Uniform spanning trees, winding, and Brownian loop measure

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Greg Lawler's 70th Birthday
July 2026 @ IMSI, Chicago

Joint work with N. Berestycki & M. Lis & M. Liu

FINNISH CENTRE OF EXCELLENCE IN RANDOMNESS AND STRUCTURES 2022–2029
ACADEMY OF FINLAND
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ML4Q MATTER AND LIGHT FOR QUANTUM COMPUTING

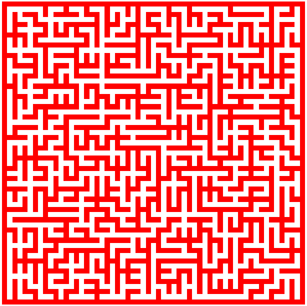
Is there anymore anything new one can say about UST in 2D?

We consider a 2D uniform spanning tree (UST) under an n-arm event. The total winding of the branches has an exact asymptotic formula, which involves the Brownian loop measure and — interestingly — essentially depends on the total number of branches only through its parity. Not surprisingly, the branches converge to multiradial SLE(2), and the law of the hitting points of the branches to the boundary coincides with that of eigenvalues of an Circular Orthogonal Ensemble (COE) random matrix.

Joint work [arXiv:2512.20540] with

Nathanaël Berestycki, Marcin Lis, and Mingchang Liu.

Uniform spanning tree model (UST)



- Consider finite, connected graph $G = (V, E)$
- **Spanning tree** on G
  = connected subgraph T of G which contains all vertices and has no cycles
- **Uniform Spanning Tree (UST)** on G
  = *uniformly chosen* such $T \subset G$

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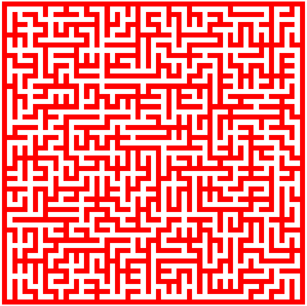

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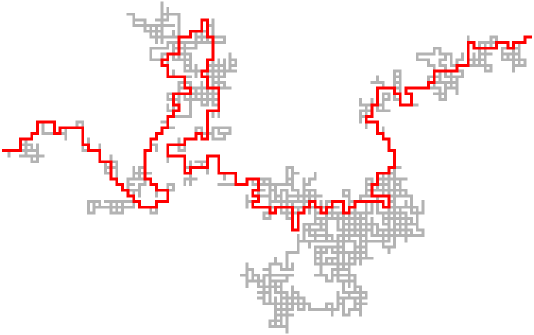
- Uniform: each tree has equal weight
- G finite $\implies$ $\exists$ finitely many spanning trees T

  (In fact, they can be counted using graph Laplacian.)
- Can impose various *boundary conditions*: wired (D), free (N)
- UST branch $\overset{(d)}{=}$ LERW [Pemantle '91, Wilson '96]

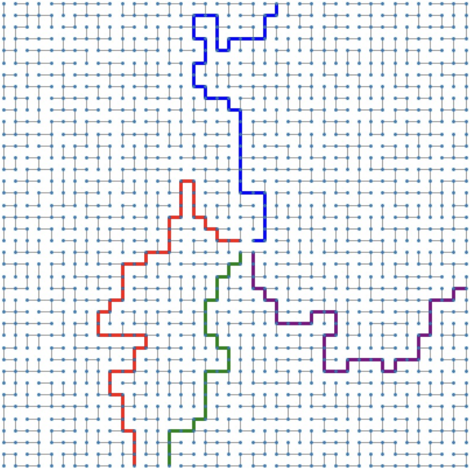
Loop-erased random walk (LERW) [Greg '80]

- Consider finite, connected graph $G = (V, E)$
- **Simple random walk** $S_n = X_0 + X_1 + X_2 + \cdots$ on G
- Loop-erasure $\rightsquigarrow$ A "self-avoiding" random walk (LERW $\neq$ SAW)



- Scaling limit $\rightsquigarrow$ SLE(2) curve [Lawler & Schramm & Werner '04]

n-ARM EVENT



Neighboring branches

$\gamma = (\gamma_1, \ldots, \gamma_n)$

do not intersect

[...; Benjamini '99; Kenyon '00, ...]

