Graph distance and effective resistance of random walk trace in four dimensions

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Several quantities generated by SRW

$S = (S_n)_{n\geq 0}$ is a simple random walk in $\mathbb{Z}^d$ started at the origin.

- $D_n =$ the shortest path graph distance (chemical distance) from 0 to S_n on $S[0,n]$.

  (Regard $S[0,n]$ as an unoriented random graph with no multiple edges.)

- $R_n =$ the effective resistance between 0 and S_n on $S[0,n]$.

- $L_n = m$, the length of LE$(S[0,n])$, where
  - $\sigma_0 = \max\{0 \leq k \leq n : S_k = 0\}$,
  - $\sigma_i = \max\{0 \leq k \leq n : S_k = S_{\sigma_{i-1}+1}\}$ for $i \geq 1$,
  - $m = \min\{i : \sigma_i = n\}$,
  - LE$(S[0,n]) := [S_{\sigma_0}, S_{\sigma_1}, \cdots, S_{\sigma_m}]$.

- Burdzy and Lawler (1990) asked the asymptotic behavior of $\mathbb{E}(D_n)$, $\mathbb{E}(R_n)$ and $\mathbb{E}(L_n)$. (Note that $R_n \leq D_n \leq L_n$.)

Rigorous exponent inequalities for random walks

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Abstract. The exponents for the resistance of a random walk path and the 'random walk on a random walk' problem are related to a number of other exponents for random walks. Some rigorous inequalities for these exponents are then established.

Let S be a simple random walk starting at the origin in $\mathbb{Z}^d$. For each n, we may consider $S[0, n]$ to be a subgraph of the integer lattice; more precisely, we let $\Gamma_n = (V_n, E_n)$ be the graph with

$$V_n = \{S_t\colon 0 \leq t \leq n\} \qquad E_n = \{\{S_t, S_{t+1}\}\colon 0 \leq t < n\}.$$

A number of papers (see Dekeyser *et al* 1987, Manna *et al* 1989 and references therein) have discussed the 'fractal nature' of the graph Γ_n. Two particular quantities of interest have been the effective resistance between 0 and S_n (assuming a unit resistance on each edge of Γ_n) and the mean-square displacement of another random walk restricted to lie on Γ_n. While numerical studies and conjectures have been made about these quantities, no rigorous mathematical statements have been made. In this letter we would like to define appropriate exponents and state what can be said rigorously about them.

The five exponents we discuss are the intersection exponent, the chemical or percolation exponent, the (chronological) loop-erasing exponent, the resistance exponent, and the exponent for mean-square displacement of a random walk on a random walk. All of these exponents have the following properties: they are dimension dependent; the value of the exponent is the same for all $d > 4$; logarithmic corrections

$d = 1$ and $d \geq 5$ are easy

- $\mathbb{E}(D_n) \asymp \mathbb{E}(R_n) \asymp \mathbb{E}(L_n) \asymp \sqrt{n}$ for $d = 1$.
- If S^1, S^2 are independent SRWs started at the origin,
$$\mathbb{P}\big(S^1[0,\infty) \cap S^2[1,\infty) = \emptyset\big) > 0 \text{ iff } d \geq 5.$$
- The number of cut times up to n,
$$C_n = |\{0 \leq k < n : S[0,k] \cap S[k+1,n] = \emptyset\}|.$$
- Note that $C_n \leq R_n \leq D_n \leq L_n$.
- $\mathbb{E}(C_n) \asymp \mathbb{E}(D_n) \asymp \mathbb{E}(R_n) \asymp \mathbb{E}(L_n) \asymp n$ for $d \geq 5$.

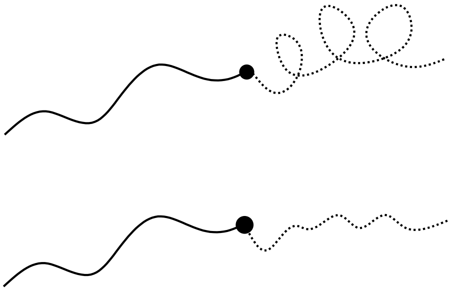
$d = 2$ and $d = 3$ are difficult

- LERW is well understood in 2D.

