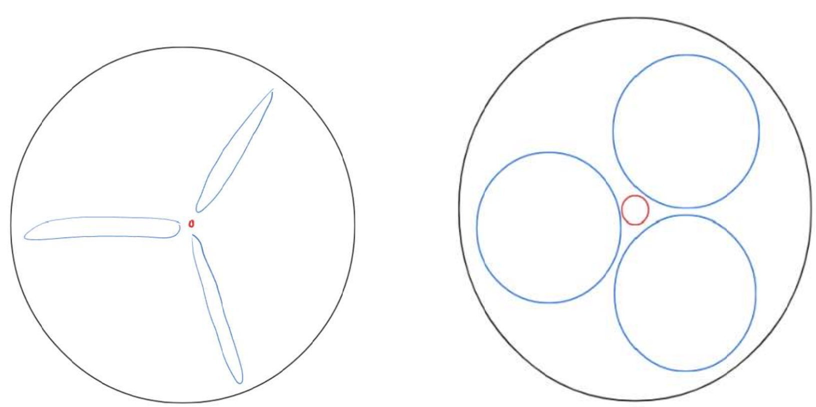


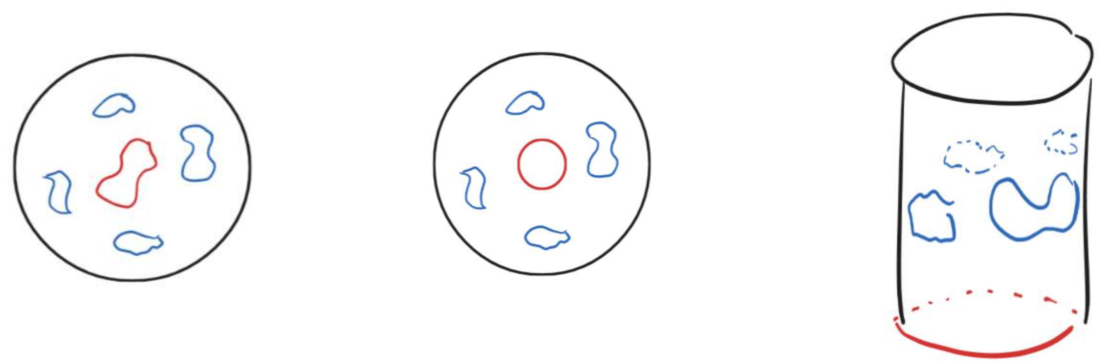
topological



"6-arm event"

Main tool: Oded Schramm's Transboundary modulus

want to measure dist between loops in conf. invariant way,
and estimate euclidean vs invariant distance.



$h = \log \frac{1}{r}$

$\frac{1}{h}$ minimizes "conformal area":

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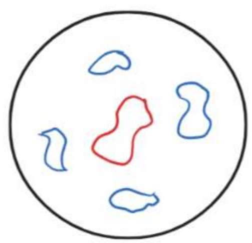
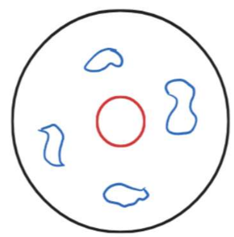
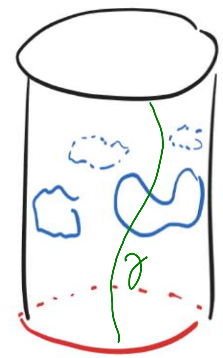






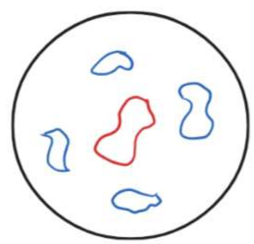
$h = \log \frac{1}{r}$

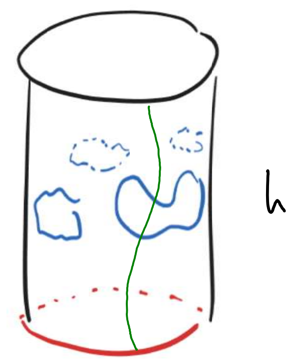
$\frac{1}{h}$ minimizes "conformal area":