Wired (absorbing) outer boundary

n-ARM EVENT

Neighboring branches

$\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$

do not intersect

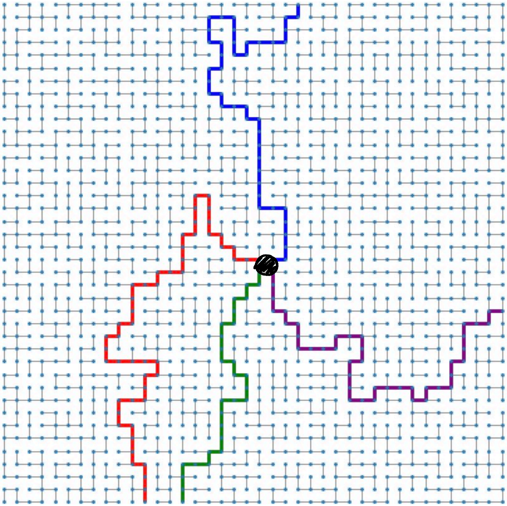
$\Longleftrightarrow$

Boundary-to-boundary

branches in annulus Ω

conformally equivalent

to $\mathbb{A}(r, 1)$, small $r > 0$



Free (reflecting) inner boundary; Wired (absorbing) outer boundary

n-ARM EVENT

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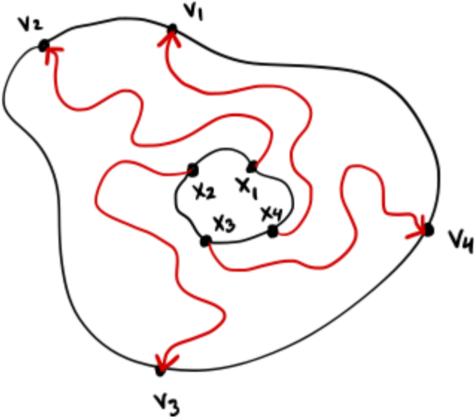
Boundary-to-boundary branches in annulus Ω conformally equivalent to $\mathbb{A}(r, 1)$, small $r > 0$



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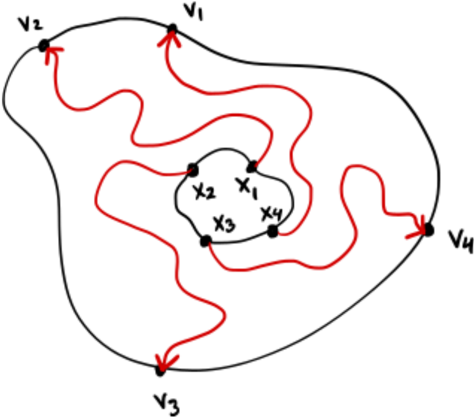
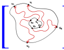
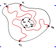
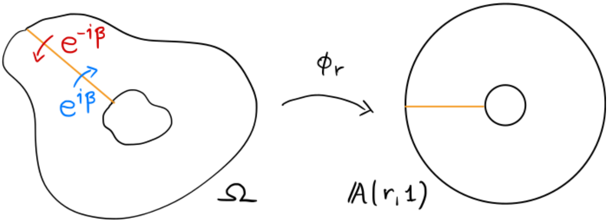


$\mathbb{P}\Big[$  $\Big] \sim r^{\frac{n^2-1}{4}}$

$\mathbb{P}\Big[$  $\Big] = \mathbb{P}\left[E_{\boldsymbol{x},\boldsymbol{v}}\right] =$?? when $\boldsymbol{v} = (v_1, \ldots, v_n)$ fixed

Keeping track of winding around $\partial_{\text{in}}\Omega$

Add weight $\beta \in \mathbb{R}$ across cut (more precisely, $e^{\pm i\beta}$):



"Excursion kernel" : $h_\beta(x,v) := \sum_{P:\, x\to v} \underbrace{w_\beta(P)}_{\text{weight of path } P}$

