Scaling limits for the geometry of eigenvectors of random matrices

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Disordered Quantum Systems

Quantum System: Configuration space $\boldsymbol{\Sigma}$ $\quad (\mathbb{R}^d, \mathbb{Z}^d, [\![\mathbf{1}, \boldsymbol{N}]\!])$

State space $\boldsymbol{\ell^2(\Sigma)}$ $\quad (\mathbb{R}^{\boldsymbol{N}})$

Wave function $\boldsymbol{\psi \in \ell^2(\Sigma)}$ with $\|\psi\|_2 = 1$ $\quad (\boldsymbol{\psi \in \mathbb{S}^{N-1}})$

Hamiltonian operator $\boldsymbol{H = H^* : \ell^2(\Sigma) \to \ell^2(\Sigma)}$

$\boldsymbol{|\psi(x)|^2}$: Probability to find the quantum particle at configuration x

Disordered Quantum System: H is a **random** Hamiltonian.

$\Sigma = [\![\mathbf{1}, \boldsymbol{N}]\!]$, H is a $\boldsymbol{N \times N}$ **random Hermitian matrix**

$H_{xx'}$ describes the **quantum transition amplitudes** from configuration x to x'

$\lambda_1 \leqslant \cdots \leqslant \lambda_N$ eigenvalues of H (**energy levels**)

$U = (\mathbf{u}_1, \ldots, \mathbf{u}_N)$, $\mathbf{u}_k \in \mathbb{S}^{N-1}$ corresponding eigenvectors (**stationary states**).

$$H\mathbf{u}_k = \lambda_k \mathbf{u}_k.$$

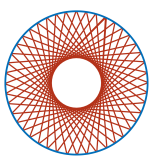
Quantum Chaos Theory

A natural quantum system to consider is a **quantum billiard**. The quantum particle is confined in a domain $\Omega \subset \mathbb{R}^2$ and the Hamiltonian is the **Laplacian** with Dirichlet boundary conditions:

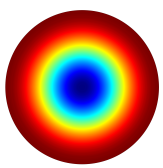
$$\boxed{-\Delta\psi = \lambda\psi \text{ on } \Omega, \quad \psi_{|\partial\Omega} = 0}.$$

We can find **two main types** of billiards:

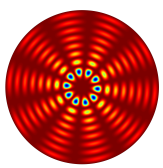
- **Integrable billiards**



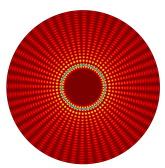
Classical



Low energy

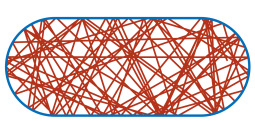


Medium energy



High energy

- **Chaotic billiards**



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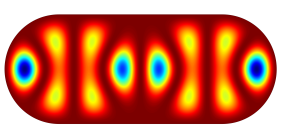
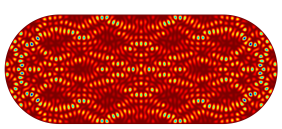
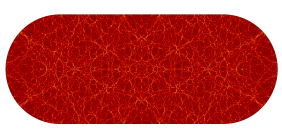


Low energy



Medium energy



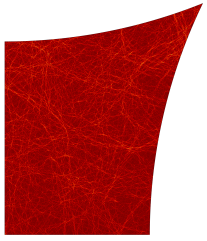
High energy

Quantum Unique Ergodicity and the Berry Conjecture

The **Quantum Unique Ergodicity** conjecture [Rudnick–Sarnak '94, Zelditch '94] states that for a class of **strongly chaotic** billiards

For any open set $A \subset \Omega$

$$\int_A |\psi_k(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \xrightarrow[k\to\infty]{} \frac{\mathrm{Area}(A)}{\mathrm{Area}(\Omega)}.$$



Proved for arithmetic surfaces [Lindenstrauss, Holowinsky, Soundararajan] or under averaging over k [Shnirel'man, Colin de Verdière, Zelditch, Anantharaman–Le Masson] but still **open** in generality.

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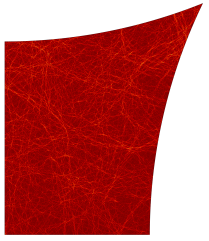

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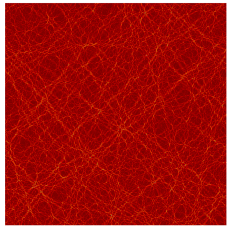
The **Berry conjecture** [Berry '77] states that the eigenfunctions of a chaotic billiard are **asymptotically Gaussian random plane waves**

$$\psi_k \xrightarrow[k\to\infty]{\text{in some sense}} \Psi,$$

$$\Psi \text{ isotropic Gaussian field with } \mathbb{E}\left[\Psi(x)\Psi(y)\right] = \int_{\mathbb{S}^{d-1}} \mathrm{e}^{\mathrm{i}\langle x-y,\theta\rangle}\sigma(\mathrm{d}\theta)$$

A **proper mathematical formulation** of the conjecture can be seen in [Ingremeau '17, Abert–Bergeron–Le Masson '18, García-Ruiz '23]

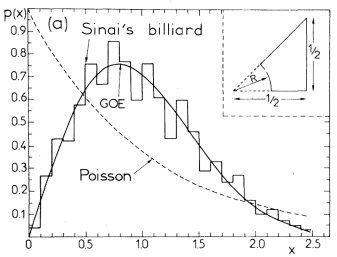


The Bohigas–Giannoni–Schmit conjecture

For **chaotic** billiards, the **Bohigas–Giannoni–Schmit conjecture** [Bohigas–Giannoni–Schmit '84] states that the **local statistics of eigenvalues** are given by those of the **Gaussian Orthogonal Ensemble**.