Among all conformal metrics $\rho\, ds$ s.th. dist (top − bottom) ≥ 1 $\left(\int_\gamma \rho\, ds \geq 1\right)$:

$$\min_\rho \text{ area} = \min \int \rho^2\, dA = \frac{1}{h^2} 2\pi h = \frac{2\pi}{h}$$

Oded Schramm's Transboundary modulus

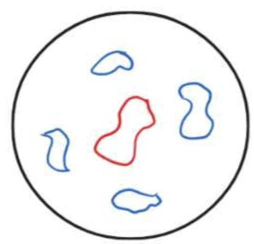


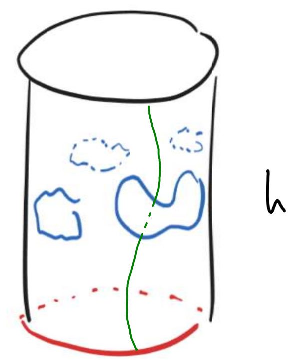


Without holes: $\min_{\rho} \int_{C} \rho^2 \, dA$

where $\forall \gamma$, $\text{length}_{\rho}(\gamma) \geq 1$

Oded Schramm's Transboundary modulus





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With holes: $\min \int_X \rho^2 \, dA + \sum_j \rho_j^2$,

$\forall \gamma$, $\text{length}_\rho(\gamma) := \int_{\gamma \cap \Omega} \rho \, ds + \sum_{\gamma \cap \ell_j \neq \emptyset} \rho_j \geq 1$

Def.: Transboundary modulus $M^{tb}(\Gamma) = \inf_{(\rho, \rho_j)} \int_X \rho^2 \, dA + \sum_j \rho_j^2$

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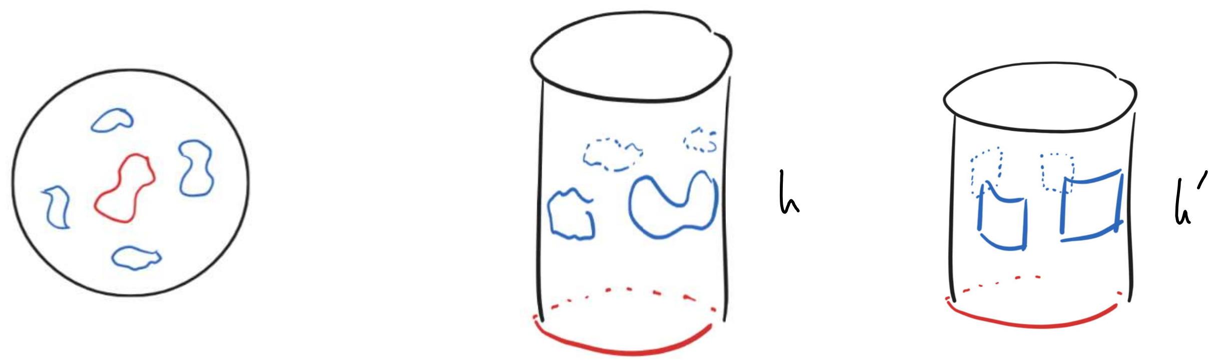

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Def.: Transboundary modulus $M^{tb}(\Gamma) = \inf_{(\rho, \rho_j)} \int_X \rho^2 \, dA + \sum_j \rho_j^2 = \frac{2\pi}{h'}$

Thm (Schramm; Bonk): The infimum is attained for squares.