Thm. Consequence of [Fomin '01] For $\beta \in \mathbb{R}$, we have

$$\det\left(h_\beta(x_i,v_j)\right)_{1\le i,j\le n} = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) \sum_{\substack{P_1,\ldots,P_n \\ P_j:\, x_j\to v_{\sigma(j)}\ \forall\, j \\ \mathrm{LE}(P_i)\cap P_j=\emptyset\ \forall\, i<j}} w_\beta(P_1)\cdots w_\beta(P_n)$$

Fomin's formula in annulus

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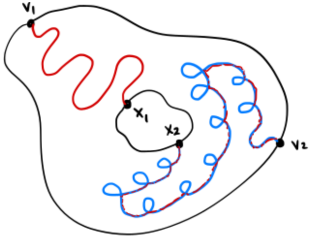

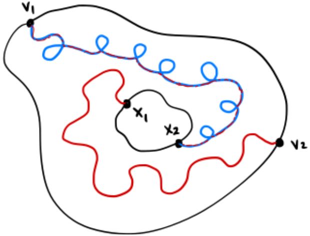

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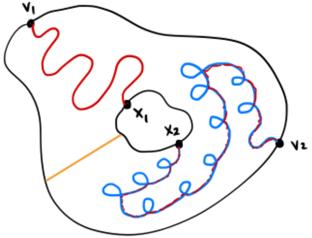

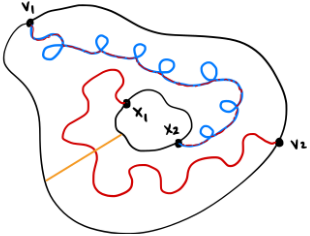

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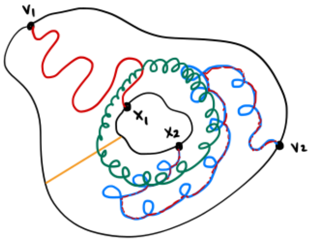

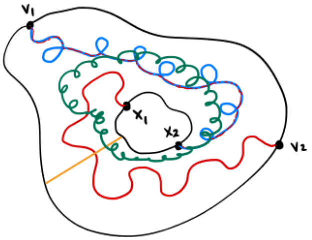

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Idea: add loops back to LERW [Lawler & Trujillo Ferreras '04; ...]

Prop. [Berestycki, Lis, P., Liu '25] Explicit formula:

- n odd:

$$\mathbb{P}\left[E_{\boldsymbol{x},\boldsymbol{v}}\right] = \det\left(h_0(x_i, v_j)\right)_{1\leq i,j\leq n}$$

- n even:

$$\mathbb{P}\left[E_{\boldsymbol{x},\boldsymbol{v}}\right] = \exp\left(\Lambda_0^{\mathrm{ND}}[\mathcal{L}_*] - \Lambda_\pi^{\mathrm{ND}}[\mathcal{L}_*]\right) \det\left(h_\pi(x_i, v_j)\right)_{1\leq i,j\leq n}$$

- "Excursion kernel" : $h_\beta(x,v) := \sum_{P:\ x\to v} \underbrace{w_\beta(P)}_{\text{weight of path } P}$
- $\Lambda_\beta^{\mathrm{ND}}[\mathcal{L}_*] := |$non-contractible RW loops w/ weight β across the cut$|$
- (Here, $\partial_{\mathrm{in}}\Omega$ has reflecting (N) and $\partial_{\mathrm{out}}\Omega$ absorbing (D) b.c.)

