Power-Law Rates of Convergence to SLE

A model-independent framework, with application to Percolation and Harmonic Explorer

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Based on joint work with L. Chayes (UCLA), D. Chelkak (University of Michigan), H. Lei, and L. Richards (University of Leeds)

Random Explorations

July 6, 2026

Roadmap

The Question

- The scheme for proving convergence of the critical interface: build an observable, use conformal invariance to identify the scaling limit *through the observable* (Smirnov for Percolation; Lawler–Schramm–Werner for LERW $\to$ SLE_2; Schramm–Sheffield for Harmonic Explorer; Chelkak–Smirnov for Ising/FK-Ising; ...).

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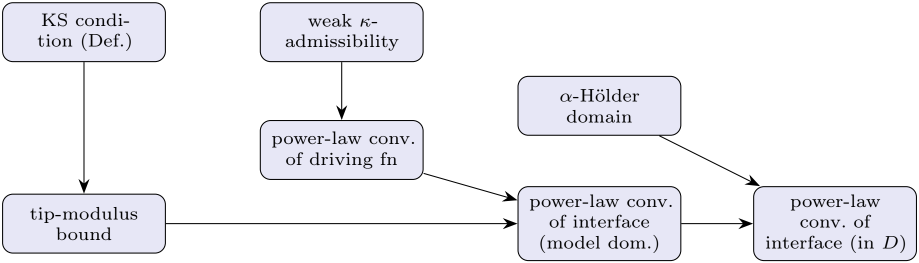
  - Beneš–Johansson Viklund–Kozdron: first power-law rate (LERW);
  - Viklund: a path rate from a *driving-function* rate, given crossing data via the *tip structure modulus*.

This talk

A model-independent machine: a power-law rate for a martingale observable, plus a geometric tightness condition, yields a power-law rate for the *interface*. The hard input in Viklund's framework — the tip-modulus bound — comes *for free* from the tightness condition (KS condition).

The Map of the Argument



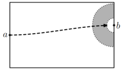
Two disjoint arguments meet at the interface.

- **Framework I** (right): observable ⇒ driving function — *no geometry*.
- **Framework II** (left): KS ⇒ tip-modulus bound ⇒ Viklund ⇒ path.
- Hölder regularity upgrades parameter-plane convergence to convergence in D.

Roadmap

The Kemppainen–Smirnov Condition

A crossing of $A(z_0, r, R)$ is *unforced* if it lies in a component of $A \cap D_\tau$ that does not disconnect $\gamma(\tau)$ from b — the curve could have avoided it.



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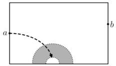

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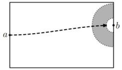

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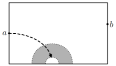

unforced

Definition (KS condition)

$\exists\, C > 1$ such that for all n, all stopping times τ, and all annuli $A(z_0, r, R)$ with $Cr < R$, $\mathbb{P}_n\big(\gamma^n[\tau, 1]$ makes an unforced crossing of $A \mid \gamma^n[0, \tau]\big) < \frac{1}{2}$.

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Theorem (Kemppainen-Smirnov)

Any family of random curves satisfying the Kemppainen-Smirnov (KS) condition is tight; all the limit points lie on Hölder continuous Löwner curves (+ some bounded exponential moments of driving functions).

Conformally Covariant Martingale Observables

Definition (martingale observable)

$F(D; a_1, \ldots, a_n)$ is a *conformally covariant martingale observable* if it transforms with covariance exponents $(\alpha, \alpha', \beta_j, \beta_j')$ under conformal maps:

$$F(\varphi(D); \varphi(a_1), \cdots, \varphi(a_n))(\varphi(z)) = \\ \varphi'(z)^{\alpha}\overline{\varphi'(z)^{\alpha'}}\varphi'(a_1)^{\beta_1}\overline{\varphi'(a_1)^{\beta_1'}}\cdots F(D; a_1, \cdots, a_n)(z).$$

and $F\big(D \setminus \gamma[0,t]; \gamma(t), a_2, \ldots\big)$ is a martingale along the Loewner flow. It is *invariant* if all exponents vanish.

Theorem (Martingale Principle; Schramm-Smirnov)

A random Loewner curve with the domain Markov property that admits a nontrivial conformally covariant martingale F is SLE_κ*. κ is determined by F.*

In the discrete world this fixes the target: build observables converging to such an F. We use *invariant* observables to keep notation light.