Proof strategy for

Thm (Aru, Bordereau, R, Werness):

a) $\sum \operatorname{dia}^2(D_j) < \infty$ (GSS) $\Rightarrow$ X_n' converge

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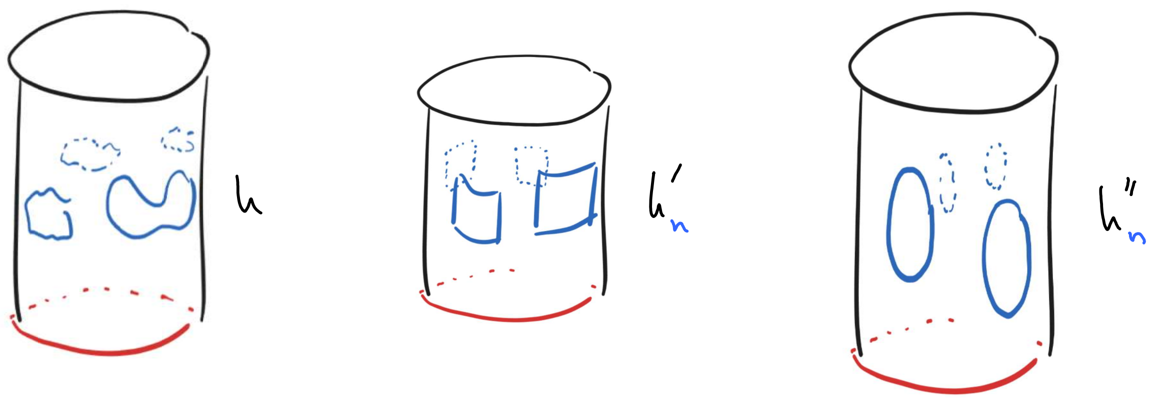
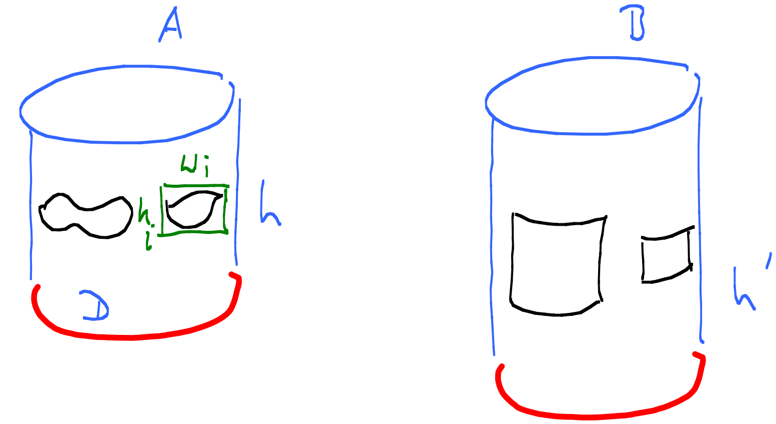
a)



Show $h \simeq h' \simeq h''$

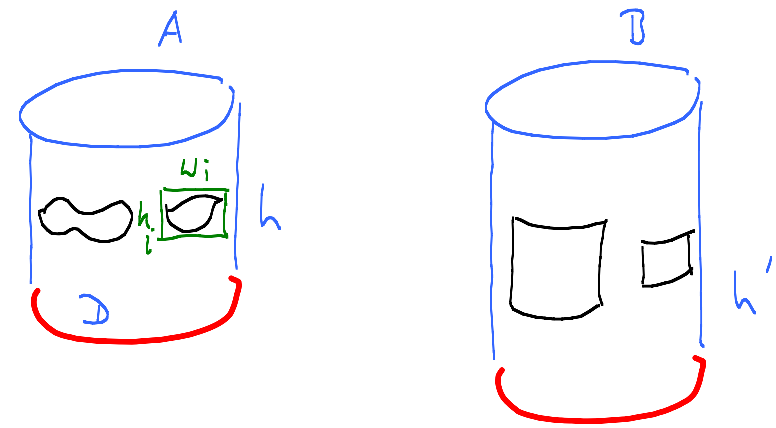
h vs h' :



<u>Lemma</u> :

$$\frac{\text{area } A}{\text{area } D + \sum h_i^2} \leq \frac{h'}{h} \leq \frac{\text{area } D + \sum w_i^2}{\text{area } A}$$

h vs h' :



