INDEPENDENT COVERS

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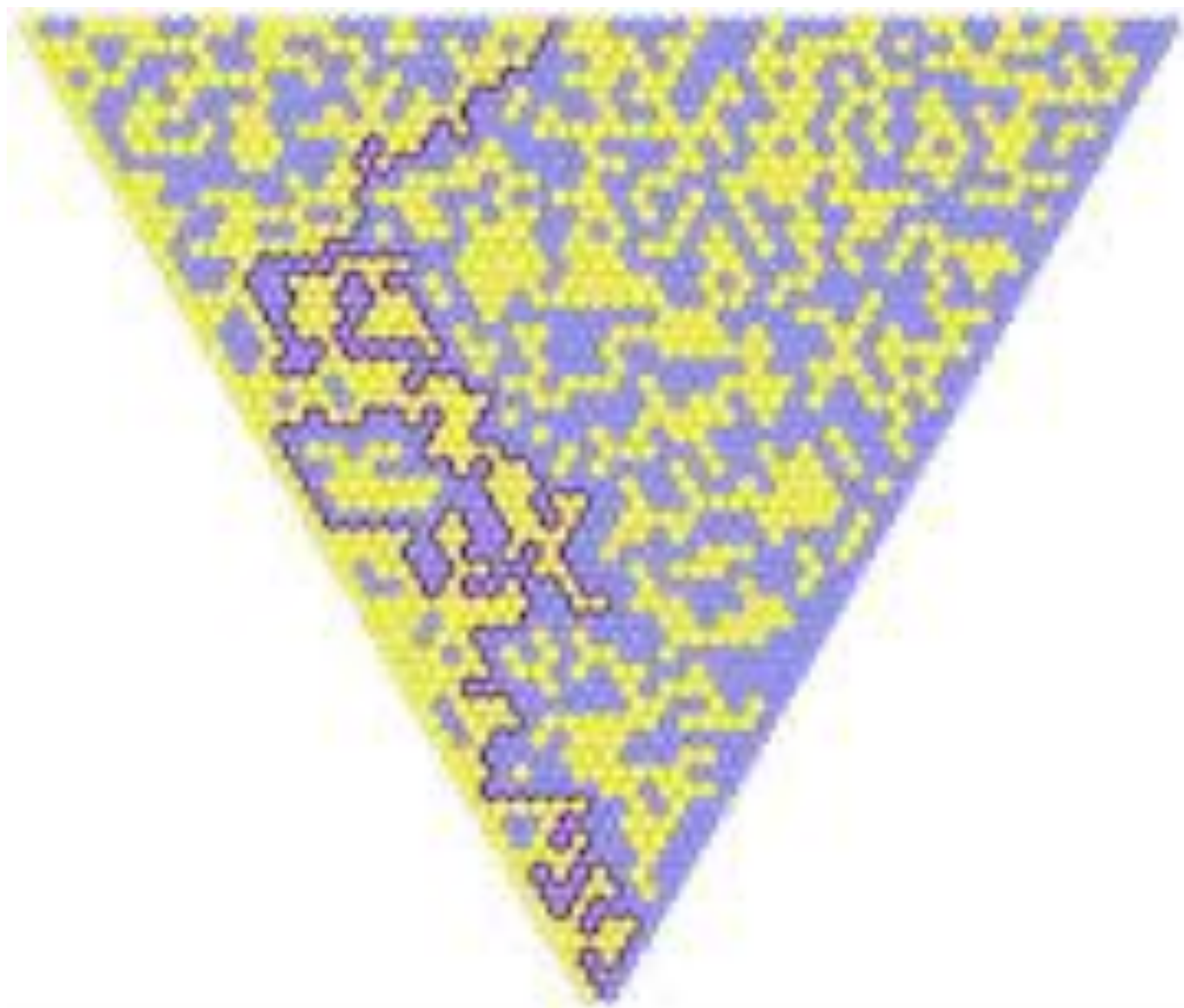
WEEDS OF THE NORT

Scaling limits of 2d lattice models

1. dimer model
2. spanning trees
3. ising model

and, to a lesser extent:

4. percolation
5. 5-vertex model



In this talk we define and study a rich family of determinantal processes in $\mathbb{Z}^d$, constructing scaling limits.

tools:
* linear algebra
* Fourier analysis

Setting: a graph (or set) G and collection $\mathcal{T}$ of subsets ("tiles").

or multisets

Let D be the $|V| \times |\mathcal{T}|$ incidence matrix:

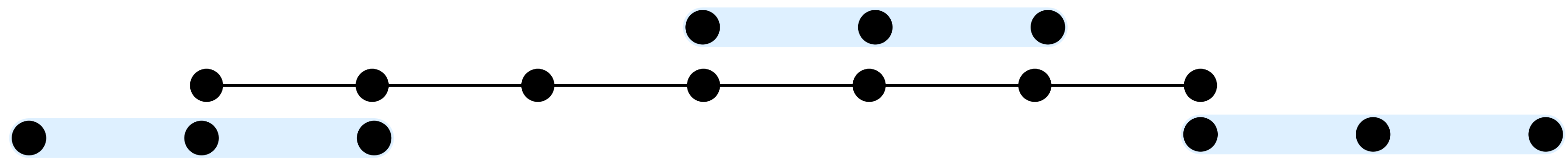
$$D_{v,t} = \begin{cases} 1 & v \in t \\ 0 & \text{else.} \end{cases}$$

An **independent covering (IC)** is a collection $m \subset \mathcal{T}$ of tiles such that

1. $|m| = |G|$
2. It covers all of G: $\bigcup_{t \in m} t = G$
3. It is independent: $\det D_m \neq 0$.

submatrix with columns m

Example: $G = \{1, 2, \ldots, n\}$



$\mathcal{T}$ consists in all translates of $\{1, 2, 3\}$ that intersect G. $\mathcal{T} = \{0, 1, \ldots, n+1\}$

$$D = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

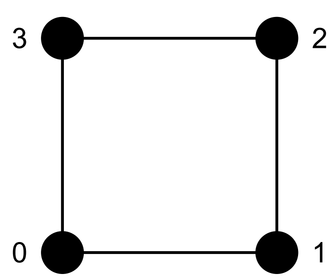
Question: Which maximal minors have full rank? (See below.)

Example: spanning trees

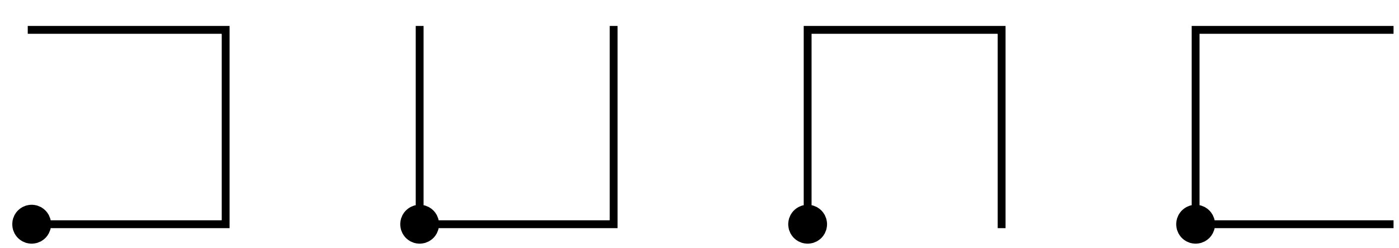
$G = (V, E)$ is a bipartite graph; $v_0 \in G$ a root vertex.

$\mathcal{T} = E \cup \{v_0\}$.