If $\lambda_1 \leqslant \lambda_2 \dots$ are the eigenvalues of $-\Delta$, after unfolding the spectrum $x_k = \overline{N}(\lambda_k)$, the **gap distribution** is given by

$$\frac{1}{K} \sum_{k=1}^{K} \mathbb{1}_{x_{k+1} - x_k \leqslant s} \xrightarrow[K \to \infty]{} F_{\mathrm{GOE}}(s).$$



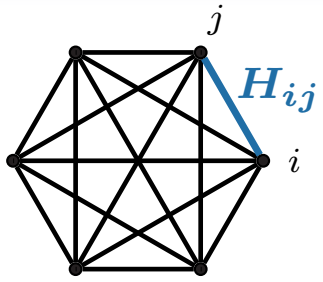
This is reminiscent of Wigner's original idea that **random matrices** can be used to model **complex quantum systems** such as heavy nuclei [Wigner '57].

Random matrices are toy models for complex chaotic systems.

From now on, we will focus on **random matrix models** and their **eigenvectors**.

The Gaussian Orthogonal Ensemble

$$\boldsymbol{H \sim \mathrm{GOE}_N}: \quad H = H^\top \in \mathbb{R}^{N\times N},$$
$$\frac{1}{\sqrt{2}} H_{ii} \sim H_{ij} \sim \mathcal{N}(0, N^{-1}) \quad \text{i.i.d. for } i > j.$$



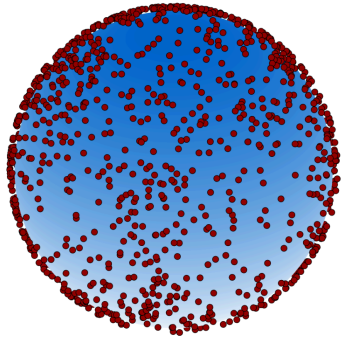
The GOE is **invariant** under orthogonal conjugation: for $O \in M_N(\mathbb{R})$, $OO^\top = \mathrm{Id}$,

$$\boxed{OHO^\top \overset{(d)}{=} H}$$

- U is **Haar distributed** on the orthogonal group and independent of λ_k's
- $\mathbf{u}_k$'s are **uniformly distributed** on the sphere $\mathbb{S}^{N-1}$

Eigenvectors are **trivial** for the GOE but we can deduce some properties from the **exact distribution** of eigenvectors.

It is a **reference** for other delocalized systems!



Some properties for the GOE

- **Eigenvector delocalization** [Ledoux–Talagrand, O'Rourke–Vu–Wang]: for any $D > 0$, there exists a $\kappa > 0$ such that

$$\forall k \in [\![1, N]\!], \quad \mathbb{P}\left(\|\mathbf{u}_k\|_\infty \geqslant \sqrt{\frac{(2+\kappa)\log N}{N}} \right) \leqslant N^{-D} \qquad \left(\sum_{\alpha=1}^{N} \mathbf{u}_k^2(\alpha) = 1 \right)$$

- **Asymptotic Gaussianity** [Mehler 1866, Diaconis–Freedman, Jiang–Ma '17]: Let I, $J \subset [N]$ with fixed cardinality,

$$\left(\sqrt{N} |\mathbf{u}_{k_i}(\alpha_j)| \right)_{(i,j) \in I \times J} \xrightarrow[N \to \infty]{} (|\mathcal{N}_{i,j}|)_{(i,j) \in I \times J}$$
$$\mathcal{N}_{i,j} \quad \text{i.i.d.} \quad \mathcal{N}(0,1).$$

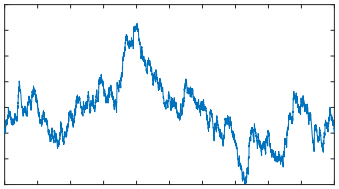
- **Quantum Unique Ergodicity**: For A deterministic with $\|A\| \leqslant 1$, for any $\varepsilon, D > 0$, with $\langle A \rangle = \frac{1}{N} \operatorname{Tr} A$ and $\mathring{A} = A - \langle A \rangle$,

$$\mathbb{P}\left(\sup_{k,\ell} |\langle \mathbf{u}_k, A \mathbf{u}_\ell \rangle - \langle A \rangle \delta_{k\ell}| \geqslant \frac{N^\varepsilon}{\sqrt{N}} \sqrt{\langle \mathring{A} \mathring{A}^* \rangle} \right) \leqslant N^{-D}$$

Eigenvector Processes

It can be shown using Donsker's theorem for $\mathbf{q} \in \mathbb{S}^{N-1}$ deterministic that,

$$\left(\sqrt{\frac{N}{2}} \sum_{k=1}^{\lfloor Nt \rfloor} \left(\langle \mathbf{q}, \mathbf{u}_k \rangle^2 - \frac{1}{N} \right) \right)_{t \in [0,1]} \xrightarrow[N \to \infty]{(d)} \mathrm{BB}$$

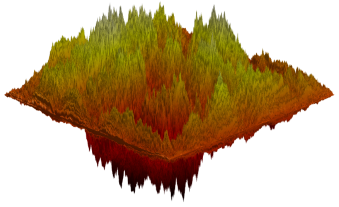


In [Silverstein '81], this type of convergence was seen as an **evidence** towards **convergence to the Haar measure** on the orthogonal group.