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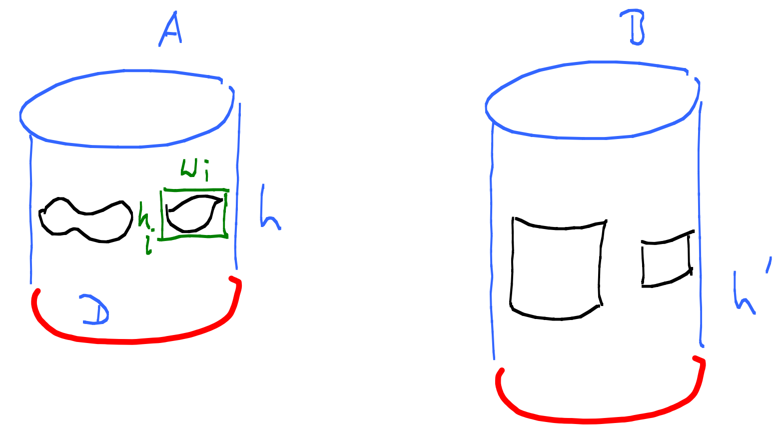
Corollary : $\frac{h'}{h}$ bdd if $\sum \text{diam}^2 \ell_j < \infty$

h vs h' :



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<u>Proof</u> : 1) Set $\rho \equiv \frac{1}{h}$ on D, $\rho_i = \frac{h_i}{h}$ on ℓ_i,

then $\ell(\gamma) \geq \frac{\Delta y}{h} \geq 1$

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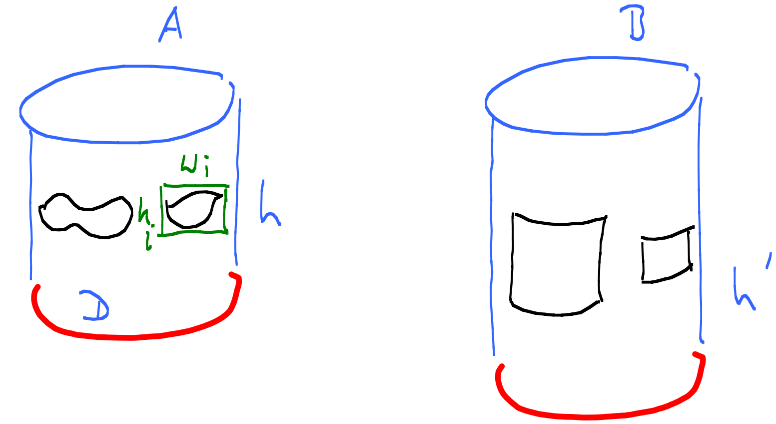

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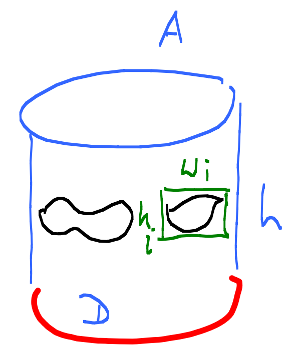
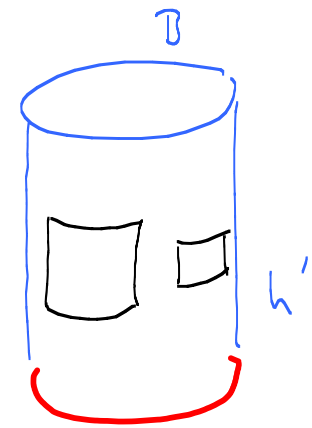
h vs h' :





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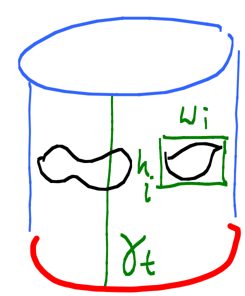