Prop. (Poincaré Lemma) An independent cover is a set of $|V| - 1$ edges and v_0 which make a spanning tree (rooted at v_0).
Conversely, any spanning tree is an IC.



$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 01 & 12 & 23 & 30 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

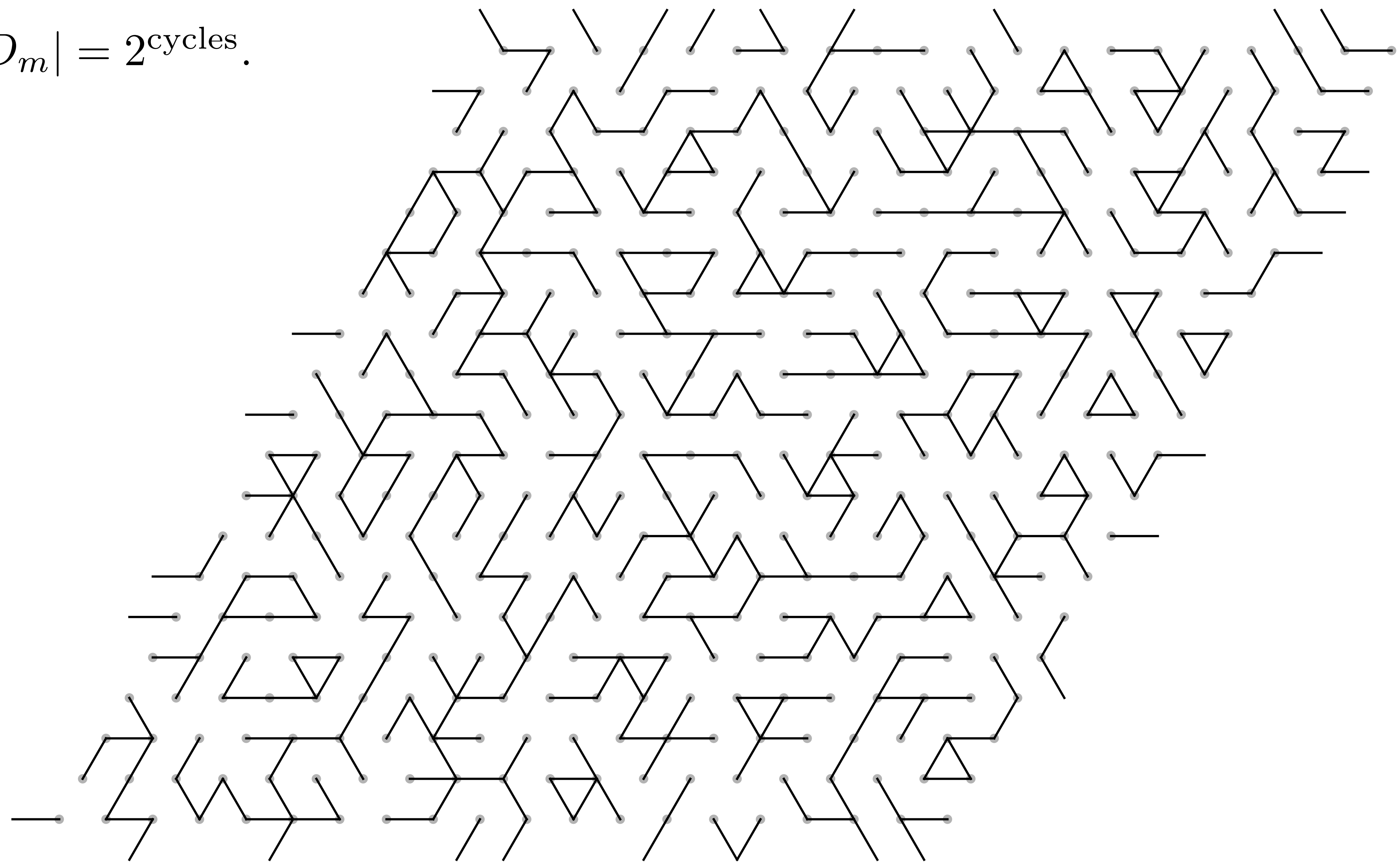


CRSF example
(cycle-rooted spanning forest)

Take G non-bipartite, and $\mathcal{T} = E$.

Thm: An independent cover is a cycle-rooted spanning forest with odd-length cycles.

Lemma: $|\det D_m| = 2^{\text{cycles}}$.



There is a natural probability measure on ICs:

$$Pr(m) = \frac{\det D_m D_m^*}{\det DD^*}.$$

This is a probability measure by the **Binet-Cauchy identity:** For an $n \times m$ matrix A and $m \times n$ matrix B,

$$\det AB = \sum_{S \in \binom{[n]}{m}} \det A_S B^S.$$

Thm: This measure on ICs is determinantal, with kernel $K = D^*(DD^*)^{-1}D$.

This generalizes a theorem of Burton and Pemantle (1990) for the spanning tree case. See also Benjamini/Lyons/Peres/Schramm (2001).

What is a determinantal probability measure?
("free fermionic" measure)

Def: A measure μ on $\{0,1\}^n$ is *determinantal* if there is an $n \times n$ matrix K, the *kernel*, such that

$$\forall S \subset [n], \qquad \Pr(x_i = 1 \ \forall i \in S) = \det(K_S^S).$$

Example:[Burton-Pemantle 1993] Edges in a uniform spanning tree.

Example: [K 1997] Edges in the dimer model for a planar bipartite graph.

(Continuous) Example: [Peres-Virag 2005] Zeros of the Gaussian analytic function.

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_k \sim N_{\mathbb{C}}(0,1)$$

+ other examples in random matrices: GUE, CUE, Ginibre, ...

Linear algebra fact:

Thm: Let W be a k-dimensional subspace of $\mathbb{R}^n$ and $w_1, \ldots, w_k$ a basis for W. Let D be the $k \times n$ matrix with rows w_i. Then $D^*(DD^*)^{-1}D$ is the matrix of orthogonal projection to W.

Cor: The kernel of the IC determinantal measure is the matrix of the orthogonal projection to $ker(D)^{\perp}$.

Def: A **comic** is the **com**plement of an **IC**: the set of tiles not used in an IC.

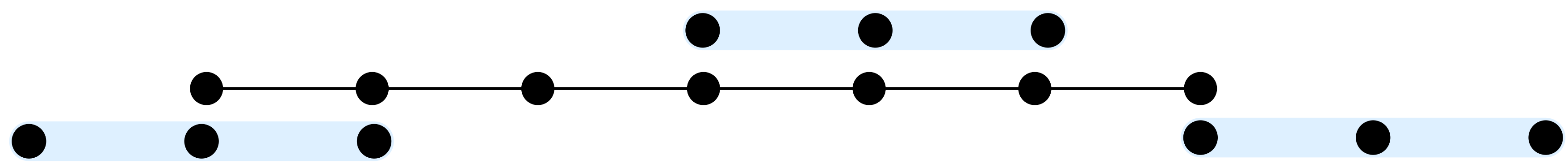
Comics have a natural determinantal measure with kernel $I - K$ (where K is the kernel for the IC). This operator $I - K$ is orthogonal projection to $\ker D$.

The size of a comic is $|\mathcal{T}| - |G| = \dim \ker D$.