This convergence was generalized in [Donati-Martin–Rouault '11] using Weingarten calculus

$$\left(\frac{1}{\sqrt{2}} \sum_{k=1}^{\lfloor Nt \rfloor} \sum_{\alpha=1}^{\lfloor Ns \rfloor} \left(\mathbf{u}_k(\alpha)^2 - \frac{1}{N} \right) \right)_{s,t \in [0,1]} \xrightarrow[N \to \infty]{(d)} \mathrm{BBB},$$

$$\mathbb{E}\left[\mathrm{BBB}_{s,t} \mathrm{BBB}_{s',t'} \right] = (s \wedge s' - ss')(t \wedge t' - tt')$$



Bivariate tied-down Brownian bridge

Universality paradigm

Universality: Do these properties hold **beyond** Gaussian ensembles?

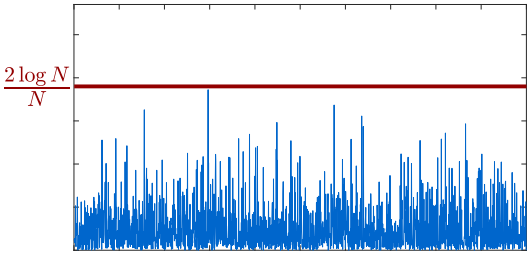
W is a **Generalized Wigner matrix** if $W = W^\top$ with independent entries (up to the symmetry), $\mathbb{E}W_{ij} = 0$, and

$$\frac{c}{N} \leqslant \mathbb{E}|W_{ij}|^2 \leqslant \frac{C}{N}, \quad \sum_{j=1}^{N} \mathbb{E}|W_{ij}|^2 = 1, \quad \mathbb{E}|W_{ij}|^p \leqslant C_p \left(\mathbb{E}|W_{ij}|^2\right)^{p/2}$$

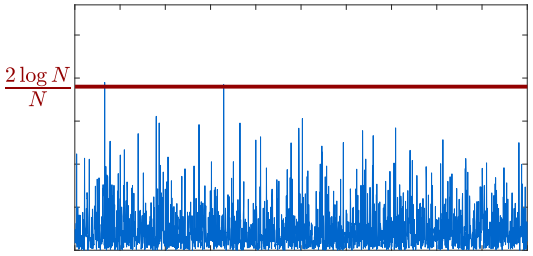
Example: Rademacher entries $W_{ij} = \pm N^{-1/2}$ with probability $\frac{1}{2}$.

No orthogonal invariance!

GOE



Bernoulli

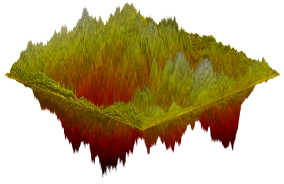


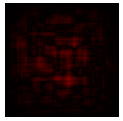
From global to local observables

Global statistics are not always universal!

Global statistics: Depend on macroscopically many or $\mathcal{O}(N)$ eigenvectors.

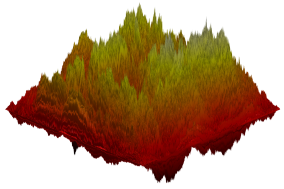
Gaussian

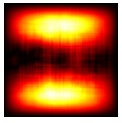




Empirical variance defect

Rademacher





Empirical variance defect

[Benaych-Georges '11] shows that the limiting process is **not universal** and depends on the fourth moment of the entries. See also [Bao–Pan–Zhou '12].

Local statistics: Depend on finitely many eigenvectors.

Local statistics are typically universal!

Eigenvector universality

There has been an **extensive program** on proving **eigenvector universality**,

$$\|\mathbf{u}_k\|_\infty \leqslant \sqrt{\frac{(2+\kappa)\log N}{N}}.$$

- $\sqrt{N}$: [Erdős–Schlein–Yau,'08–'10]
- $\sqrt{\log N}$: [Vu–Wang '13, B.–Lopatto '20]
- $\sqrt{2}$ [B.–Lopatto '20]

Asymptotic Gaussianity of entries has also been studied,

$$\left(\sqrt{N}|\mathbf{u}_{k_i}(\alpha_j)|\right)_{(i,j)\in I\times J} \xrightarrow[N\to\infty]{} (|\mathcal{N}_{i,j}|)_{(i,j)\in I\times J}$$

[Knowles–Yin '11, Tao–Vu '11, Bourgade–Yau '13, Marcinek–Yau '20, B. '26].

Quantum Unique Ergodicity

$$\mathbb{P}\left(\sup_{k,\ell}|\langle \mathbf{u}_k, A\mathbf{u}_\ell\rangle - \langle A\rangle\delta_{k\ell}| \geqslant \frac{N^\varepsilon}{\sqrt{N}}\sqrt{\langle \mathring{A}\mathring{A}^*\rangle}\right) \leqslant N^{-D}$$

[Bourgade–Yau–Yin '18, B.–Lopatto '21, Cipolloni–Erdős–Schröder '20 '22 '23]

Berry conjecture for Wigner matrices

We can also get **Gaussian fluctuations** in the QUE property of eigenvectors. This is somewhat reminiscent of the **Berry conjecture** for chaotic billiards.

