Quantum Loewner evolution in quantum natural time

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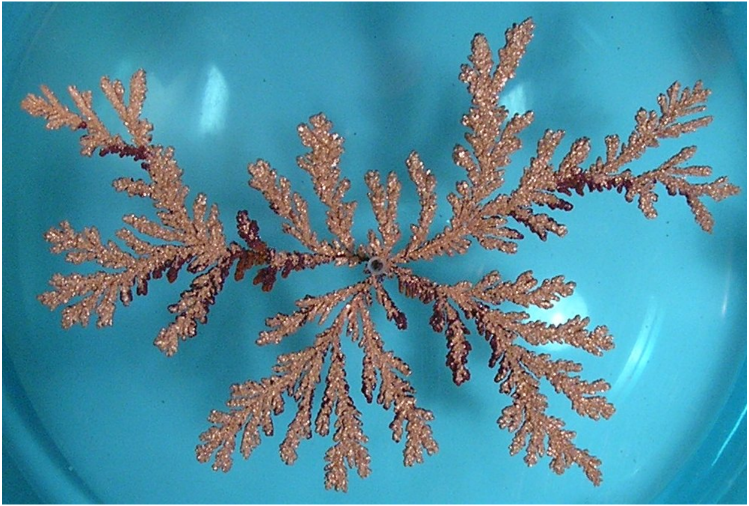
UC San Diego

In honor of Greg Lawler's 70th birthday

Institute for Mathematical and Statistical Innovation, July 2026

Based on joint work with Deven Mithal (University of Chicago)

Copper aggregate grown by electrodeposition



Diffusing particles/ions attach to an existing cluster.
Image by Kevin R Johnson

Laplacian growth in nature

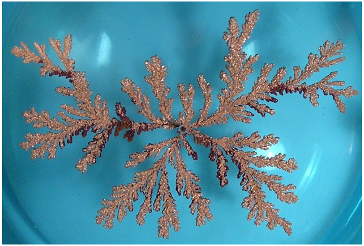


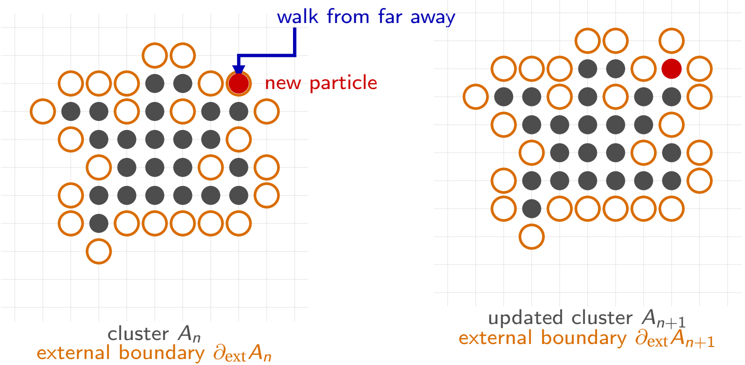

image 1: Kevin R Johnson, image 2: Bert Hickman, image 3: Anthony Hall, image 4: Mark A Wilson.

Diffusion-limited aggregation (Witten–Sander 1981)

Start from $A_0 = \{0\} \subset \mathbb{Z}^2$.
Given A_n, run simple random walk from ∞ until it hits

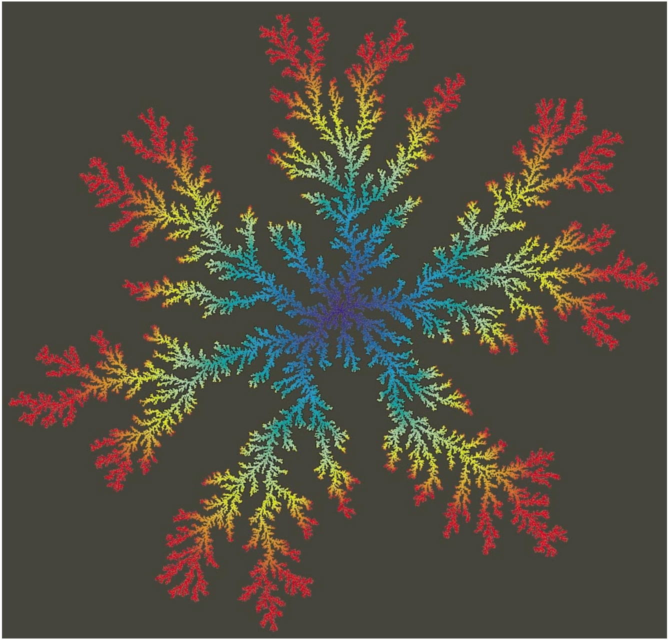
$$\partial_{\text{ext}} A_n := \{x \notin A_n : \exists y \in A_n,\ x \sim y\};$$

add that site to the cluster.