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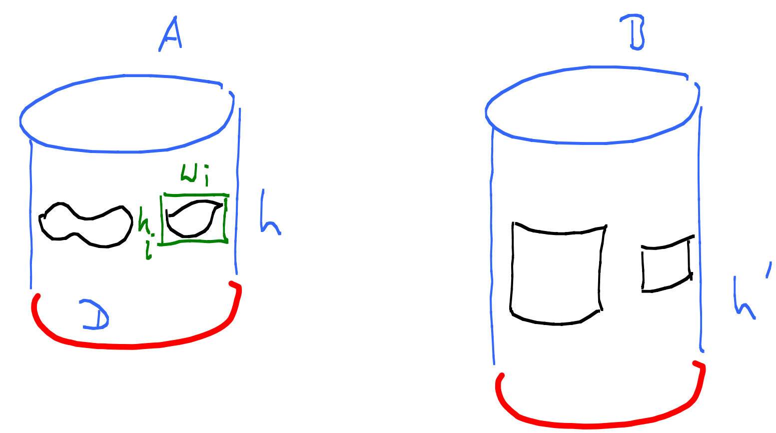

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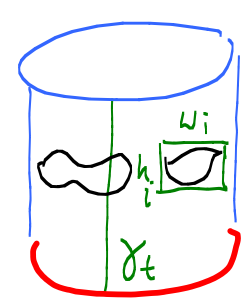

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$$\leq \left(\int \rho^2 dA + \sum \rho_i^2\right)^{1/2} \left(\text{area } D + \sum w_i^2\right)^{1/2}$$

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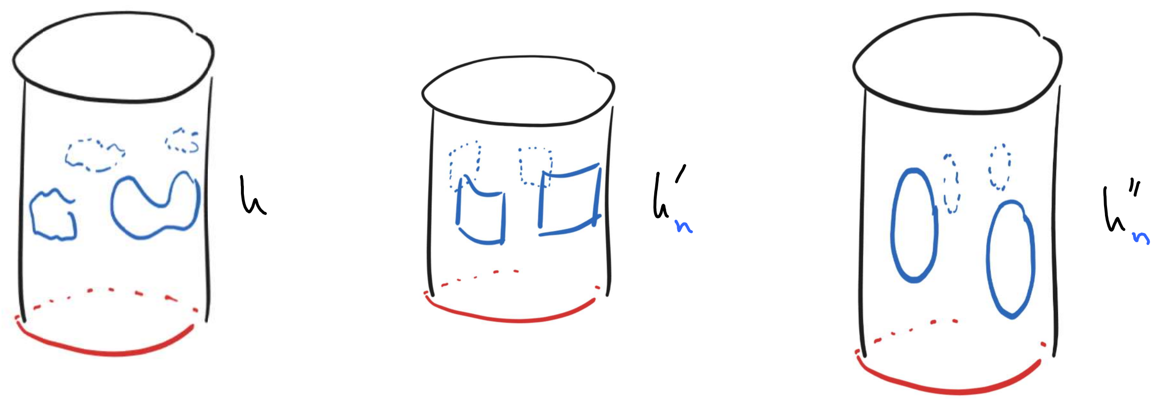
h' vs h'' :