Theorem (B.–Cipolloni '24)

Fix $p \in \mathbb{N}$. For $A_1, \dots A_p$ deterministic matrices with $\|A_i\| \leqslant C$ and $\langle \mathring{A}_i \mathring{A}_i^* \rangle \geqslant N^{-1+\varepsilon}$ and for $i_1, \dots i_{2p} \in [\![\alpha N, (1-\alpha)N]\!]$ if

$$\Phi(A,k,\ell) = \sqrt{\frac{N}{\langle \mathring{A}\mathring{A}^* \rangle}} \left(\langle \mathbf{u}_k, A\mathbf{u}_\ell \rangle - \langle A \rangle \delta_{k\ell} \right)$$

then $(\Phi(A_1, i_1, i_2), \dots, \Phi(A_p, i_{2p-1}, i_{2p}))$ converges to a Gaussian field with the covariance structure

$$\mathbb{E}\left[\Phi(A,i,j)\Phi(B,k,\ell)\right] = \left(\delta_{jk}\delta_{i\ell} + \delta_{ik}\delta_{j\ell}\right) \frac{\langle \mathring{A}\mathring{B}^* \rangle}{\sqrt{\langle \mathring{A}\mathring{A}^* \rangle \langle \mathring{B}\mathring{B}^* \rangle}}$$

The limiting Gaussian process is **simpler** than random plane waves!

Eigenvector process for Wigner matrices

The previous theorem gives the **finite-dimensional convergence** of general eigenvector process. By proving **tightness**, we can get **functional convergence**.

Theorem (B.–Rezaie Feyzabady '26)

Let $(A_t)_{t\in[0,1]}$ *be symmetric matrices with* $A_0 = 0$, $\mathrm{Tr}(A_t) = 0$, $\|A_t\| \leqslant C$, *and* $\langle A_t^2 \rangle \geqslant N^{-1+\varepsilon}$ *for all* $t \in [0,1]$. *Suppose that there exists* $\gamma > 0$ *and* $L > 0$ *such that*

$$\langle (A_t - A_s)^2 \rangle \leqslant L|t-s|^\gamma \quad \text{and} \quad \langle A_s A_t \rangle \xrightarrow[N\to\infty]{} K(s,t) \quad \text{for any } s,t \in (0,1]$$

with K *continuous and positive semidefinite. Then for any* $k,\ell \in [\![\alpha N, (1-\alpha)N]\!]$,

$$\left(\sqrt{\frac{N}{1+\delta_{k\ell}}} \langle \mathbf{u}_k, A_t \mathbf{u}_\ell \rangle \right)_{t\in[0,1]} \xrightarrow[N\to\infty]{(d)} \mathrm{G}$$

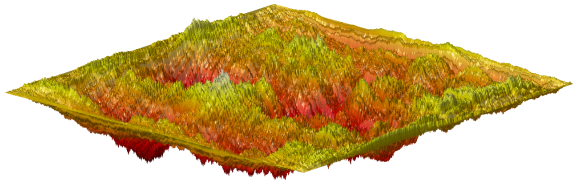
where G *is a centered Gaussian process with covariance* K *such that* $G(0) = 0$.

Similar results hold when considering different eigenvector processes jointly.

Local processes: First examples

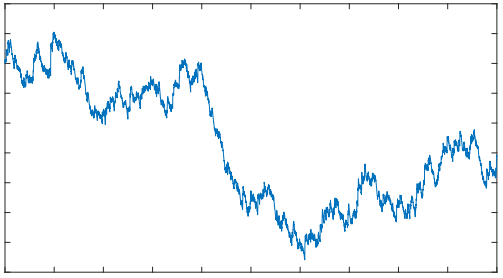
Take $(\mathbf{q}_\alpha)_{\alpha=1}^N$ an orthonormal basis of $\mathbb{R}^N$ and consider the **local process**,

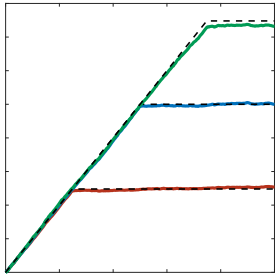
$$\mathbf{X}(\mathbf{q},k) := \left(\sqrt{\frac{N}{2}}\sum_{\alpha=1}^{\lfloor Nt\rfloor}\left(\langle \mathbf{q}_\alpha,\mathbf{u}_k\rangle^2-\frac{1}{N}\right)\right)_{t\in[0,1]} \xrightarrow[N\to\infty]{(d)} \text{BB}.$$



Other limiting processes are possible! If $(\mathbf{q}_\alpha)_{\alpha=1}^N$ follows $\langle \mathbf{q}_\alpha,\mathbf{q}_\beta\rangle = \delta_{\alpha\beta}+\frac{1-\delta_{\alpha\beta}}{\sqrt{N}}$

$$\mathbf{X}(\mathbf{q},k) \xrightarrow[N\to\infty]{(d)} \text{BM}.$$





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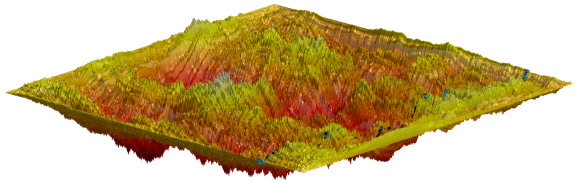

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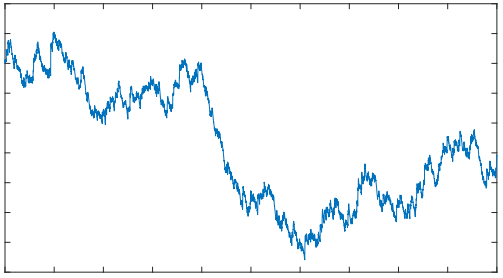
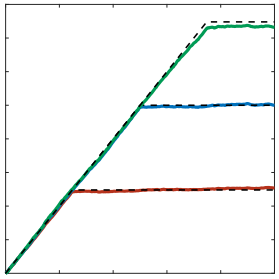