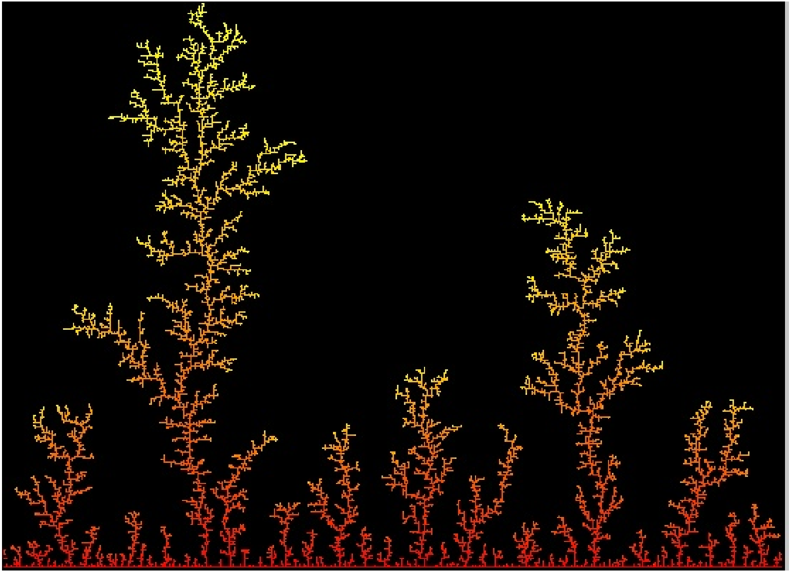


Also have an edge variant, where we add edges to the cluster.

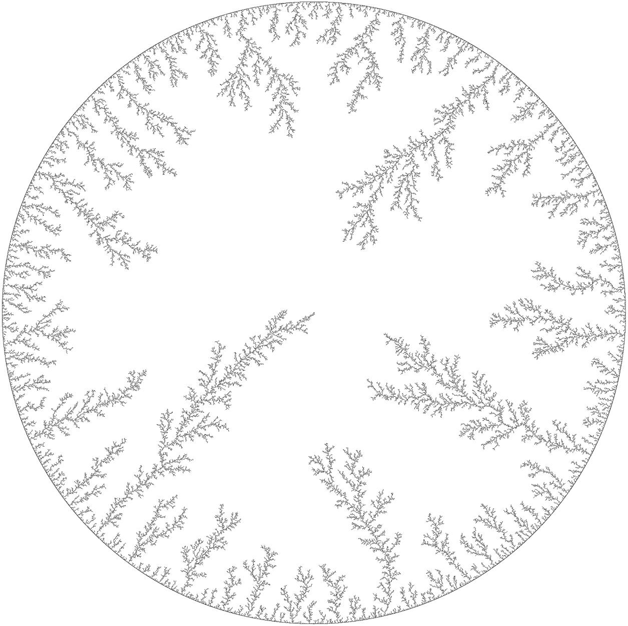
Simulation of DLA (Mandelbrot–Evertsz)



Simulation of DLA (physics stackexchange)



Simulation of DLA (softologyblog.wordpress.com)



Rigorous results on DLA

Upper bound (Kesten 1987)

For DLA on $\mathbb{Z}^d$ with $d \geq 2$, a.s. for all sufficiently large n,

$$\operatorname{diam}(A_n) \leq \begin{cases} C_d\, n^{2/(d+1)}, & d \neq 3, \\ C_d\, (n \log n)^{1/2}, & d = 3. \end{cases}$$

Upper bound (Lawler 1996)

For DLA on $\mathbb{Z}^d$ with $d = 3$, a.s. for all sufficiently large n,

$$\operatorname{diam}(A_n) \leq C_3\, n^{1/2} (\log n)^{1/4}.$$

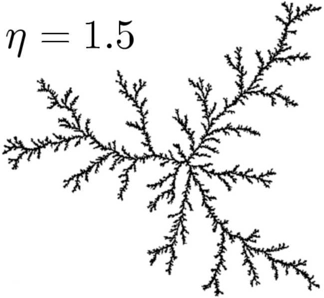
Only trivial lower bound $\operatorname{diam}(A_n) \gtrsim n^{1/d}$.