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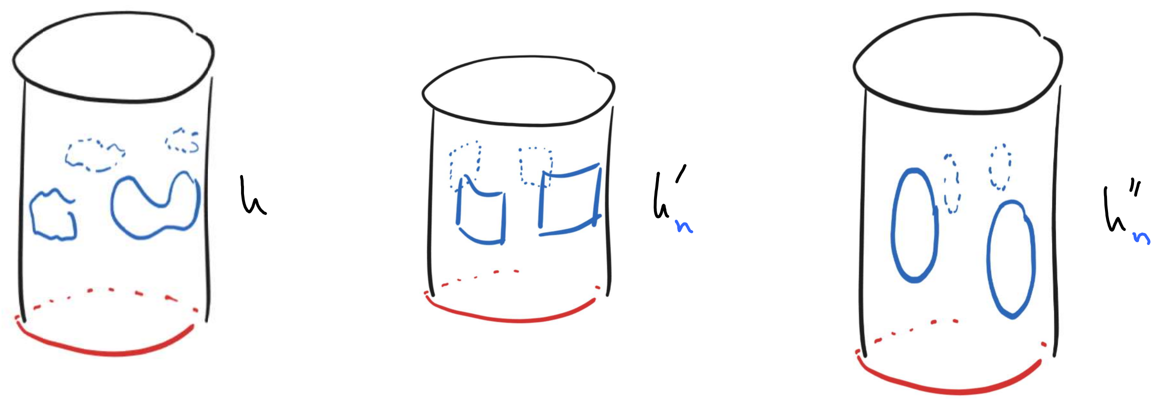
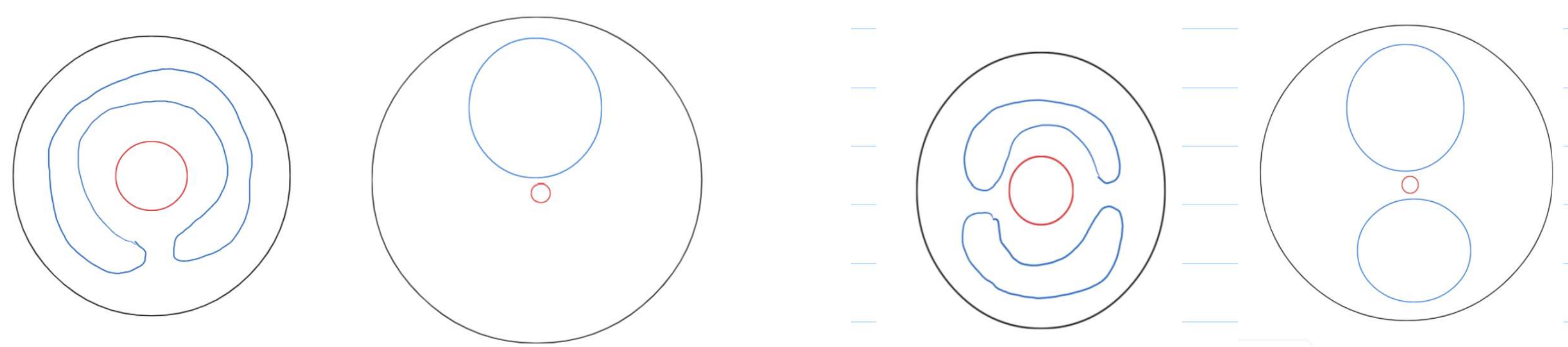
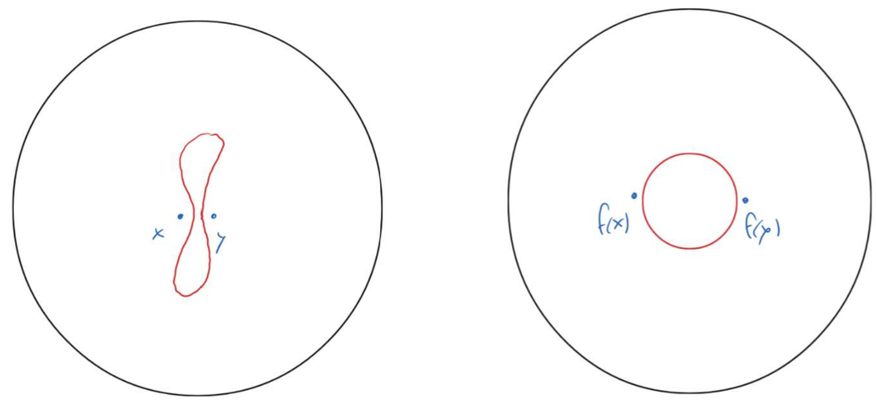


Lemma: $h'' \lesssim_C h'$ if $h''(B_j) \le C \ \forall j$

Lemma: $h''(B) \gg 1$ only in one of these situations:



b) (f_n) equicontinuous, f homeo :



Quantitatively : 4-arm event Wu, Gao - Nolin - Qian, $\xi_4(\kappa) = \frac{(12-\kappa)(\kappa+4)}{8\kappa} > 2 \quad (\kappa < 4)$