Thm: Let B be a matrix whose rows form a basis for $ker D$. Then for a comic S,

$$Pr(S) = \frac{\det(B_S B_S^*)}{\det BB^*}.$$

Example: $G = \{1, 2, \ldots, n\}$



$\mathcal{T}$ consists in all translates of $\{1, 2, 3\}$ that intersect G.

$$D = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$\ker D$ is 2-dimensional,

$$B = \begin{pmatrix} 1 & \omega & \omega^2 & 1 & \ldots & \omega^{n+1} \\ 1 & \omega^{-1} & \omega^{-2} & 1 & \ldots & \omega^{-n-1} \end{pmatrix}. \qquad \omega = e^{2\pi i/3}$$

The comic is supported on two points...

$$\begin{aligned} Pr(x_1, x_2) &= \frac{1}{Z} |\det B_{x_1,x_2}|^2 \\ &= \frac{1}{Z} \left| \begin{pmatrix} \omega^{x_1} & \omega^{x_2} \\ \omega^{-x_1} & \omega^{-x_2} \end{pmatrix} \right|^2 \\ &= \frac{1}{Z} \sin^2(\frac{2\pi(x_1 - x_2)}{3}) \\ &= \begin{cases} 0 & x_1 \equiv x_2 \bmod 3 \\ c & \text{else.} \end{cases} \end{aligned}$$

$$D = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Question: Which maximal minors have full rank?

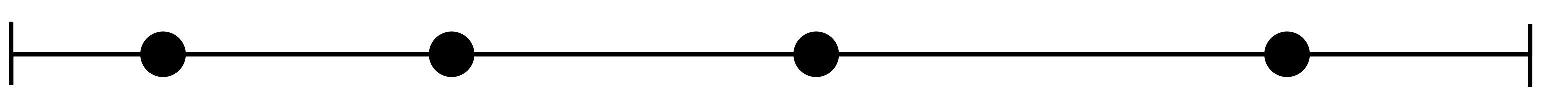
Answer: Remove two columns which don't differ by 0 mod 3.

A small generalization of ICs allows us to add multiplicities to tiles: $t_v > 1$.

Example 2. $G = [n]$, $\mathcal{T}$ consists of all translates of $(1\ 4\ 6\ 4\ 1)$. Then the comic is supported on 4 points, and

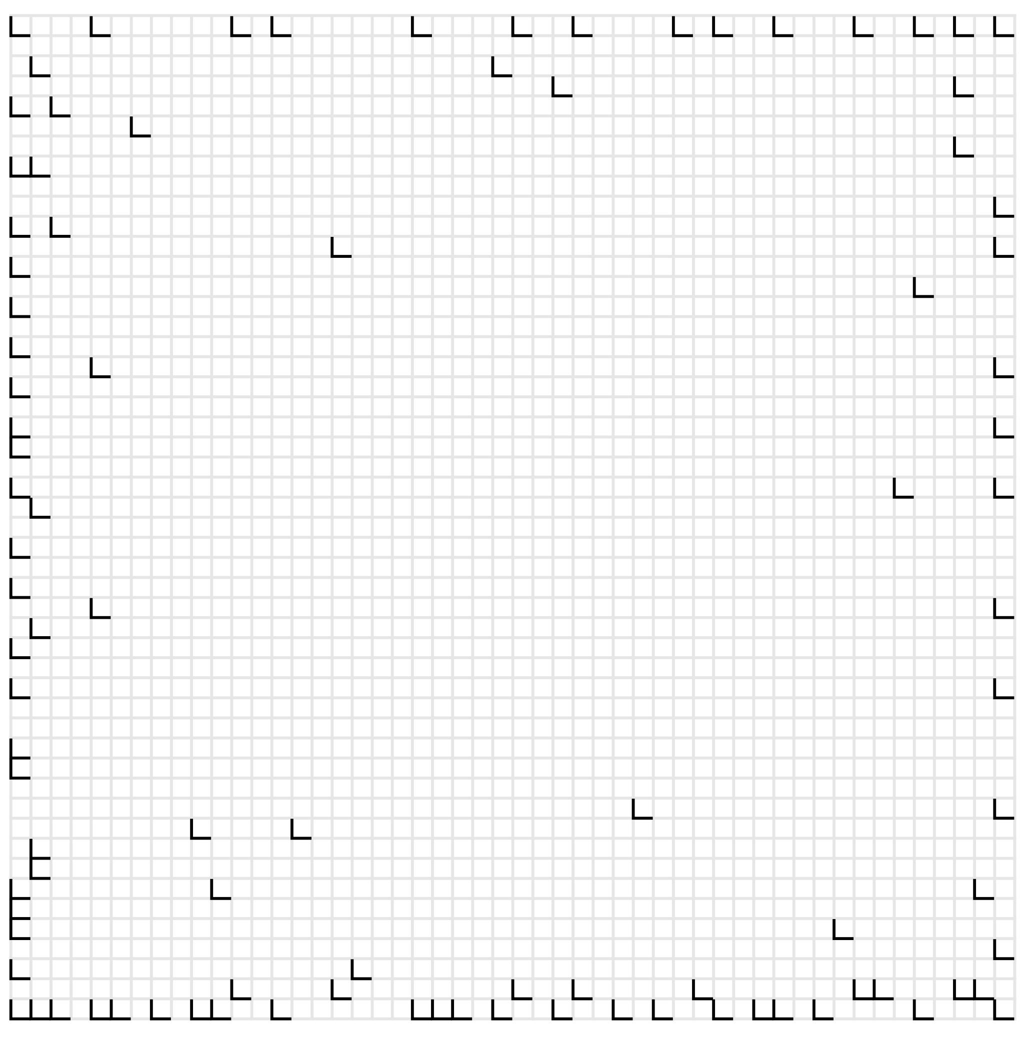
$$D = \begin{pmatrix} 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & 1 & -1 & \\ 1 & -2 & 3 & -4 & \dots \\ 1 & -4 & 9 & -16 & \\ 1 & -8 & 27 & -64 & \end{pmatrix}$$