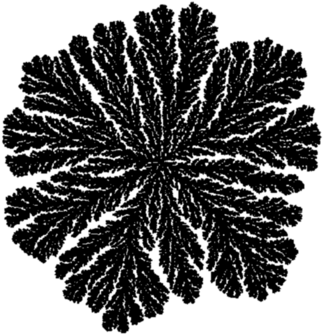
Dielectric breakdown model (η-DBM)

Parameter $\eta \in \mathbb{R}$. Same as DLA, but attach at site with probability proportional to (harmonic measure)$^\eta$.

$\eta = 0$: Eden model (exponential FPP). $\eta = 1$: DLA.

$\eta = 1.5$





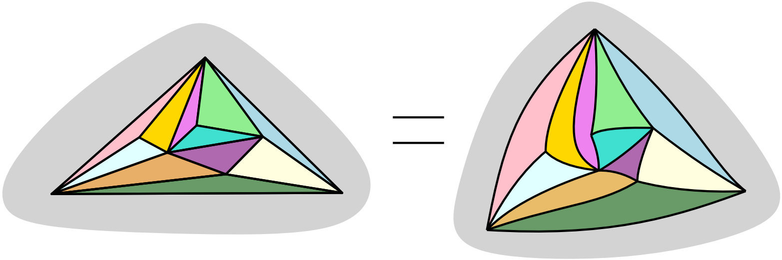
$\eta = 0.5$

Upper bounds on η-DBM diameter on $\mathbb{Z}^2$ and $\mathbb{Z}^3$ by [Losev–Smirnov 2023]! For $\eta = 1$ recovers DLA bounds of Kesten and Lawler.

Outline

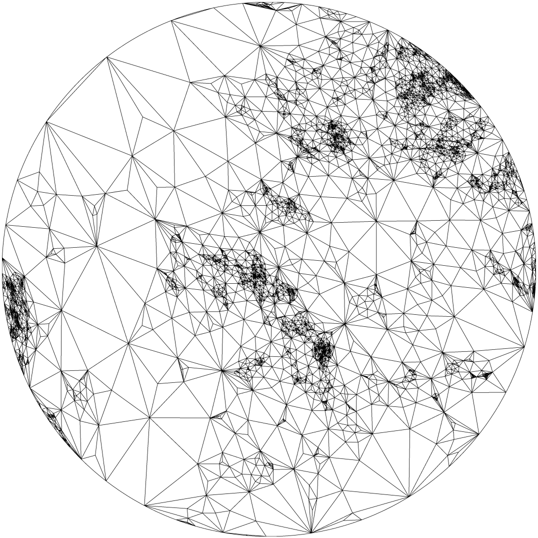
- Discuss DLA and η-DBM in random environments called random planar maps.
- Conjectural scaling limit is **quantum Loewner evolution (QLE)** introduced by Miller–Sheffield 2016.
- We give a different construction of QLE (possibly gives the same object?), which has certain advantages.
- Phases of QLE.
- Markov properties of QLE.

What is a planar map?



Embedding of planar graph into Riemann sphere $\mathbb{C} \cup \{\infty\}$,
modulo homeomorphisms of $\mathbb{C} \cup \{\infty\}$.

Random planar map simulation



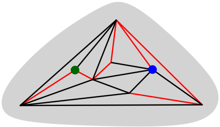
Random planar map, harmonic embedding in $\mathbb{D}$ (Jason Miller).

Spanning-tree weighted random planar maps

$\mathcal{MT}^{\text{bulk}}_{n,\ell}$: set of disk triangulations having

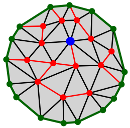
- n vertices • boundary length 2ℓ • marked interior vertex
- spanning tree with wired boundary conditions.

$\mathcal{MT}^{\text{bulk}}_{n,0}$



n vertices, 0 boundary length, two marked vertices

$\mathcal{MT}^{\text{bulk}}_{n,\ell}$



n vertices, 2ℓ boundary length, marked interior vertex

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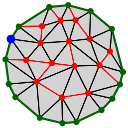
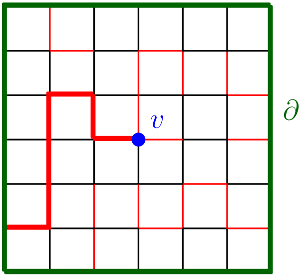


n vertices, 2ℓ boundary length, marked boundary vertex

A **spanning-tree weighted random planar map** is a uniform sample from one of these after forgetting the tree.

By combining two ingredients, we will see that DLA on these random environments is nice!

(1) Uniform spanning tree and loop-erased random walk



Deterministic graph with a boundary ∂ and interior vertex v.