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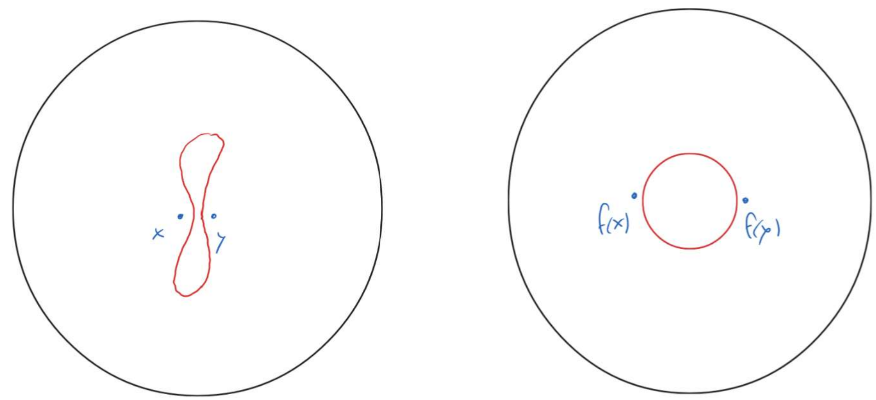

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Or : $X = \mathbb{D} \setminus \bigcup_j D_j$ locally connected , u.t.s. :

Claim : $K \subset X$ connected , small $\Rightarrow$ dia $f_n(K)$ small

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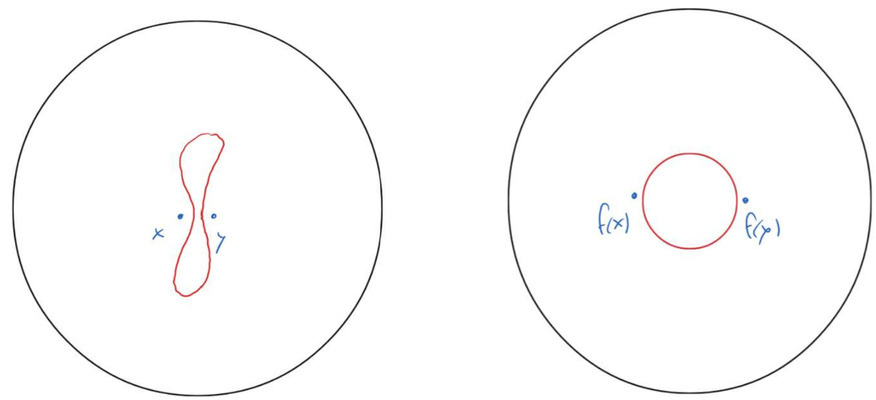

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Claim : $K \subset X$ connected , small $\Rightarrow$ dia $f_n(K)$ small

$\Gamma = \{\gamma \text{ around } K\}$: $M^{tb}(\Gamma) \overset{\text{if no loops cross}}{\geq} \sum_{j \in (\text{LSS})} M^{tb}(\Gamma \cap A(x, r_j, 2r_j)) \to \infty$ as $\text{dia}(K) \to 0$

$\Longrightarrow M^{tb}(f_n \Gamma) \to \infty \Longrightarrow \text{dia}(f_n K) \to 0$.

Injectivity of limit $f_n \to f \iff f^{-1}(y)$ is a point $\forall y \in Y$.

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$\Gamma = \{\gamma \text{ around } K\}$: morally, $C \geq M^{tb}(\Gamma) = M^{tb}(f_n\Gamma) \to \infty$ $\lightning$

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Finally, showing that $\sum_j \operatorname{dia}^2 \ell_j < \infty$ a.s. (ℓ_j CLE_κ in $\mathbb{D}$)

amounts to estimating $E \sum_j \operatorname{dia}^2 \ell_j = \int_{\mathbb{D}} E\left[\frac{\operatorname{dia}^2 \ell(z)}{\operatorname{area} \ell(z)} \right] d^2 z$

via Markovian exploration of $\ell(z)$.

Happy birthday, Greg!