$$Pr(x_1, x_2, x_3, x_4) \propto \prod_{1 \le i < j \le 4} (x_i - x_j)^2.$$

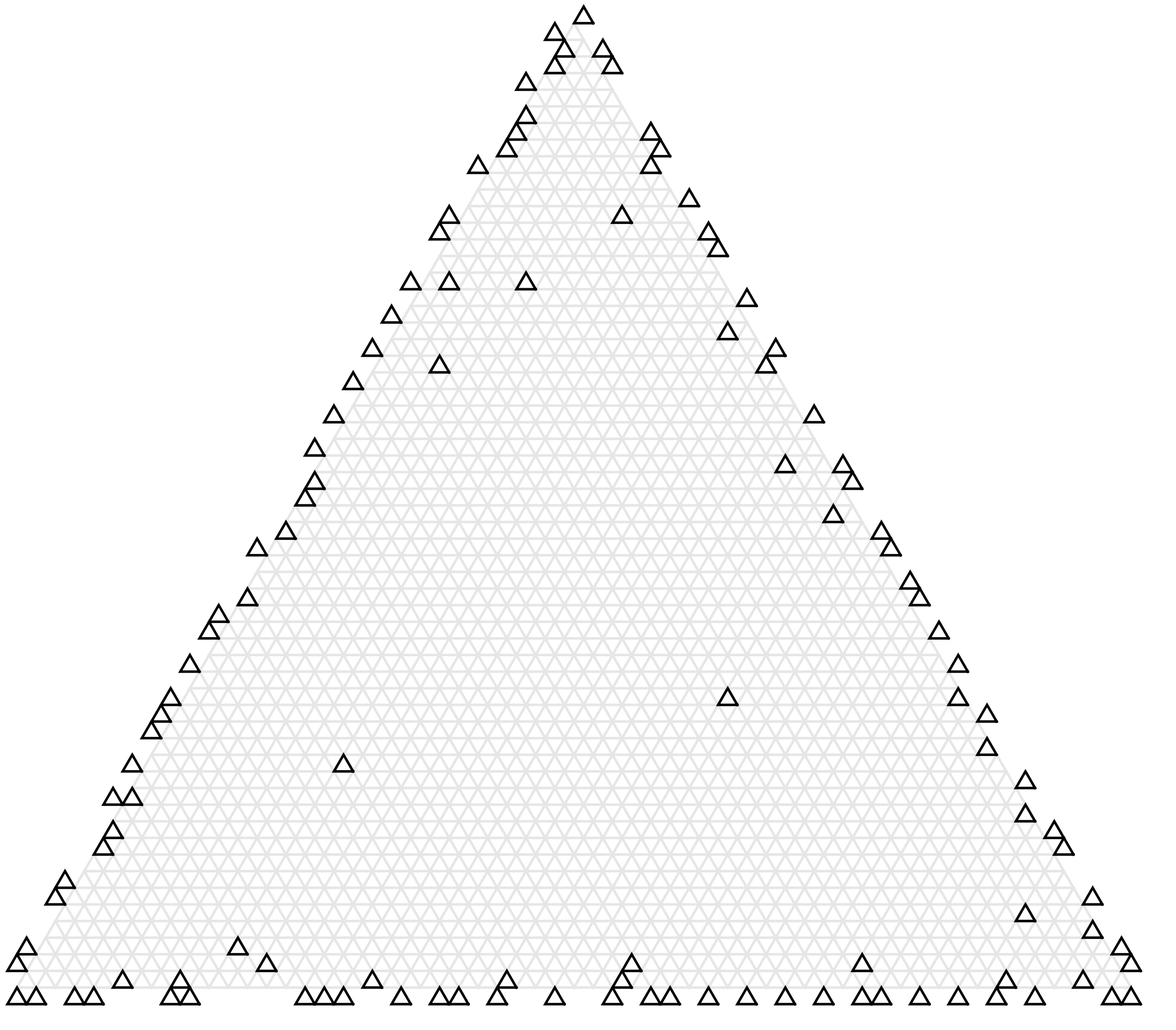


Because of their determinantal nature, we can *exactly sample* large systems.

Example 3: $G \subset \mathbb{Z}^2$ and $\mathcal{T}$ consists of translates of the L tile $\{(0,0),(1,0),(0,1)\}$.

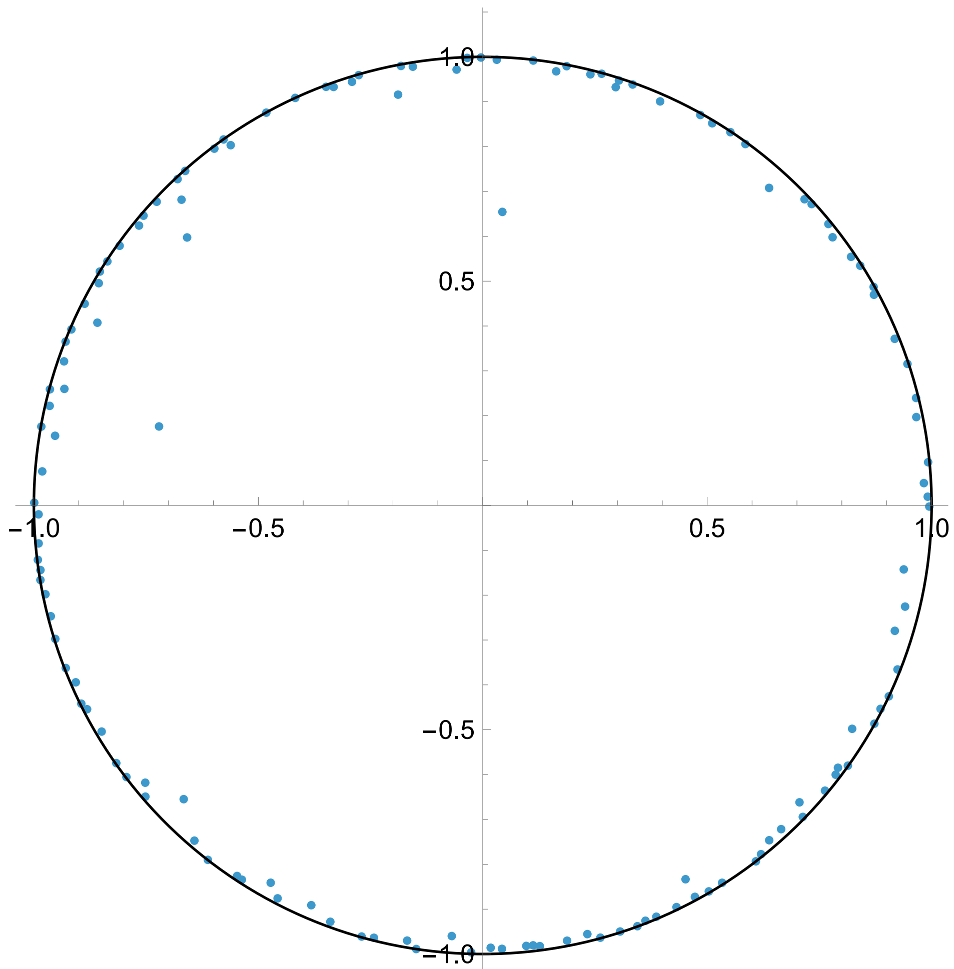


A comic on the $n \times n$ square has $2n$ points.