Theorem (Pemantle 1991)

Sample a uniform spanning tree (UST).

$$\text{path from } v \text{ to } \partial \overset{d}{=} \text{loop-erased random walk from } v \text{ to } \partial.$$

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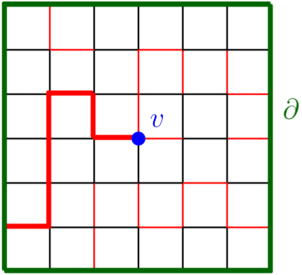

Deterministic graph with a boundary ∂ and interior vertex v.

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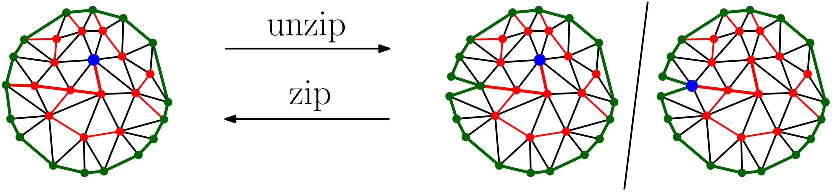
Single DLA step from v can be carried out by sampling a UST and adding last LERW edge.

(2) A trivial bijection



$\mathcal{MT}^{\text{bulk}}_{n,\ell}$

$\mathcal{MT}^{\text{bulk}}_{n,\ell+1} \cup \mathcal{MT}^{\text{bdy}}_{n,\ell+1}$

DLA on spanning-tree weighted random planar map

Consider a spanning-tree weighted random planar map with n vertices and boundary length 2ℓ. Call the marked interior vertex v.

Run DLA on this map from v to the boundary.

Stationarity of DLA on ST weighted RPM (Miller–Sheffield 2016)

For any $k \geq 0$, on the event that DLA has not terminated at step k, the complement of DLA has the law of a spanning-tree weighted random planar map with n vertices and boundary length $2(\ell + k)$.

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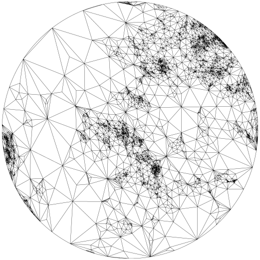
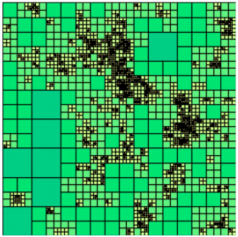
Proof: Suppose it holds for some $k \geq 0$.
Sample a uniform spanning tree; the configuration is uniform in $\mathcal{MT}^{\mathrm{bulk}}_{n,\ell+k}$. Unzip once.
By (1), this corresponds to a single DLA step.
By (2), the complement is uniform in $\mathcal{MT}^{\mathrm{bulk}}_{n,\ell+k+1} \cup \mathcal{MT}^{\mathrm{bdy}}_{n,\ell+k+1}$. $\square$



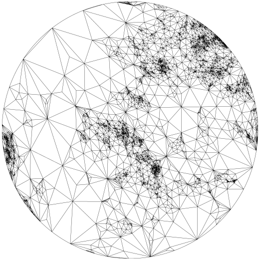


n-face random planar map
Conformally embedded in $\mathbb{D}$.

Liouville quantum gravity
Discretized: squares are same size

Discrete area measure A_n, boundary measure L_n, distance D_n

General conjecture:
(A_n, L_n, D_n) converges to a continuum triple $(A_\infty, L_\infty, D_\infty)$ called **Liouville quantum gravity**.



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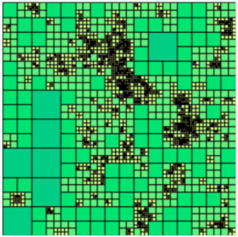

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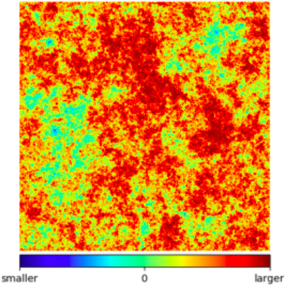
Proved for very few cases. [Gwynne-Miller-Sheffield '17], [Holden-Sun '19], [Bertacco-Gwynne-Sheffield '23].

Liouville quantum gravity area measure

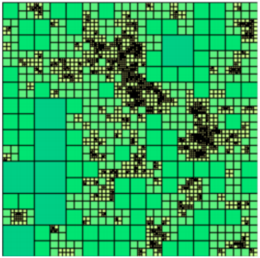
Let h be a Gaussian free field.
Let $h_\varepsilon(z)$ be the average of h on the radius-ε circle around z.
Let $\gamma \in (0,2)$. The γ-**LQG area measure** is

$$A_h^\gamma(dz) := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}\, dz.$$

This is an example of **Gaussian multiplicative chaos (GMC)**.
Kahane '85, Robert-Vargas '08, Duplantier-Sheffield '08,
Berestycki '15, Shamov '16.