Branches: total winding $\mathrm{wind}(\boldsymbol{\gamma}) := \sum_{j=1}^{n} \mathrm{wind}(\gamma_j)$

Thm. [Berestycki, Lis, P., Liu '25]

Characteristic function $\mathbb{E}_{\boldsymbol{x},\boldsymbol{v}}[e^{\mathrm{i}\beta\, \mathrm{wind}(\boldsymbol{\gamma})}]$ has an explicit formula:

- n odd:

$$\exp\left(\Lambda_0^{\mathrm{ND}}[\mathcal{L}_*] - \Lambda_\beta^{\mathrm{ND}}[\mathcal{L}_*]\right) \frac{\det\left(h_{2\pi\beta}(x_i, v_j)\right)_{1\le i,j\le n}}{\det\left(h_0(x_i, v_j)\right)_{1\le i,j\le n}}$$

- n even:

$$\exp\left(\Lambda_\pi^{\mathrm{ND}}[\mathcal{L}_*] - \Lambda_{2\pi\beta+\pi}^{\mathrm{ND}}[\mathcal{L}_*]\right) \frac{\det\left(h_{2\pi\beta+\pi}(x_i, v_j)\right)_{1\le i,j\le n}}{\det\left(h_\pi(x_i, v_j)\right)_{1\le i,j\le n}}$$

See also [Kenyon '00] for arguments using relation to dimers.

Scaling limit: Take $\delta \to 0$, then $r \to 0$

Thm. [Berestycki, Lis, P., Liu '25]

For each $\beta \in (0, 1/2)$, the limit $\lim_{\delta\to 0} \mathbb{E}^{\delta}_{\boldsymbol{x},\boldsymbol{v}}[e^{\mathrm{i}\beta\,\mathrm{wind}(\boldsymbol{\gamma}^{\delta})}]$ with $r \sim 0$ is expressed using **Brownian loop measure** $\mu^{\mathrm{ND}}_{\Omega}$:

- n odd:

$$\left(e^{\mathrm{i}\beta\, c(\Omega,\boldsymbol{x},\boldsymbol{v})} + o_r(1)\right) r^{\beta} \exp\left(\mu^{\mathrm{ND}}_{\Omega}\left[\left(1 - e^{\mathrm{i}\beta\,\mathrm{wind}(\ell)}\right) \mathbf{1}\{\ell \in \mathcal{L}_*\}\right]\right)$$

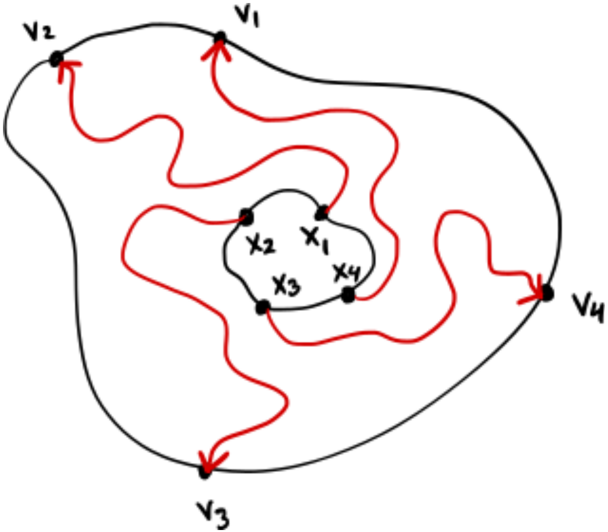
- n even:

$$\left(e^{\mathrm{i}\,(\beta+\frac{1}{2})\, c(\Omega,\boldsymbol{x},\boldsymbol{v})} + o_r(1)\right) \exp\left(\mu^{\mathrm{ND}}_{\Omega}\left[e^{\mathrm{i}\,\mathrm{wind}(\ell)/2}\left(1 - e^{\mathrm{i}\beta\,\mathrm{wind}(\ell)}\right) \mathbf{1}\{\ell \in \mathcal{L}_*\}\right]\right)$$

- RW loops $\xrightarrow{\delta\to 0}$ Brownian loops cf. [Lawler & Trujillo Ferr. '04]
- Computation of scaling limit of h_{β} [Arista & O'Connell '19]
- $c(\Omega, x, v) = \sum_{j=1}^{n}\left(\arg\phi(x_j) - \arg\phi(v_j)\right)$ $\qquad$ $\phi\colon \mathrm{fill}(\Omega) \to \mathbb{D}$ conf. bij.