tilted square lattice

Zeroes of a Gaussian analytic function

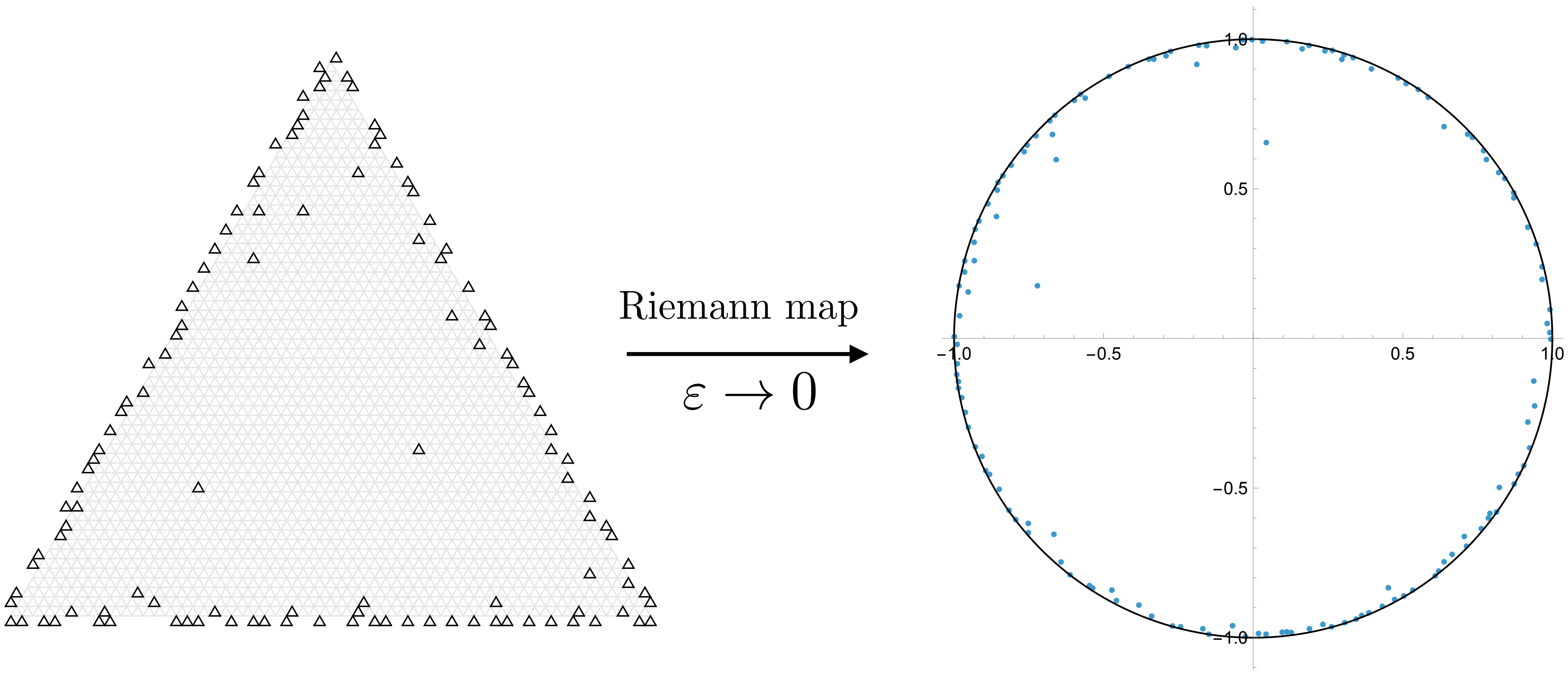


zeros of $f(z) = \sum_{j=0}^{\infty} a_j z^j$ where $a_j = N_{\mathbb{C}}(0,1)$

Thm [Peres-Virag] The zeros of the GAF are a determinantal process on the unit disk with kernel given by the Bergman kernel $B(z,w) = \frac{1}{\pi(1-z\bar{w})^2}$.

The L polyomino comic scaling limit

Thm: Given a domain $U \subset \mathbb{C}$, and let $G_\epsilon = U \cap \epsilon\mathbb{Z}^2$. In the limit $\epsilon \to 0$ the point process for the comic for the L tile on G_ϵ converges in law (after linear map) to the zeros of the Gaussian analytic function on U (determinantal point process defined by the Bergman kernel on U).



Cor: Comics for the L tile are conformally invariant.

Proof ideas: For ε small, a function in $\ker D$ and in $L^2(U)$ is of the form

$$f(x,y) = \omega^{(x-y)/\varepsilon} F(x,y) + \omega^{(y-x)/\varepsilon} G(x,y) + O(\varepsilon^2)$$

where $\omega = e^{2\pi i/3}$ and $F, \bar{G}$ are analytic functions of $z = x - \omega^2 y$.

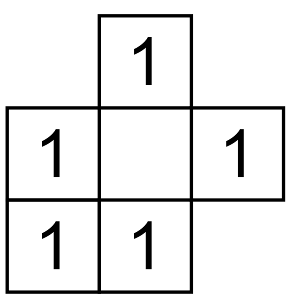
$$\begin{array}{cccccc}
1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\
\omega & \omega^2 & 1 & \omega & \omega^2 & 1 \\
\omega^2 & 1 & \omega & \omega^2 & 1 & \omega \\
1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\
\omega & \omega^2 & 1 & \omega & \omega^2 & 1 \\
\omega^2 & 1 & \omega & \omega^2 & 1 & \omega
\end{array}$$

After conjugation by the oscillating prefactor, and the linear change of coordinates $(x,y) \mapsto x - \omega^2 y$, the projection D is a discrete version of the *Bergman kernel*, the projection operator from $L^2(U)$ to $L^{2,h}(U)$. $\square$

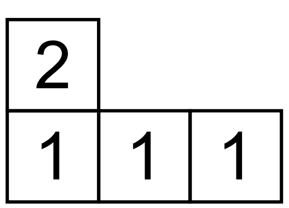
What property of the L tile did we use?

The L tile has Fourier transform $1 + z + w$ which has two simple zeros $(z, w) = (\omega, \omega^2), (\omega^2, \omega)$ on the unit torus.

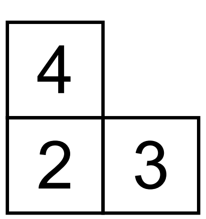
Similar results hold for any comic when $\mathcal{T}$ consists of translates of a single tile whose Fourier transform $p(z, w)$ has only a single pair of complex conjugate simple zeros on $S^1 \times S^1$.



$$p(z, w) = 1 + z + w + z^2w + zw^2$$

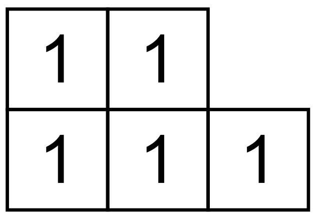


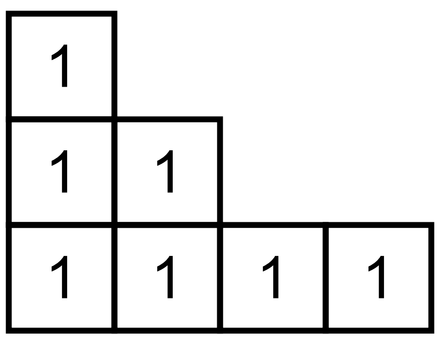
$$p(z, w) = 1 + z + z^2 + 2w$$



$$p(z, w) = 2 + 3z + 4w$$

When tile polynomial has multiple simple root pairs, get a linear superposition of (stretched, oscillated) Bergman kernels.