The BPZ-type Equation

Lemma (necessary PDE)

A C^2 conformally invariant martingale observable $F(\mathbb{H}; 0, \infty, z_1, \ldots, z_m; c_1, \ldots, c_l)$ for SLE_κ $(z_j = x_j + iy_j)$ satisfies

$$\sum_{j=1}^{m} \frac{x_j \partial_{x_j} F - y_j \partial_{y_j} F}{x_j^2 + y_j^2} + \sum_{j=1}^{l} \frac{1}{c_j}\, \partial_{c_j} F + \frac{\kappa}{2}\Big(\sum_{j=1}^{m} \partial_{x_j}^2 F + \sum_{j=1}^{l} \partial_{c_j}^2 F \Big) = 0.$$

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$F(\hat{\xi}_t)$ a martingale $\Rightarrow$ the dt-drift vanishes. □

Non-degeneracy: the first-order part is not a constant multiple of the second-order part — exactly what makes the key estimate work.

Discrete Approximation

$\Gamma^n = (V^n, E^n)$ – an embedded planar graph with vertices $V^n \subset D$ and straight edges E^n; the interior of the union of faces, D^n, is a simply connected subdomain of D.

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The domain $D^n \subset D$, created from a graph Γ^n with two fixed vertices $a^n,\ b^n \in \partial D^n \cap V^n$ is called *n-th discrete approximation of D with respect to* $w_0 \in D$ if

1. $w_0 \in D^n$
2. For any edge $e \in E^n$, $\operatorname{diam}(e) \leq 1/n$.
3. For any face f of Γ^n, $\operatorname{diam}(f) \leq 1/n$.
4. For any $z \in \partial D^n$ one can find $w \in \partial D$ with $|z - w| \leq 1/n$.
5. a^n and b^n are $\frac{1}{n}$ close to a and b respectively, i.e. one can find two crosscuts of D, γ_a, γ_b of diameter at most $1/n$ such that γ_a separates a and a^n from w_0 and γ_b separates b and b^n from w_0.

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A *discrete curve* in D^n will be a simple curve that consists of edges from E^n, starts at a^n, ends in b^n, and contains no other boundary vertices, i.e, no vertices $v \in V^n \cap \partial D^n$.

Weak κ-Admissibility

Definition (weak κ-admissibility)

A domain-Markov family of discrete curves $\{\gamma^n_{(D,a,b)}\}$ is *weakly κ-admissible* if $\exists\, m, \ell \in \mathbb{N}, s \in (0,1), n_0(R)$ so that for $\operatorname{crad}(\phi) > R$ there is a multivariable observable $H^n_D \colon (D^n)^m \times (\partial D^n)^l \to \mathbb{C}$ with

1. **almost martingale:** $\left|H^n_{D_t}(\hat{v}) - \mathbb{E}[H^n_{D_{t'}}(\hat{v}) \mid D_t]\right| \le n^{-s}$, $t' \ge t$;
2. **continuous limit:** a non-degenerate C^3 solution h of the BPZ-type equation with parameter κ, defined on an open set $\mathcal{V} \subset \mathbb{H}^m \times (\mathbb{R} \setminus \{0\})^l$;
3. **power-law closeness:** $\left|H^n_D(\hat{v}) - h(\hat{\phi}^n(\hat{v}))\right| \le n^{-s}$ for $\hat{\phi}^n(\hat{v}) \in \mathcal{V}$.

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By the previous lemma, clause (2) is not restrictive: a genuine SLE observable already solves BPZ.

Main Theorem

Setup. $\tilde{\gamma}^n = \phi^n(\gamma^n)$ in $(\mathbb{H}, 0, \infty)$, capacity parameterization; d_* makes $\mathbb{H} \cup \{\infty\}$ compact; $\tilde{\gamma}(t)$ SLE_κ in $(\mathbb{H}, 0, \infty)$, $T_\rho = T \wedge \min\{t : \tilde{\gamma}(t) > \rho\}$.

Theorem (B.–Richards)

Let $\{\gamma^n_{(D,a,b)}\}$ be κ-admissible, $\kappa \in (0, 8)$. There is a coupling of $\tilde{\gamma}^n$ with $\sqrt{\kappa}\, B$ such that for $n \geq N$,

$$\mathbb{P}\left(\sup_{t \in [0, T_\rho]} d_*\big(\tilde{\gamma}^n(t), \tilde{\gamma}(t)\big) > n^{-u} \right) < n^{-u},$$

with $u \in (0, 1)$ depending only on s and the κs-modulus bound.