  Kenyon (2000), Schramm (2000), Lawler-Schramm-Werner (2004), Lawler-Viklund (2021), $\cdots$

- LERW is analyzable even in 3D!
  - Lawler (1999): $cn^{\frac{1}{2}+\epsilon} \leq \mathbb{E}(L_n) \leq Cn^{\frac{5}{6}}$.
  - Kozma (2007): Existence of the scaling limit (as a set).
  - S. (2018), Li-S. (2019): $\exists \alpha \in (1/2, 5/6]$ such that $\mathbb{E}(L_n) \asymp n^{\alpha}$.
  - Li-S. (2025), Hernández-Torres-Li-S. (2025): Convergence in the natural parametrization.
    - Angel-Croydon-Hernández-Torres-S. (2021): Existence of the scaling limit of the 3D UST under a Gromov-Hausdorff-Prokhorov-type topology.

$d = 2$ and $d = 3$ are difficult



- The domain Markov property of LERW helps a lot.
- Much less is known about D_n and R_n!

What about $d = 4$?

- A standard Brownian motion in $\mathbb{R}^4$ is a simple curve.
- SRW has a "logarithmic loop". That is, $\exists c \in (0,1)$ s.t.

$$\mathbb{P}\Big(\exists 0 \le i < j \le n \text{ s.t. } S_i = S_j \text{ and } j - i \ge n(\log n)^{-a}\Big) \\ = \begin{cases} o(1) & (0 < a < 1), \\ c + o(1) & (a = 1), \\ 1 - o(1) & (a > 1). \end{cases} \tag{1}$$

- Lawler (1991): $\exists \delta > 0$ s.t. $\forall n, k \ge 2$,

$$\mathbb{P}\big(S[0,n] \cap S[kn,(k+1)n] \neq \emptyset\big) \\ = \frac{1}{2\log n} \log\left[1 + \frac{1}{k^2 - 1}\right] \left\{1 + O\left(\frac{1}{k(\log n)^\delta}\right)\right\}. \tag{2}$$

Questions for $d = 4$

- Burdzy-Lawler (1990) conjectured that $\exists a_1, a_2 > 0$ such that

$$\mathbb{E}(D_n) = n(\log n)^{-a_1 + o(1)},$$
$$\mathbb{E}(R_n) = n(\log n)^{-a_2 + o(1)}.$$

  Here, the exact values of a_1 and a_2 were not predicted.

  — Lawler (1992): $\mathbb{E}(C_n) \asymp n(\log n)^{-1/2}$.

  — Lawler (1993): $\mathbb{E}(L_n) \asymp n(\log n)^{-1/3}$.

- Let X_n be either D_n or R_n.

Q1 Determine the logarithmic correction to $\mathbb{E}(X_n)$.

Q2 Study the fluctuations of $X_n - \mathbb{E}(X_n)$.

Main Results

Theorem (S.-Watanabe 2026)

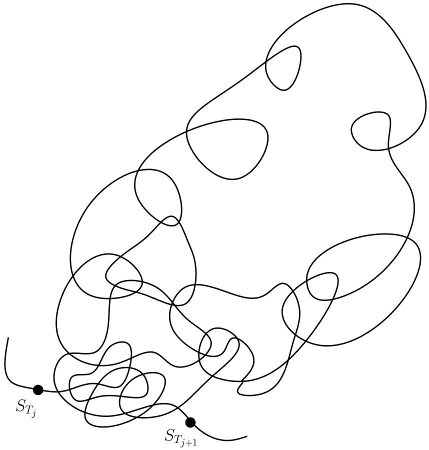
Let $d = 4$ and let X_n be either D_n or R_n. $\exists b, \delta \in (0, \infty)$ s.t.

$$\mathbb{E}(X_n) = b\, n(\log n)^{-\frac{1}{2}} \Big\{ 1 + O\Big(\big(\log n \big)^{-\delta} \Big) \Big\}. \tag{3}$$

- The constant b depends on whether we choose D_n or R_n.
- As a corollary, for fixed $N \geq 1$, by "Taylor expansion".

$$N\, \mathbb{E}(X_{n/N}) - \mathbb{E}(X_n) = \frac{b \log N}{2} \cdot n(\log n)^{-\frac{3}{2}} \Big\{ 1 + O\Big((\log n)^{-\delta} \Big) \Big\}. \tag{4}$$

Intuition for the first theorem



A shape of $S[T_j, T_{j+1}]$ when $T_{j+1} - T_j$ is large.