Gaussian free field h



Discretization of A_h,
squares have comparable A_h-mass.

Images by Minjae Park.

Liouville quantum gravity parameter γ

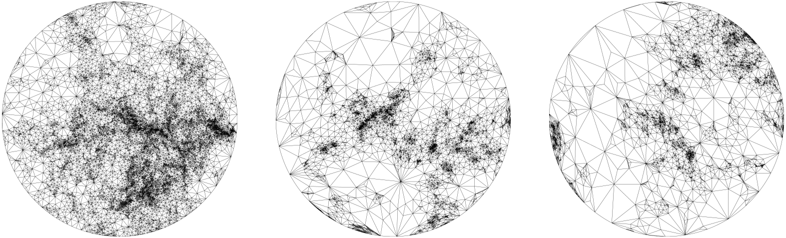
"Larger $\gamma \implies$ rougher surface".

$\gamma = \sqrt{2}$: Tree-decorated random planar map.

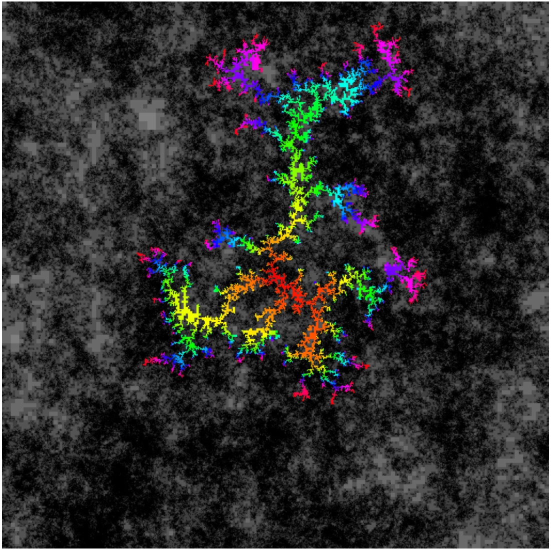
$\gamma = \sqrt{8/3}$: Uniform random planar map.

$\gamma = \sqrt{3}$: Ising-decorated random planar map.

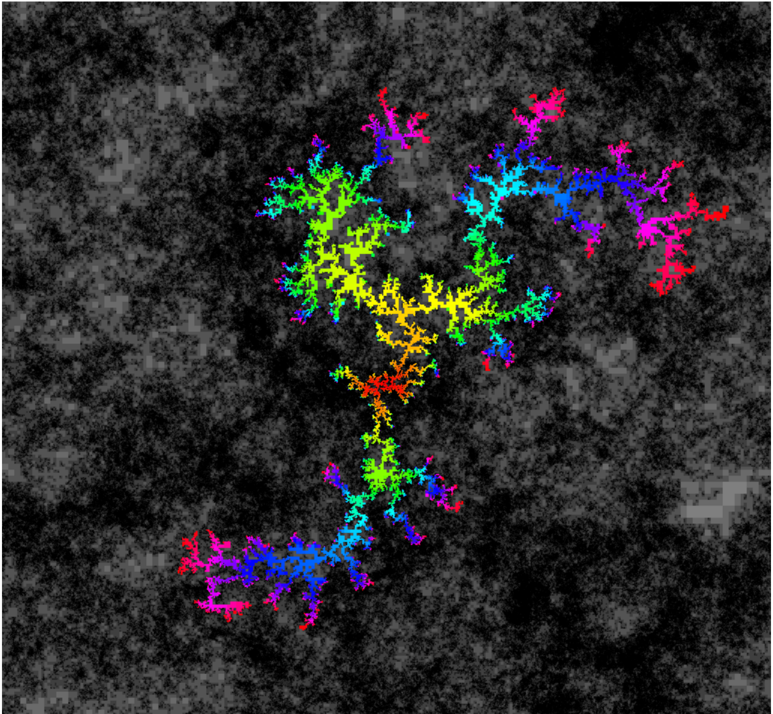
$\gamma = 2$: GFF-decorated random planar map

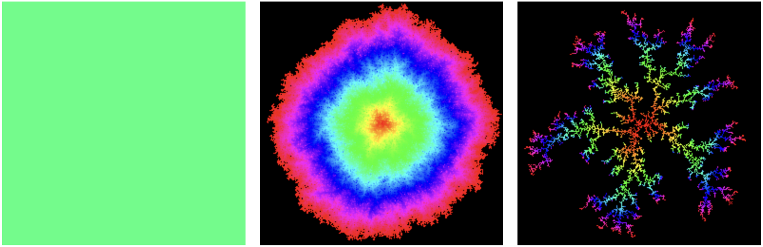


Quantum Loewner evolution (γ^2, η): conjectural scaling limit of η-DBM on RPM in γ-LQG universality class.