Corollary (Hölder refinement)

If D is α-Hölder, the same coupling gives
$\mathbb{P}\big(\sup_{[0,T_\rho]} |\gamma^n(t) - \phi^{-1}(\tilde{\gamma}(t))| > n^{-\lambda}\big) < n^{-\lambda}$, *with $\lambda = \lambda(\alpha, u)$.*

Upshot: a path rate follows from any observable that converges as fast.

Roadmap

From Observable to Driving Term (no KS)

Three scales compound into the driving-term rate:

- **microscopic** — observable error at edge size n^{-1};
- **mesoscopic** — transferred to the driving function over capacity pieces of size $n^{-2/3}$;
- **Skorokhod** — coupling with Brownian motion costs an extra $\frac{1}{3}$ in the exponent, $\sim n^{-(2/3)(1/3)}$: the price of converting L^2 (quadratic-variation) control of the increments into a *uniform-in-time* coupling.

Theorem (B.–Richards)

For a weakly κ-admissible family ($\kappa > 0$) there is $r = r(s) > 0$ and a coupling of γ^n with Brownian motion such that

$$\mathbb{P}\left(\sup_{t\in[0,T_\rho]} |W^n(t) - W(t)| > n^{-r}\right) < n^{-r}, \qquad W(t) = B(\kappa t).$$

The KS condition is not used here, and the statement holds for all $\kappa > 0$.

The Key Estimate

Stop a mesoscopic piece at

$$m = \min\Big\{K(D),\ \min\{j \geq k:\ t_j - t_k \geq n^{-2s/3} \text{ or } |W(t_j) - W(t_k)| \geq n^{-s/3}\}\Big\}.$$

Proposition (key estimate)

Under weak κ-admissibility, for $n > n_0$ and $k < K(D)$,

$$\begin{aligned} K(D, \hat{p}) &\geq C(R, U)\, n, \\ \Big|\mathbb{E}[W(t_m) - W(t_k) \mid \gamma[0,k]]\Big| &\leq c\, n^{-s}, \\ \Big|\mathbb{E}[(W(t_m) - W(t_k))^2 - \kappa\,(t_m - t_k) \mid \gamma[0,k]]\Big| &\leq c\, n^{-s}. \end{aligned}$$

$K(D)$ is the first time a marked point leaves the chart $\mathcal{V}$; the bound $K \gtrsim n$ (Beurling projection) lets us iterate the estimate.

Why the Key Estimate Holds

Sketch.

Almost-martingale + closeness transport the observable:

$$\mathbb{E}[h(\hat{\phi}_m(\hat{p}_0) - \hat{W}_{t_m}) \mid D_k] = h(\hat{\phi}_k(\hat{p}_0) - \hat{W}_{t_k}) + O(n^{-s}).$$

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Taylor-expand $h \in C^3$ (1st order in $\hat{z}$, since $|\Delta z| = O(n^{-2s/3})$; 2nd order in W, since $|\Delta W| = O(n^{-s/3})$) and take $\mathbb{E}[\cdot \mid D_k]$:

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Two facts finish the argument:

- h solves BPZ $\Rightarrow A(\xi) = -\kappa\, C(\xi)$;
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Solving the 2×2 system isolates $\mathbb{E}[\Delta W]$ and $\mathbb{E}[(\Delta W)^2 - \kappa \Delta t]$. □

Skorokhod Embedding

Lemma (Skorokhod)

For a martingale (M_k) *with* $\|M_{k+1} - M_k\|_\infty \le \delta$, $M_0 = 0$, *there are Brownian stopping times* $0 = \tau_0 \le \cdots \le \tau_K$ *with* $(M_k) \overset{d}{=} (B(\tau_k))$ *and* $\mathbb{E}[\tau_{k+1} - \tau_k \mid \mathcal{F}_k] = \mathbb{E}[(B(\tau_{k+1}) - B(\tau_k))^2 \mid \mathcal{F}_k]$.

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- embed by Skorokhod — the τ_k track capacities $\kappa\, t_{m_k}$ w.h.p.;
- a Brownian modulus-of-continuity bound (Csőrgő–Révész) upgrades closeness at the embedding points to all times.