Other examples

We can also construct explicitly limiting processes with separable covariances

$$K(s,t) = \int_0^1 f_u(s) f_u(t) \mathrm{d}u.$$

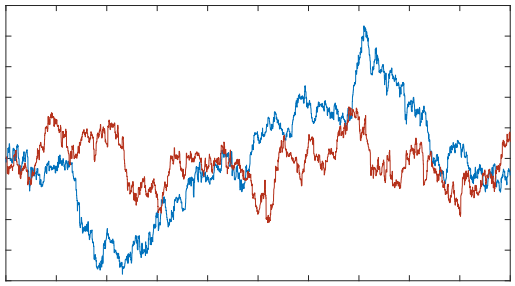
Then we can construct

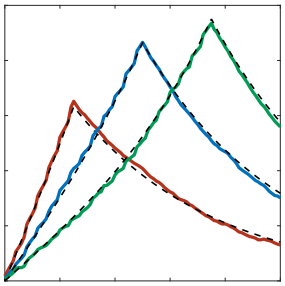
$$A_t = \sum_{\alpha=1}^{\lfloor \frac{N}{2} \rfloor} f_{\frac{2(\alpha-1)}{N}}(t) \left(\mathrm{e}_{2\alpha} \mathrm{e}_{2\alpha}^\top - \mathrm{e}_{2\alpha-1} \mathrm{e}_{2\alpha-1}^\top \right)$$

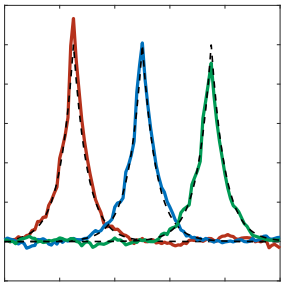
so that

$$\langle A_t A_s \rangle = \frac{2}{N} \sum_{\alpha=1}^{\frac{N}{2}} f_{\frac{2(\alpha-1)}{N}}(t) f_{\frac{2(\alpha-1)}{N}}(s) \xrightarrow[N\to\infty]{} \int_0^1 f_u(s) f_u(t) \mathrm{d}u.$$

Ornstein–Uhlenbeck process: $f_u(t) = \sigma \mathrm{e}^{-\theta(t-u)} \mathbb{1}_{u \leqslant t}$ with $\theta, \sigma > 0$.







Limiting processes

Let $\mathbf{G}$ be a Gaussian process with continuous covariance K. Define $(\psi_k, \lambda_k)_{k\geqslant 1}$

$$\int_0^1 K(s,t)\psi_k(s)\mathrm{d}s = \lambda_k \psi_k(t).$$

Karhunen–Loève decomposition: $G_t = \sum_{k=1}^{+\infty} \sqrt{\lambda_k} Z_k \psi_k(t)$ with Z_k i.i.d. $\mathcal{N}(0,1)$.

Theorem (B.–Rezaie Feyzabady '26)

Suppose that $\sum_{k\geqslant 1} \lambda_k \|\psi_k\|_\infty^2 < \infty$ *and that there exists* $L, \gamma > 0$ *such that*

$$|K(s,s) + K(t,t) - 2K(s,t)| \leqslant L|t-s|^\gamma \quad \textit{for any } s,t \in [0,1].$$

Then there exists $(A_t)_{t\in[0,1]}$ *such that for any* $k \in [\![\alpha N, (1-\alpha)N]\!]$,

$$\left(\sqrt{\frac{N}{2}}\langle \mathbf{u}_k, A_t \mathbf{u}_k\rangle\right)_{t\in[0,1]} \xrightarrow[N\to\infty]{(d)} \mathbf{G}.$$

Brownian motion

$$B_t = \sqrt{2}\sum_{k=1}^{\infty} Z_k \frac{\sin((k-1/2)\pi t)}{(k-1/2)\pi}$$

Brownian bridge

$$\mathrm{BB}_t = \sqrt{2}\sum_{k=1}^{\infty} Z_k \frac{\sin(k\pi t)}{k\pi}$$

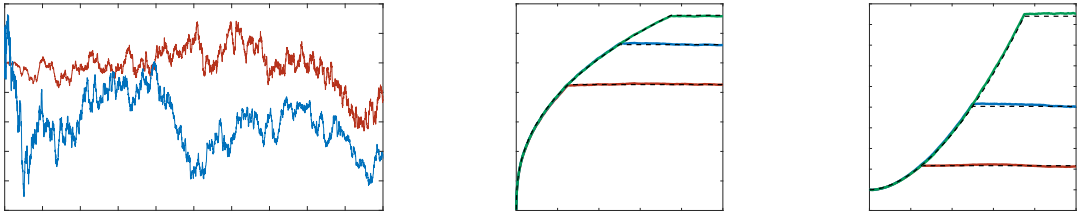
More Examples

From this, we can construct many other limiting processes!