Main Results

Theorem (Adhikari-Okada-S. 2026+)

Let X_n be either D_n or R_n. For $d = 5$,

$$\frac{X_n - \mathbb{E}(X_n)}{n^{\frac{2}{3}}} \Rightarrow \frac{3}{2}\text{-stable (skewness } -1). \tag{5}$$

For $d = 4$, $\exists c_0 > 0$, s.t.

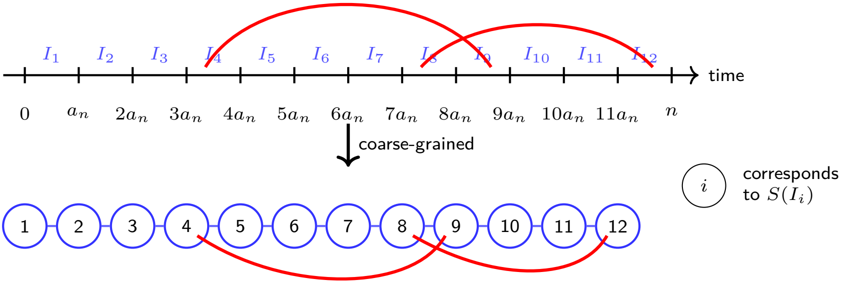
$$\frac{X_n - \mathbb{E}(X_n) - c_0 n(\log n)^{-3/2} \log\log n}{n(\log n)^{-3/2}} \Rightarrow 1\text{-stable (skewness } -1). \tag{6}$$

- Adhikari-Okada (2026): For $d \geq 6$, $X_n - \mathbb{E}(X_n)$, after appropriate rescaling, converges to a normal distribution.

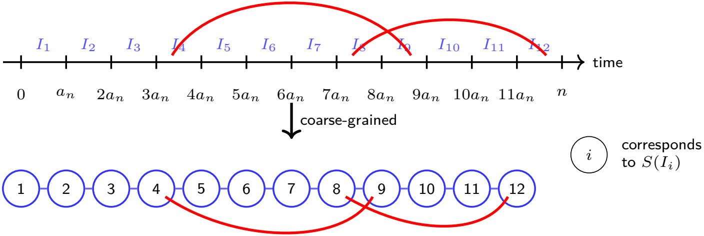
Proof sketch for $d = 4$: coarse-graining

- Consider the case $X_n = D_n$.
- The proof relies on a coarse-graining argument.
- Need to estimate

$$D_n - \sum_{i=1}^{N} \{\text{local cost in } S(I_i)\}.$$



Proof sketch for $d = 4$: main tasks



Main tasks:

(i) The local cost required to move within each block $S(I_i)$ can be approximated by its expectation.

(ii) Estimate the graph distance between 1 and N in the coarse-grained graph.

The rigorous justification of (i) is delicate, but I will focus on (ii).

Coarse-grained graph G_1 and LRP G_2

- Choose $a_n = \epsilon n(\log n)^{-1}$ and $N = \dfrac{n}{a_n} = \epsilon^{-1}\log n$.
- Define the coarse-grained graph G_1 by
  - vertex set $\{1, 2, \ldots, N\}$; vertex j corresponds to the block $S(I_j)$;
  - (j, k) is an edge if $S(I_j) \cap S(I_k) \neq \emptyset$ and $j \neq k$;
  - nearest-neighbor edges $(j, j+1)$ are always included.
- Let $\mathcal{D}_1$ be the graph distance between 1 and N in G_1.
- Define independent LRP G_2 on the same vertex set $\{1, 2, \ldots, N\}$ with

$$p_{j,j+1} = 1, \qquad p_{j,j+k} = \frac{1}{2\log n}\log\left(1 + \frac{1}{k^2 - 1}\right), \quad k \geq 2.$$

- Let $\mathcal{D}_2$ be the graph distance between 1 and N in G_2.