DLA on discretization of $\gamma = \sqrt{2}$ LQG. Image by Jason Miller





Squares | Eden model | DLA

Figure 1.5: $\gamma = 0$

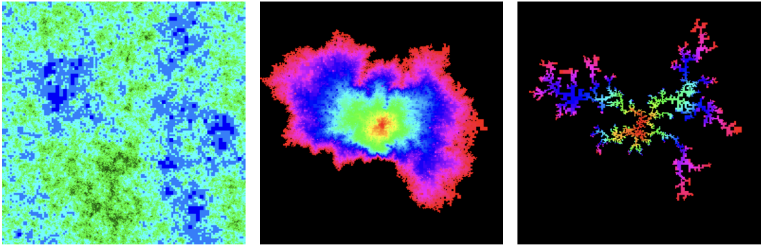


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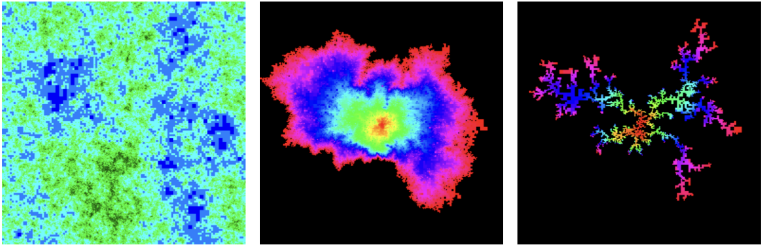
Figure 1.8: $\gamma = 3/4$



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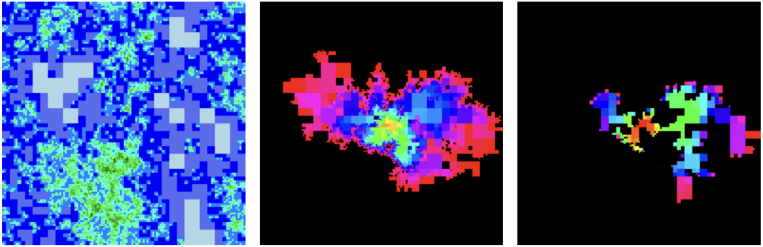

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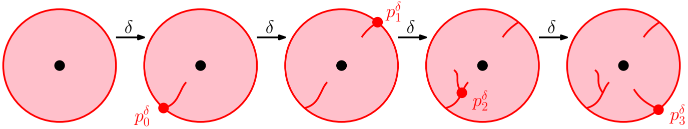
Figure 1.11: $\gamma = 3/2$

(Capacity time) QLE(γ^2, η): random growth process on a γ-LQG environment.



Let $\gamma \in (0,2)$ and $\kappa \in \{\gamma^2, 16/\gamma^2\}$. Sample a random γ-LQG environment in the disk, with field h. Iteratively:

- Sample boundary point from **GMC measure** depending on h;
- Grow an independent radial SLE$_\kappa$ segment towards the origin for **capacity time** δ. (Conformal radius at time t is e^{-t}.)

This gives δ-QLE(γ^2, η) with $\eta = \frac{3}{\kappa} - \frac{1}{2}$.

Subsequential limit as $\delta \to 0$ gives capacity time QLE(γ^2, η).

Quantum Loewner evolution (Miller–Sheffield 2016)

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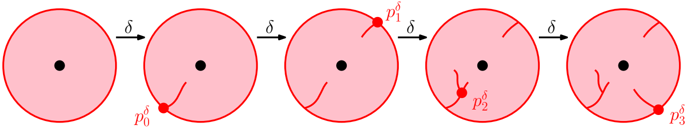

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Time parametrization does not depend on γ-LQG geometry!