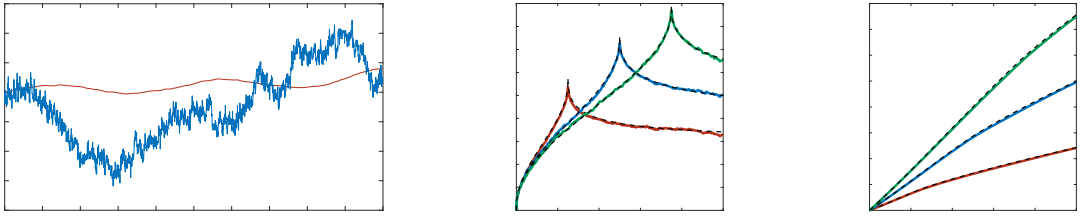
"Historical" fractional Brownian motion [Lévy '53] : For $H \in (0,1)$, and B Brownian motion

$$B_H(t) = \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} \mathrm{d}B_s, \quad \mathbb{E}\left[B_H(t)B_H(s)\right] = \frac{(t \wedge s)^{2H}}{2H\Gamma(H+\frac{1}{2})^2}.$$

Karhunen–Loève decomposition in terms of Bessel functions [Maccone '81].



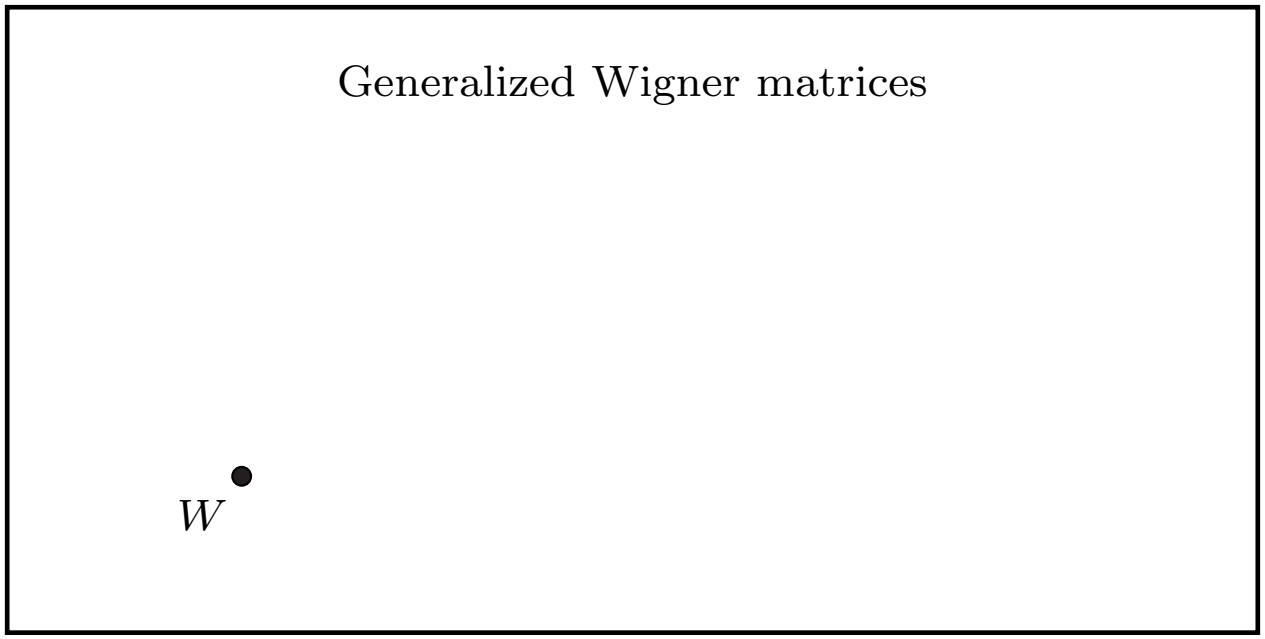
Actual fractional Brownian motion [Mandelbrot–Van Ness '68]: No comparably simple explicit Karhunen–Loève decomposition but can be done numerically



Sketch of proof

How can we **prove** these results **without** orthogonal invariance?

Dynamical method [Erdős–Schlein–Yau]



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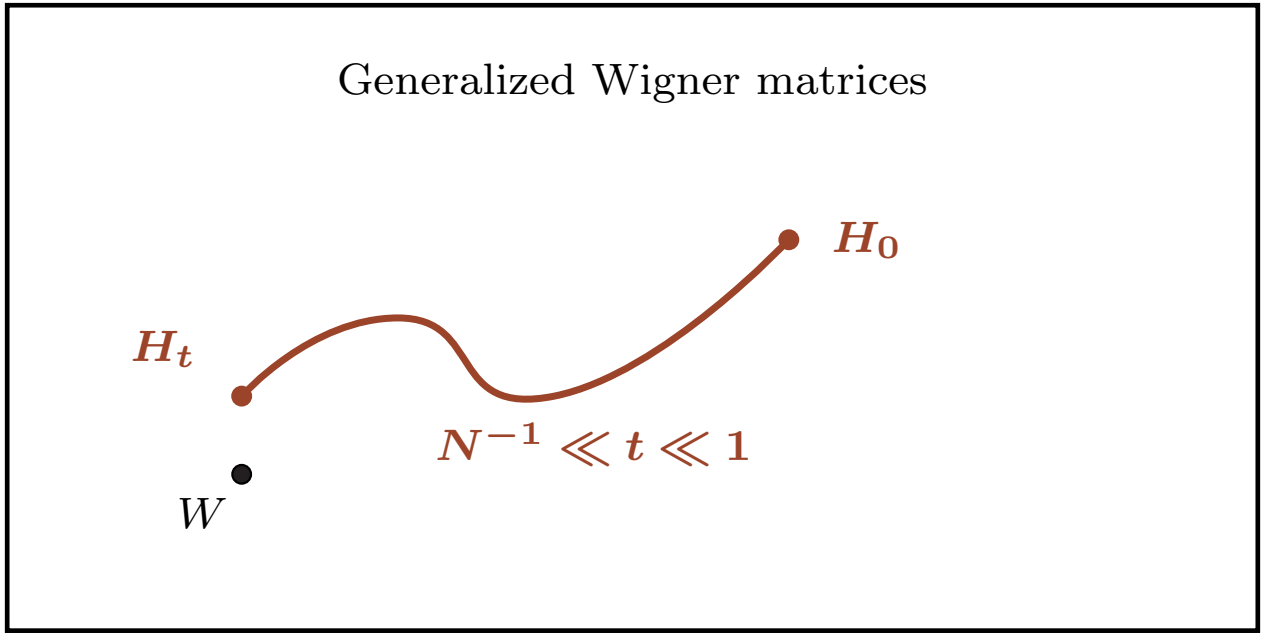