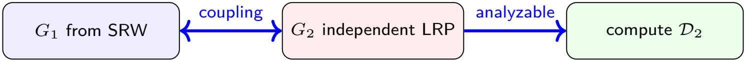
Coupling G_1 with G_2

Theorem (Adhikari–Okada–S. 2026+)

Fix $\epsilon \in (0,1)$. There exists a coupling of G_1 and G_2 such that, as $n \to \infty$,

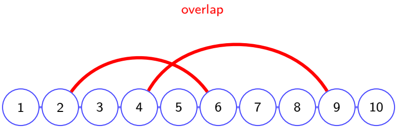
$$\mathbb{P}(G_1 = G_2) \longrightarrow 1.$$

Thus it suffices to analyze the graph distance $\mathcal{D}_2$ in the independent LRP G_2.

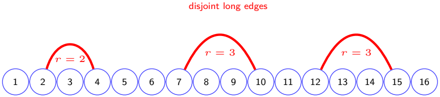


Typical shape of G_2: long edges do not overlap

Atypical: overlapping long edges



Typical: disjoint long edges



- With high probability, the bad configurations on the left do not occur. Thus,

$$\mathcal{D}_2 = (N-1) - \sum_{r=2}^{N-1} (r-1) N_r, \qquad N_r := \#\{\text{edges of length } r \text{ in } G_2\}.$$

- Let $p_r := p_{1,1+r}$. In the LRP model, for fixed $r \geq 2$,

$$N_r \overset{d}{=} \mathrm{Bin}\left(N - r, p_r\right) \approx \mathrm{Poi}\left(\frac{1}{2\epsilon r^2}\right), \qquad N_2, N_3, \ldots \text{ are independent.}$$

(Recall that $p_r = \frac{1}{2 \log n} \log\{1 + 1/(r^2 - 1)\}$ and $N = \epsilon^{-1} \log n$.)

- Let $n \to \infty$ first, and then let $\epsilon \downarrow 0$. The desired result can be obtained.

- One also needs to estimate the expectation of $D_n - \sum_{i=1}^{N} \{\text{local cost in } S(I_i)\}$. We use

$$N\,\mathbb{E}(X_{n/N}) - \mathbb{E}(X_n) = \frac{b \log N}{2} \cdot n(\log n)^{-\frac{3}{2}} \Big\{1 + O\Big((\log n)^{-\delta}\Big)\Big\}.$$

A brief comment on $d = 5$

- The proof follows the same coarse-graining strategy as in $d = 4$.
- Choose
$$a_n = \epsilon n^{2/3}, \qquad N = \epsilon^{-1} n^{1/3}.$$
The scale $n^{2/3}$ is the intersection borderline.
- Define the coarse-grained graph and couple it with an independent LRP model, exactly as in the four-dimensional case.
- In the LRP picture,
$$\mathcal{D}_2 \simeq N - \sum_{r \geq 2} r N_r,$$
and the numbers N_r have heavy-tailed behavior of order $r^{-5/2}$.
- Therefore,
$$\frac{X_n - \mathbb{E}(X_n)}{n^{2/3}}$$
converges to a left-skewed $\frac{3}{2}$-stable law.

Open problems

- In $d \geq 4$, the LRP approximation works at the scales used above. However, if a_n is taken much smaller, intersections between nearby blocks form dependent clusters, and the coarse graph G_1 can no longer be approximated by LRP. (Once an intersection occurs, nearby intersections become more likely.)
- Q: For $d \geq 4$, where is the breakdown scale? More precisely, determine the threshold for a_n at which the approximation $G_1 \approx G_2$ stops being valid.
- In $d = 2, 3$, intersections are too dense: no choice of a_n gives an independent-LRP approximation of this type. At present it is unclear how to analyze the coarse graph G_1 and hence D_n and R_n in these dimensions.

  The above question is a natural warm-up for the $d = 2, 3$ problem, since the same local-dependence mechanism becomes even stronger in lower dimensions.
- Q: What are the fluctuations of the LERW length L_n? In particular, is there a stable-law picture analogous to D_n and R_n in $d = 4, 5$, or does loop-erasure lead to a different fluctuation behavior?

Greg, happy 70th birthday!