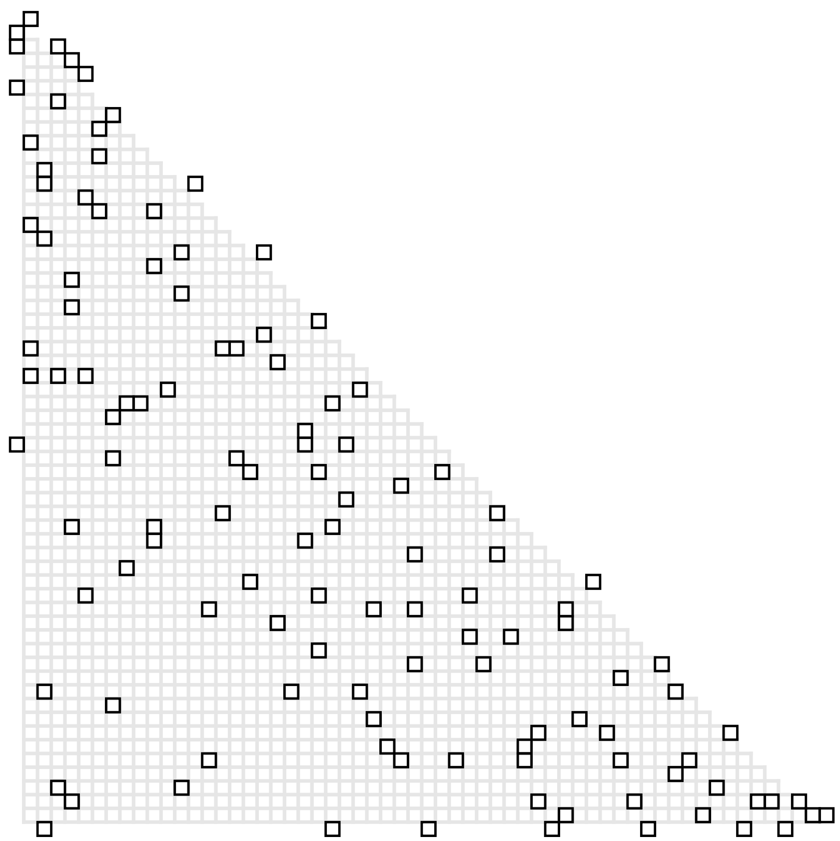




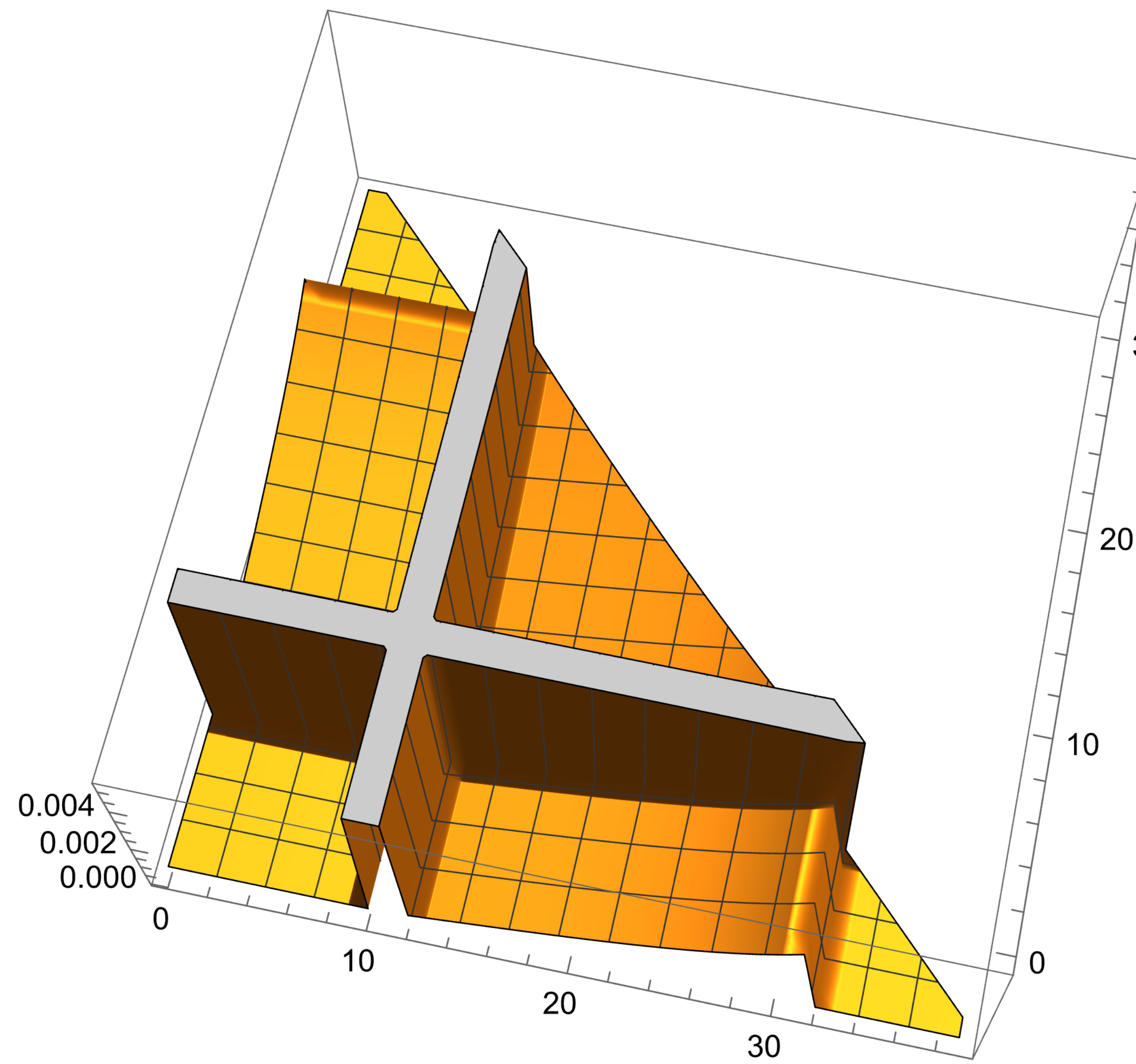
Sometimes tile polynomial can intersect the unit torus in a whole curve, leading to more weird behavior.