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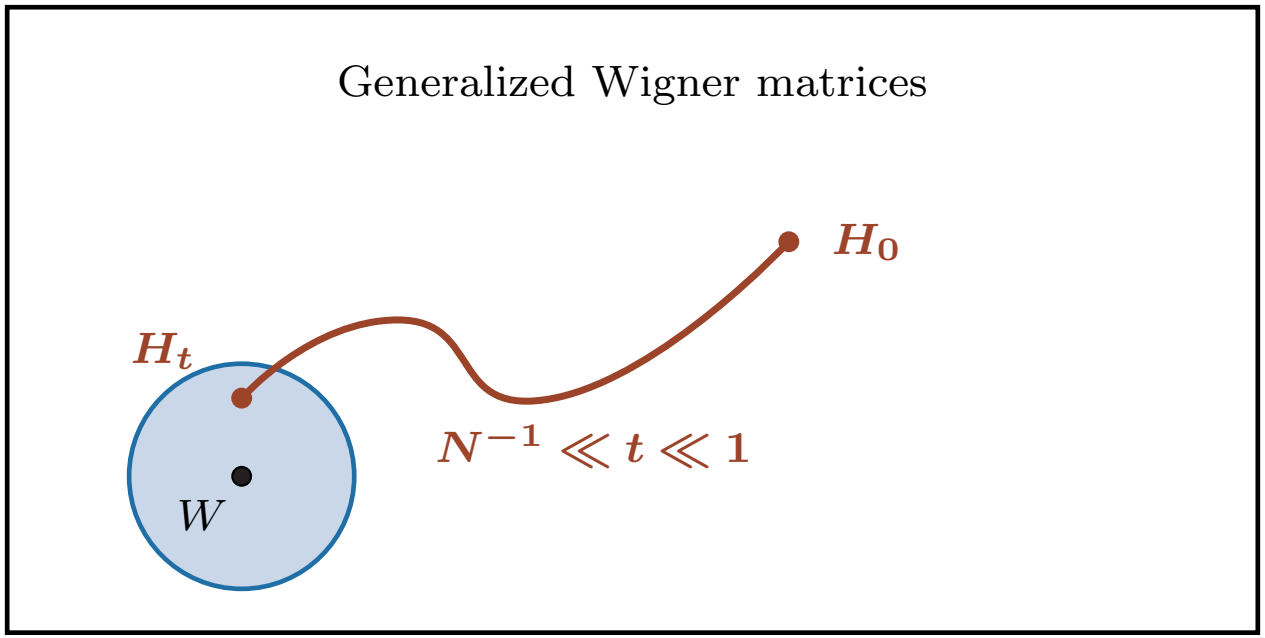

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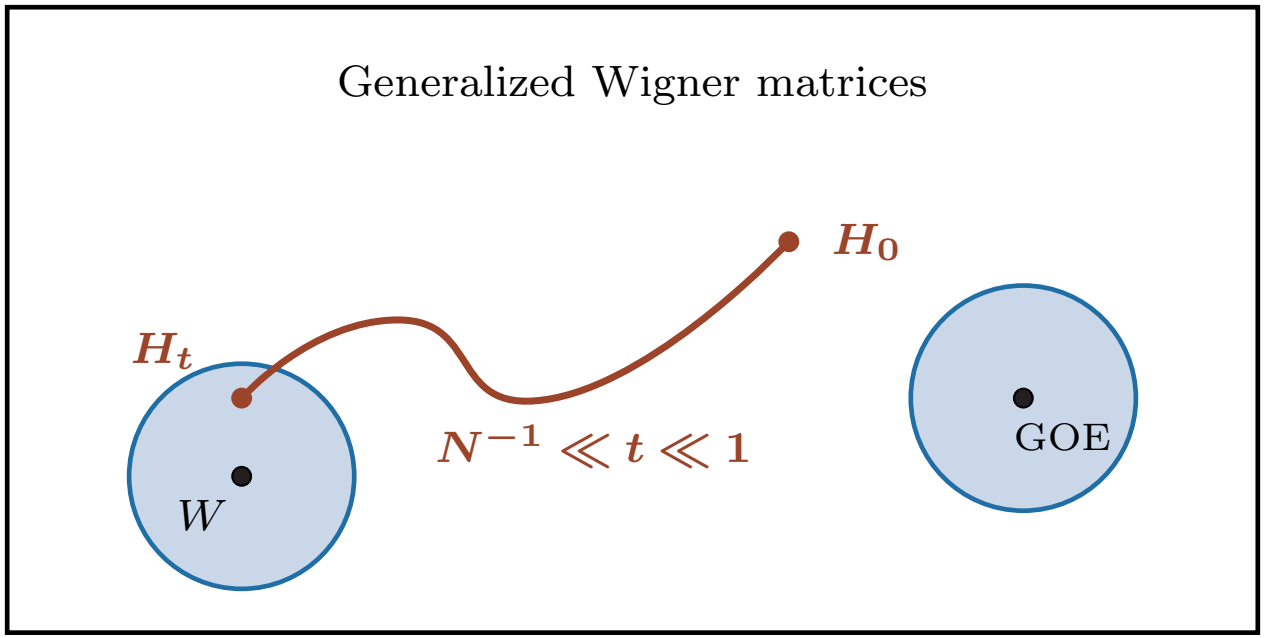