QLE in quantum natural time (A.-Mithal 2026)

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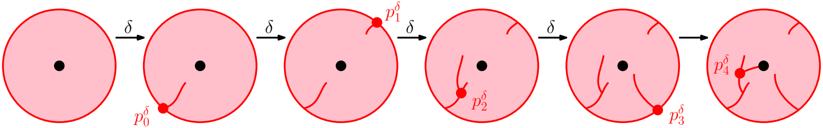

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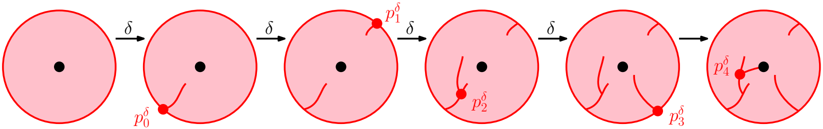

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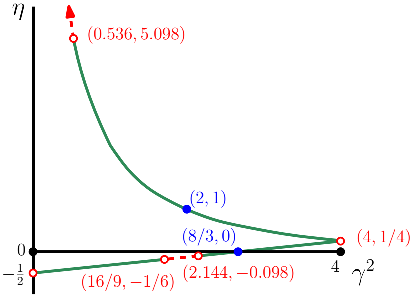
Subseq. limit as $\delta \to 0$ gives quantum natural time QLE(γ^2, η).

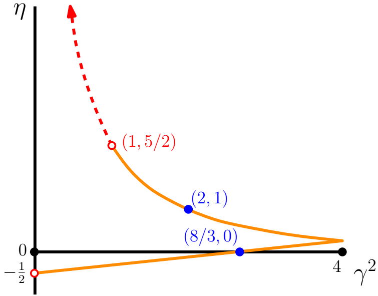
Previously constructed for $(\gamma^2, \eta) = (8/3, 0)$ by Miller-Sheffield '16.

Heuristically, each SLE is a size-δ particle in an η-DBM process.

Comparison of QLEs



Quantum natural time QLE (this paper)



Capacity time QLE (Miller-Sheffield '16)

Construction applies for different subsets of the same curve $\eta \in \{\frac{3}{\gamma^2} - \frac{1}{2}, \frac{3\gamma^2}{16} - \frac{1}{2}\}$.

(Are these two descriptions of the same object?)

Comparison of QLEs

Capacity time QLE solves an SDE that scaling limit should satisfy.

Quantum natural time QLE is closely aligned with discrete models, has nicer Markov properties.

- Its time parametrization should be the scaling limit of discrete growth model's intrinsic time parametrization.

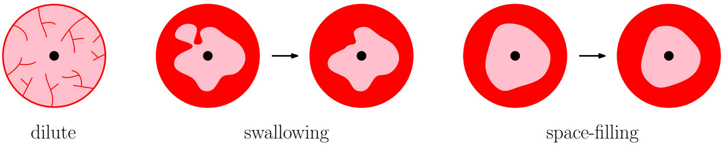
Comparison of QLEs

Capacity time QLE solves an SDE that scaling limit should satisfy.

Quantum natural time QLE is closely aligned with discrete models, has nicer Markov properties.

- Its time parametrization should be the scaling limit of discrete growth model's intrinsic time parametrization.
- Miller-Sheffield '16 used quantum natural time QLE(8/3,0)'s Markov properties to construct $\sqrt{8/3}$-LQG metric, prove agreement with Brownian map.

Let $(K_s)_{0\le s\le \mathcal{S}}$ be the quantum natural time QLE(γ^2,η) process, and $\mathcal{A}$ the γ-LQG area measure.



- (Dilute phase) For $\gamma \in (\sqrt{3}-1, 2)$ and $\eta = \frac{3}{\gamma^2} - \frac{1}{2}$, we have $\mathcal{A}(K_s) = 0$ for all s.
- (Swallowing phase) For $\gamma \in (2\sqrt{3}-2, 2)$ and $\eta = \frac{3\gamma^2}{16} - \frac{1}{2}$, the function $s \mapsto \mathcal{A}(K_s)$ is strictly increasing but has derivative zero almost everywhere.
- (Space-filling phase) For $\gamma \in (0, 4/3)$ and $\eta = \frac{3\gamma^2}{16} - \frac{1}{2}$, the function $s \mapsto \mathcal{A}(K_s)$ is continuous and strictly increasing.

Phases not yet established for capacity time QLE(γ^2,η) (question of Miller–Sheffield '16).

Markov properties of quantum time QLE (A.-Mithal 2026)

Let $(K_s)_{0 \leq s \leq \mathcal{S}}$ be a quantum natural time QLE(γ^2, η) process, and let L_s be the γ-LQG boundary length of $\mathbb{D} \backslash K_s$ at time s.

Stationarity

Let $s > 0$. Conditioned on the boundary length process up until time s, the unexplored surface has the same law as the original surface except with boundary length L_s.

Continuum analog of stationarity of DLA on ST weighted RPM!

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For capacity time QLE(γ^2,η), similar Markov property except that unexplored surface is *modulo additive constant* (Miller-Sheffield '16).