Scaling limit of endpoints

Let now $\boldsymbol{v} = (v_1, \ldots, v_n)$ be random but fix still $\boldsymbol{x} = (x_1, \ldots, x_n)$.



Scaling limit of endpoints: COE

Let now $\boldsymbol{v}$ be random but fix still $\boldsymbol{x}$. Suppose $\Omega = \mathbb{A}(r, 1)$ for ease.

Thm. [Berestycki, Lis, P., Liu '25] Law of hitting points $\boldsymbol{v} = e^{\mathrm{i}\boldsymbol{\theta}}$ converges as $\delta, r \to 0$ and has density on $\partial\mathbb{D} = \mathbb{S}$ given by

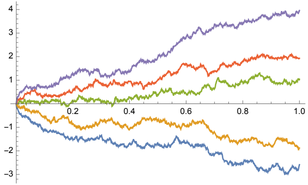
$$\rho(e^{\mathrm{i}\boldsymbol{\theta}}) := \frac{1}{\mathcal{Z}_n} \mathbf{1}\{e^{\mathrm{i}\boldsymbol{\theta}} \in \mathcal{X}_n\} \prod_{1 \leq j < k \leq n} \left| e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k} \right| \prod_{j=1}^{n} \mathrm{d}\theta_j$$

$$\mathcal{X}_n := \left\{ (e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_n}) \in \mathbb{S}^n \mid \theta_1 < \theta_2 < \cdots < \theta_n < \theta_1 + 2\pi \right\}.$$

- Coincides with **eigenvalue distribution** of $(n \times n)$-random matrix from **Circular Orthogonal Ensemble** (COE). [Dyson '62, ...]
- Given the endpoints, the law of the UST branches $\boldsymbol{\eta}$ (with common parameterization) converges as $\delta, r \to 0$ to **n-sided radial SLE(2)** whose driving function is DBM (on $\mathbb{S}$). [Cardy '03, ...]

DYSON BROWNIAN MOTION (DBM) ON THE CIRCLE

Eigenvalues of Circular β-Ensemble



- Repelling diffusions on $\mathbb{S}$
  $(\exp(\mathrm{i}\Theta_1(t)), \ldots, \exp(\mathrm{i}\Theta_n(t)))$

- Angle process $(\Theta_1(t), \ldots, \Theta_n(t))$

$$\mathrm{d}\Theta_i(t) = \sqrt{\kappa}\,\mathrm{d}W_i(t) + \beta\,\frac{\kappa}{4}\sum_{j\neq i}\cot\Big(\frac{\Theta_i(t)-\Theta_j(t)}{2}\Big)\,\mathrm{d}t + \mu\,\mathrm{d}t, \qquad 1 \le i \le n,$$

- $W_1, \ldots, W_N$ independent standard Brownian motions

- Variance parameter $\kappa > 0$, *repulsive* interaction when $\beta \ge 1$.

- Spiraling rate $\mu \in \mathbb{R}$ (usually $\mu = 0$)

- **n-sided radial SLE(2): $\kappa = 2$ and $\beta = 4$** (NB: $\beta = 8/\kappa$)

- **Endpoint distribution:** $\beta = 1$ (so secretly $\kappa = 8$, dual to $\kappa = 2$)

Asymptotic winding of n-sided radial SLE(κ)

Consider the curves $\boldsymbol{\eta}$ in $\mathbb{D}$ with common parameterization:

$$-\log \mathrm{CR}(0; \mathbb{D} \setminus \boldsymbol{\eta}[0,t]) = nt$$

Thm. [Berestycki, Lis, P., Liu '25] Winding of each curve η^j around the origin satisfies

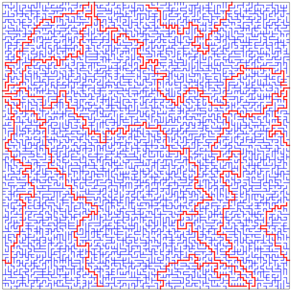
$$\frac{\mathrm{wind}(\eta^j[0,t])}{\sqrt{\kappa t/n}} \to N(0,1) \quad \text{as } t \to \infty.$$

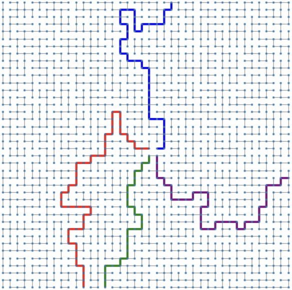
Thus, **variance of the winding** as $t \to \infty$ is $\sim \kappa/n^2$ (between two radial scales e^{-s} and e^{-s-1}).