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Dyson Brownian motion

Integrable dynamics on symmetric matrices [Dyson '62]: define B_t a symmetric matrix whose upper triangular entries are **independent Brownian motions**.

$$\boxed{H_t = H_0 + \frac{1}{\sqrt{N}} B_t.} \qquad B_{ij} \overset{(d)}{=} \frac{B_{ii}}{\sqrt{2}} \overset{(d)}{=} \text{BM} \quad \text{for } i \neq j.$$

It induces dynamics on **eigenvalues and eigenvectors**: $\widetilde{B}_t \overset{(d)}{=} B_t$,

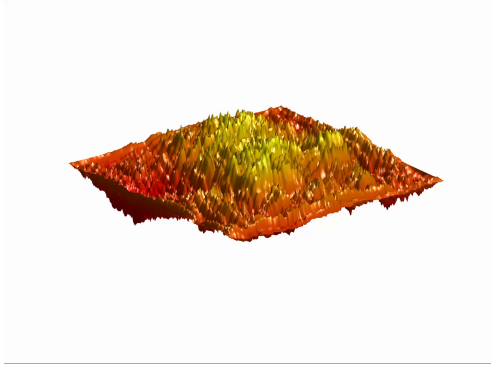
$$\mathrm{d}\lambda_k = \frac{\mathrm{d}\widetilde{B}_{kk}}{\sqrt{N}} + \frac{1}{N} \sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell} \mathrm{d}t$$

$$\mathrm{d}\mathbf{u}_k = \frac{1}{\sqrt{N}} \sum_{\ell \neq k} \frac{\mathrm{d}\widetilde{B}_{k\ell}}{\lambda_k - \lambda_\ell} \mathbf{u}_\ell - \frac{1}{2N} \sum_{\ell \neq k} \frac{\mathrm{d}t}{(\lambda_k - \lambda_\ell)^2} \mathbf{u}_k$$

Important observation: The noise driving the eigenvector dynamics is **independent** of the noise driving the eigenvalue dynamics.

They reach **local equilibrium** in **short-time**: $t \gg N^{-1}$

We obtain our results for H_t



Integrable dynamics

Denote

$$f_t(k) = \mathbb{E}\left[N\mathbf{u}_k^2(\alpha)\middle|\boldsymbol{\lambda}_{0\to 1}\right].$$

Then [Bourgade–Yau '13]

$$\partial_t f_t(k) = \sum_{\ell\neq k} \frac{f_t(\ell) - f_t(k)}{N(\lambda_k - \lambda_\ell)^2}.$$

Using **local laws** and a **maximum principle** we can get for $t \sim N^{-\omega}$,

$$\sup_{k\in[\![1,N]\!]} |f_t(k) - 1| \leqslant N^{-c(\omega)}.$$

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This generalizes to, **for a fixed** α,

$$f_t(\boldsymbol{\xi}) = \frac{1}{\prod_{k=1}^N (2\xi_k - 1)!!}\mathbb{E}\left[\prod_{k=1}^N (\sqrt{N}\mathbf{u}_k(\alpha))^{2\xi_k}\middle|\boldsymbol{\lambda}_{0\to 1}\right]$$

and to [Bourgade–Yau–Yin '18, Cipolloni–Erdős–Schröder '21], for $\langle A\rangle = 0$,

$$g_{t,n}(k) = \frac{1}{(2n-1)!!}\mathbb{E}\left[\left(\sqrt{\frac{N}{2\langle AA^*\rangle}}\langle \mathbf{u}_k, A\mathbf{u}_k\rangle\right)^{2n}\middle|\boldsymbol{\lambda}_{0\to 1}\right]$$

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$$\textbf{Not to } g_{t,n}(k,\ell) = \frac{1}{(2n-1)!!}\mathbb{E}\left[\left(\sqrt{\frac{N}{\langle AA^*\rangle}}\langle \mathbf{u}_k, A\mathbf{u}_\ell\rangle\right)^{2n}\middle|\boldsymbol{\lambda}_{0\to 1}\right]$$

Comparison

Density of the Dyson Brownian motion [Erdős–Yau–Yin '10]:

For any $t \ll 1$, we can find H_0,

$$\mathbb{E}[(H_t)_{ij}^k] = \mathbb{E}[W_{ij}^k] \quad \text{for } k = 1, 2, 3$$
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Lindeberg exchange [Tao–Vu '09, Knowles–Yin '11]:
Exchange each matrix entry one by one

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Need to understand how eigenvectors react to change in matrix entries

Difficulty: $$\partial_{ab}\mathbf{u}_k(\alpha) = \sum_{\ell \neq k} \frac{\mathbf{u}_\ell(a)\mathbf{u}_k(b) + \mathbf{u}_k(a)\mathbf{u}_\ell(b)}{\lambda_k - \lambda_\ell}\mathbf{u}_\ell(\alpha).$$

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