1	1
1	1

$$p(z,w) = (1+z)(1+w)$$



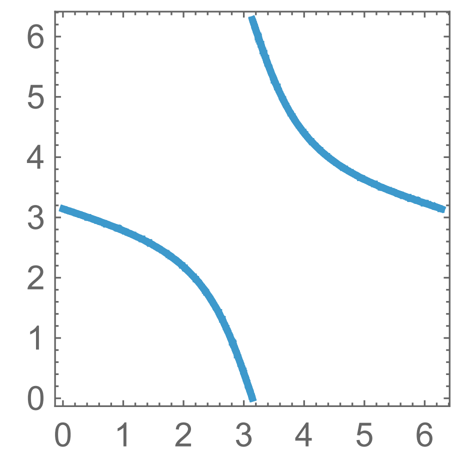
comic sample



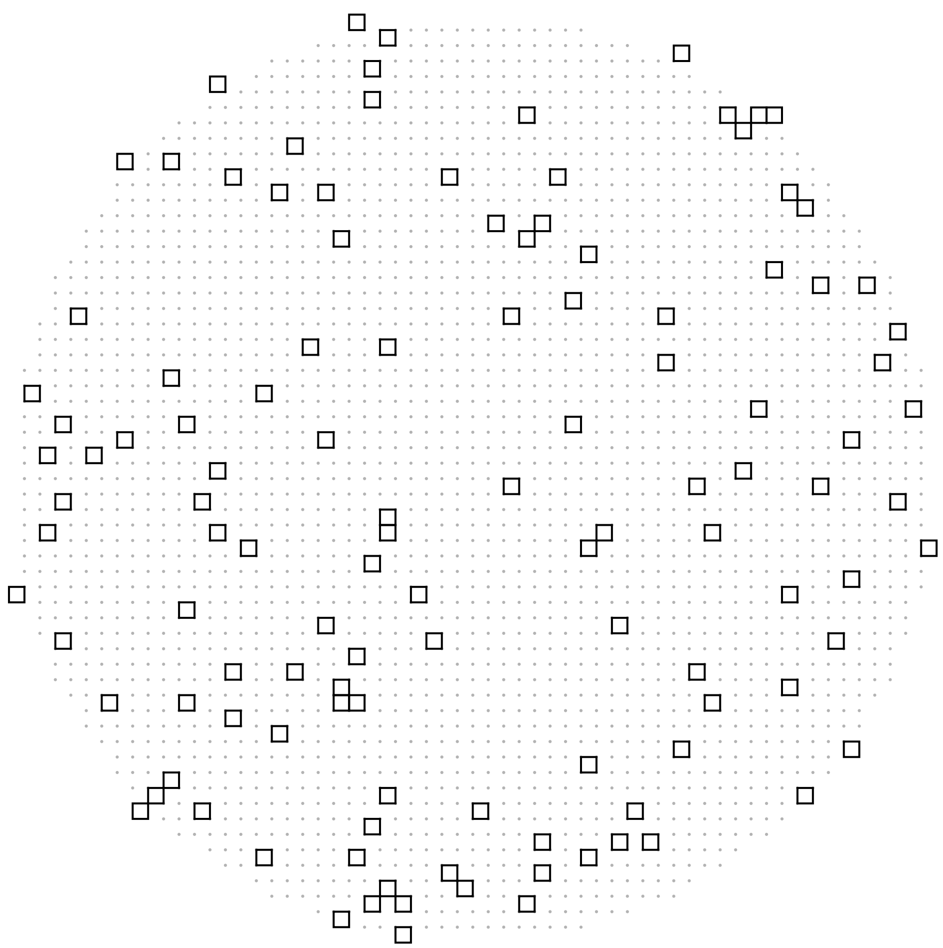
covariance function

1	2
2	1

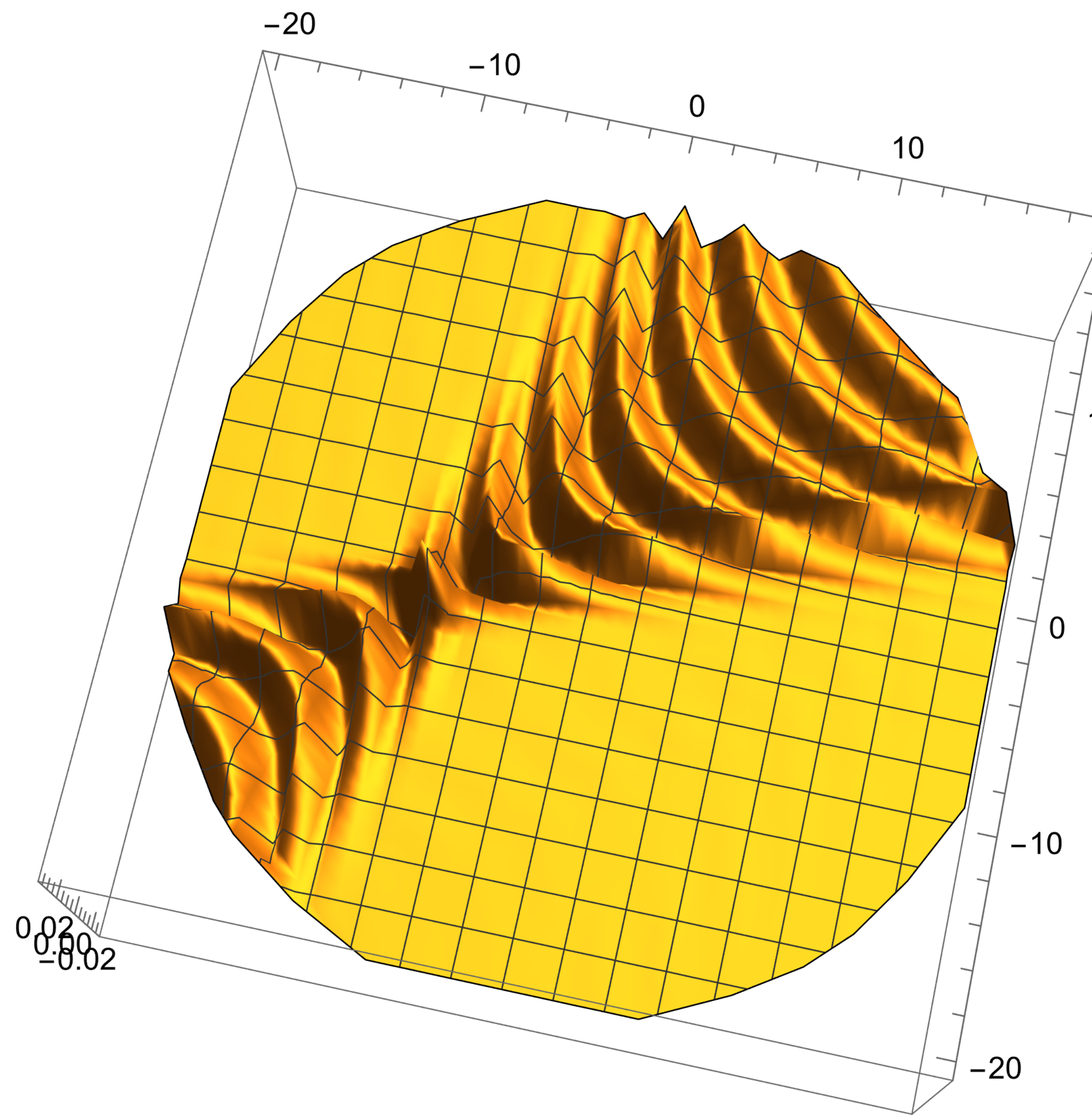
$$p(z,w) = 2 + z + w + 2zw$$



zero set of p in torus

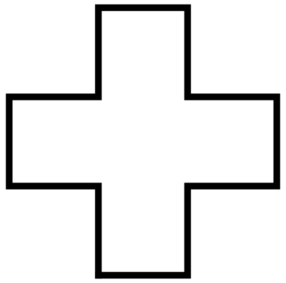


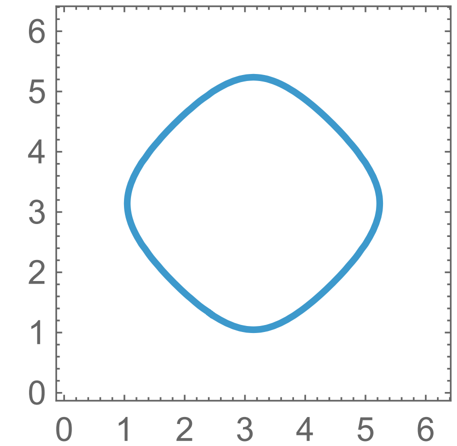
comic sample



covariance function

“Plus” polyomino





zero set of p
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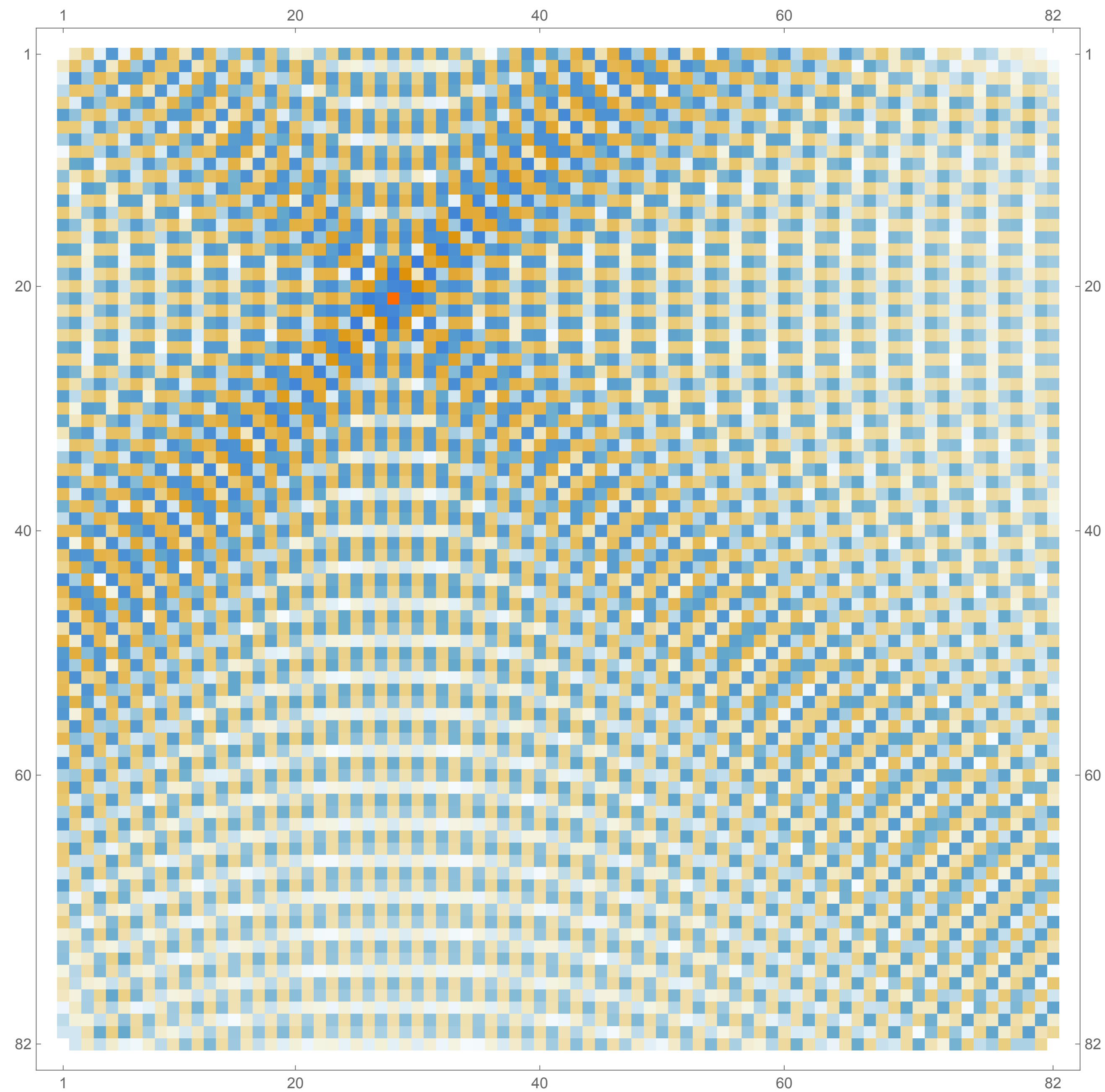