Cannot keep track of areas and lengths as the process evolves.

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Dilute phase: terminal state

The unexplored region at the end of the QLE exploration has the law of a γ-LQG quantum disk with boundary length $L_{\mathcal{S}}$.

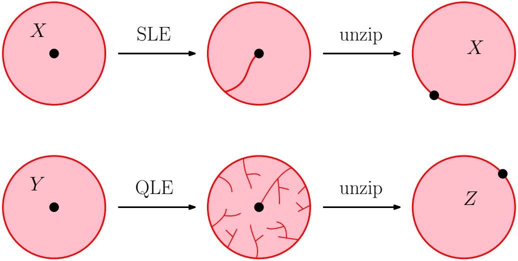
Swallowing phase: swallowed regions

Each jump of the boundary length process corresponds to a swallowed region. Conditioned on these jump sizes, the swallowed regions are conditionally independent γ-LQG disks with boundary lengths given by the jump sizes.

These Markov properties also have discrete analogs:
For dilute phase, DLA on spanning-tree weighted RPM.
For swallowing phase, is Eden model on uniform RPM.

Proof of dilute phase for quantum natural time QLE

Areas X, Y, Z.



Same initial state: $X \stackrel{d}{=} Y$.

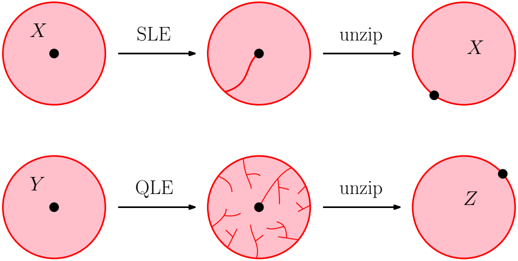
SLE has zero area, so unzipping SLE curve gives area X.

Markov property for QLE implies same final state in law: $X \stackrel{d}{=} Z$.

Therefore $Y \stackrel{d}{=} Z$.
Combining with $Y \geq Z$ a.s., the QLE trace has zero area.

Proof of dilute phase for quantum natural time QLE

Areas X, Y, Z.



Same initial state: $X \stackrel{d}{=} Y$.

SLE has zero area, so unzipping SLE curve gives area X.

Markov property for QLE implies same final state in law: $X \stackrel{d}{=} Z$.

Therefore $Y \stackrel{d}{=} Z$.
Combining with $Y \geq Z$ a.s., the QLE trace has zero area.

Harder to construct quantum natural time QLE than capacity time QLE, but given that proof of phase structure is relatively clear.

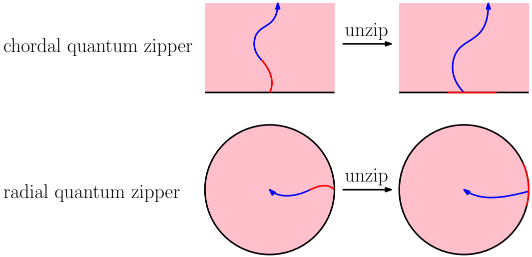
Quantum natural time QLE construction input

Sheffield's chordal **quantum zipper**:

stationarity under zipping and unzipping by SLE curve.

Capacity time zipper: Tricky martingale argument.

Quantum time zipper: Unzip for quantum time 1 and "zoom in".



Miller-Sheffield construction of capacity time QLE uses radial capacity time quantum zipper.

We build a radial quantum time quantum zipper.

Radial quantum time quantum zipper

Conformal weldings of random surfaces: SLE and the quantum gravity zipper — Sheffield '10

Liouville quantum gravity as a mating of trees — Duplantier-Miller-Sheffield '14

Liouville conformal field theory and the quantum zipper — A. '23

Reversibility of whole-plane SLE for $\kappa > 8$ — A.- Yu '23

Radial quantum time quantum zipper for $\kappa > 8$

Radial conformal welding in Liouville quantum gravity — A.- Yu '24

leads to radial quantum time quantum zipper for $\kappa \in (0,4) \cup (4,8)$

Open problems

Prove any scaling limit result for QLE.

• Only known for $(\gamma^2, \eta) = (8/3, 0)$: metric ball growth on uniform random planar map $\to$ QLE$(8/3, 0)$ on $\sqrt{8/3}$-LQG.

• DLA on spanning-tree weighted random planar map $\to$ QLE$(2, 1)$ on $\sqrt{2}$-LQG?

Does the construction for capacity time or quantum natural time QLE depend on the subsequential limit? Are the two QLEs the same?

Is QLE(γ^2, η) measurable with respect to the γ-LQG environment?

Happy 70th Birthday Greg!