- Multi-curve version of Schramm's result with $\kappa = 2$ [Schramm '00]
- Predicted also by [Wieland & Wilson '03]
- Cases $n = 2, 3$ (up to a factor): [Kenyon '00]
- Proof: Easy computation of variance of DBM

UST BRANCHES — SLE(2) VARIANTS ($c = -2$ CFT)

- **Partition functions** from probas of topological events (or from Coulomb gas), CFT primary fields of types $\phi_{1,2}, \phi_{1,s}$ or $\phi_{0,n/2}$



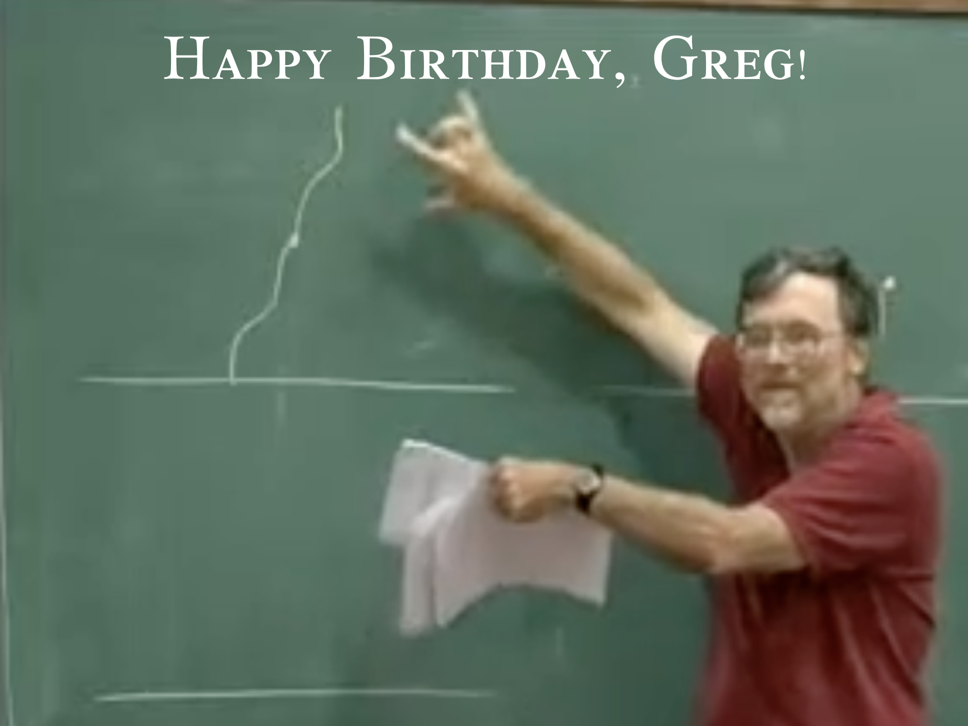


- GFF couplings? For n-SLE(κ) with spiral μ, we have the field

$$\Gamma(\cdot) + \left(\frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2} + \frac{n}{\sqrt{\kappa}}\right)\arg(\cdot) + \mathfrak{h}_{\theta}(\cdot) + \frac{\mu}{\sqrt{\kappa}}\log|\cdot|$$

where $\mathfrak{h}_{\theta}(w) = -\frac{2}{\sqrt{\kappa}}\sum_{j=1}^{n}\arg(e^{i\theta_j} - w) + \frac{1}{\sqrt{\kappa}}\sum_{j=1}^{n}\theta_j$

- Relation to dimer height function on surfaces... etc.


Happy Birthday, Greg!