covariance function

What about 3d?

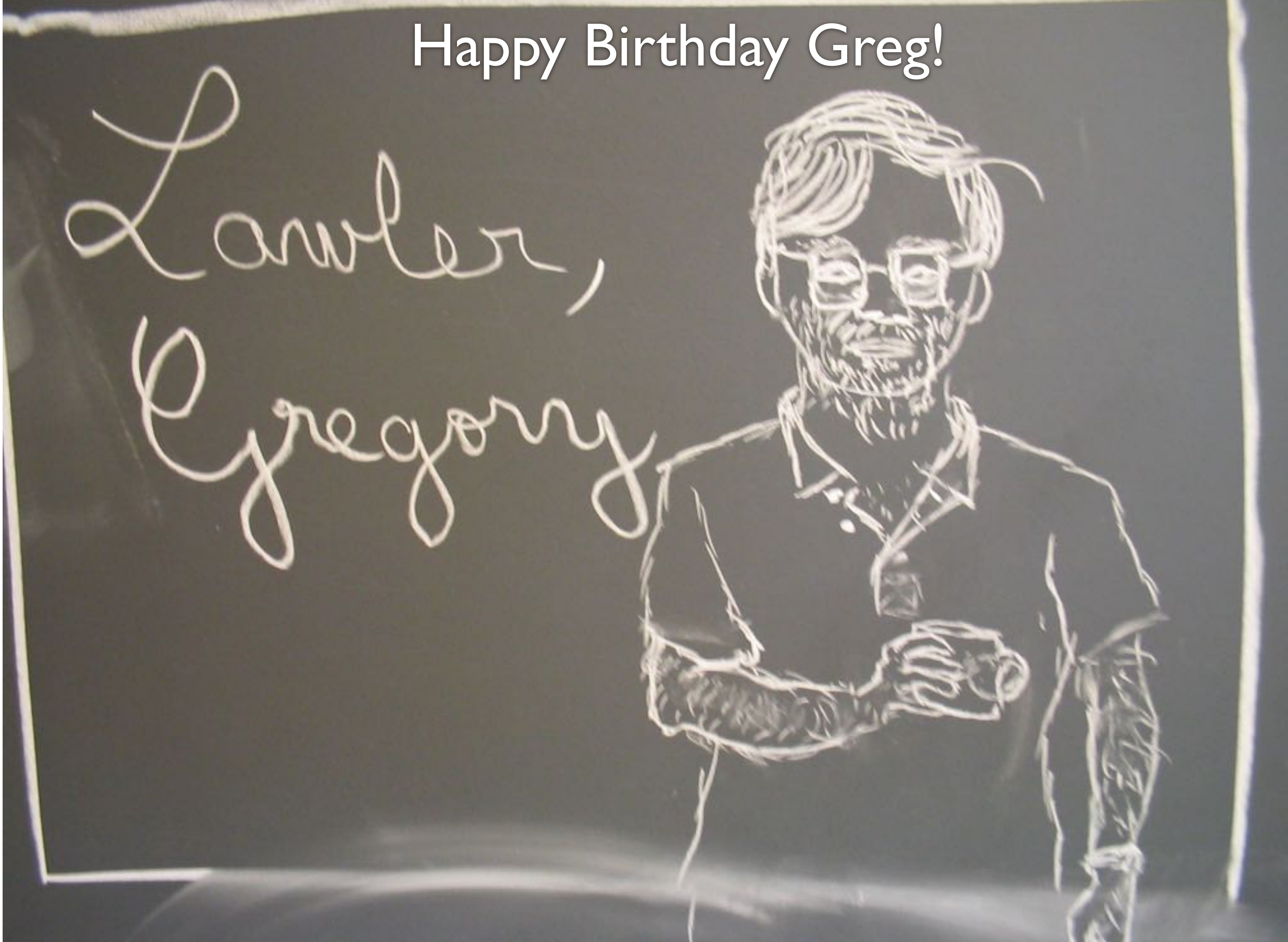
Typically, $p(z, w, u) = 0$ intersects the unit torus $\{|z| = |w| = |u| = 1\}$ in a 1d curve. We don't typically know how to describe the scaling limit.

(Needs Fourier analysis...)

With two tile types, can get isolated points, e.g. $\langle 1 + z + w, 1 + u \rangle$. Then get (stretched) rotational invariance. But not conformal invariance(?)

With $\mathcal{T} = E \cup \{v_0\}$, get exactly the 3D spanning tree as the IC.

Happy Birthday Greg!



THANK YOU

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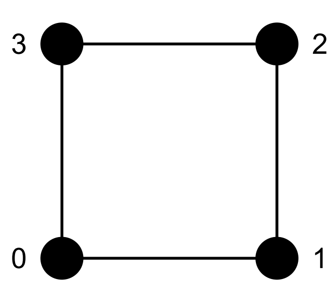

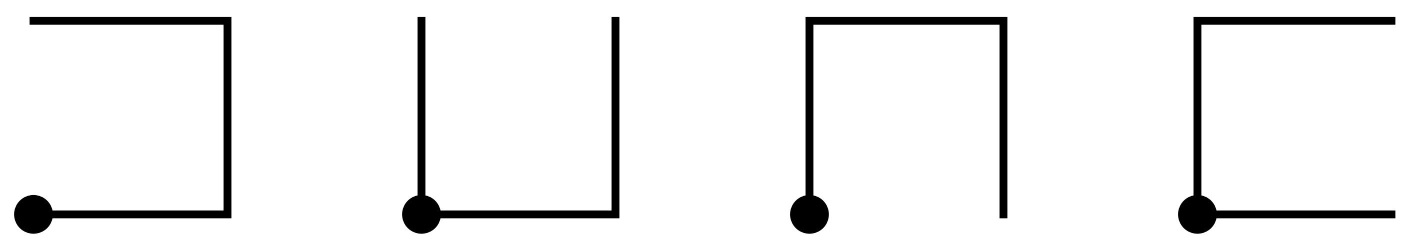

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