Roadmap

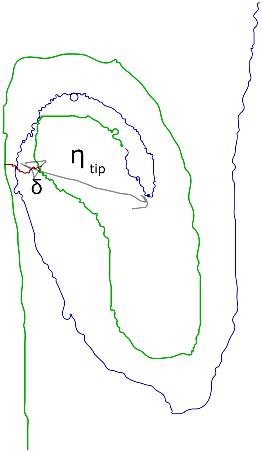
The Tip Structure Modulus

Definition (tip structure modulus)

For a crosscut C of $\mathbb{H}_t$ of diameter $\leq \delta$ separating $\gamma(t)$ from ∞, let γ_C be the arc cut off. Then

$$\eta_{\mathrm{tip}}(\delta) = \max\Big(\delta, \sup_{t \leq T} \sup_{C \in S_{t,\delta}} \operatorname{diam} \gamma_C\Big).$$

The maximal depth the curve reaches into a fjord of opening $< \delta$, seen from the growth direction.



Viklund's gauge of curve regularity in capacity parameterization. Bounding it is the technical crux — and where model-specific labour normally lives.

Viklund's Path Estimate

Lemma (Viklund)

For two $\mathbb{H}$*-Loewner pairs* (f_j, W_j)*, suppose for some* $\beta < 1$*,* $r, p, \epsilon > 0$*:*

1. *driving functions close:* $\sup_{[0,T]} |W_1 - W_2| \le \epsilon$*;*
2. *SLE derivative bound:* $\sup_t d\,|f_2'(t, W_2(t) + id)| \le c' d^{1-\beta}$ *for* $d \le \epsilon^p$*;*
3. *tip-modulus bound:* $\eta_{\text{tip}}(\epsilon^p) \le c\,\epsilon^{pr}$*.*

Then $\sup_{[0,T]} |\gamma_1 - \gamma_2| \le c'' \max\{\epsilon^{p(1-\beta)r}, \epsilon^{(1-\lambda p)r}\}$.

- (2) is the Rohde–Schramm derivative-moment estimate for SLE_κ; the explicit exponents $\lambda_C, q(\beta), \beta_+$ come from their bounds on $\mathbb{E}|f_t'(W_t + id)|^{\cdot}$, which hold precisely for $\kappa < 8$ — exactly the range of the Main Theorem (percolation $\kappa = 6$, HE $\kappa = 4$).

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For two $\mathbb{H}$-Loewner pairs (f_j, W_j), suppose for some $\beta < 1$, $r, p, \epsilon > 0$:

1. *driving functions close:* $\sup_{[0,T]} |W_1 - W_2| \le \epsilon$;
2. *SLE derivative bound:* $\sup_t d\,|f_2'(t, W_2(t) + id)| \le c' d^{1-\beta}$ *for* $d \le \epsilon^p$;
3. *tip-modulus bound:* $\eta_{\mathrm{tip}}(\epsilon^p) \le c\,\epsilon^{pr}$.

Then $\sup_{[0,T]} |\gamma_1 - \gamma_2| \le c'' \max\{\epsilon^{p(1-\beta)r}, \epsilon^{(1-\lambda p)r}\}$.

- (2) is the Rohde–Schramm derivative-moment estimate for SLE_κ; the explicit exponents $\lambda_C, q(\beta), \beta_+$ come from their bounds on $\mathbb{E}|f_t'(W_t + id)|^{\cdot}$, which hold precisely for $\kappa < 8$ — exactly the range of the Main Theorem (percolation $\kappa = 6$, HE $\kappa = 4$).
- (1) is *Framework I.* (3) is the only piece left — and it is geometric.

The Bridge: KS $\Rightarrow$ Tip-Modulus Bound

Proposition (B.–Richards)

Under the KS condition (modulus M) there are $\epsilon, \alpha, \delta_0 > 0$ such that if $\delta \le \eta^{1+\epsilon}$ and $\delta < \delta_0$,

$$\mathbb{P}\big(\eta_{\mathrm{tip}}(\delta) > \eta\big) \le 2\,\eta^{\alpha}.$$

Corollary

If $\{\gamma^n\}$ is κ-admissible, there are $p \in (0,1)$, $\alpha > 0$ (independent of ρ, n) with

$$\mathbb{P}_n\big(\eta_{\mathrm{tip}}^{(n)}(n^{-s}) > n^{-ps}\big) \le 2\,n^{-\alpha ps}.$$

The central observation: a *global* crossing bound yields the *local* tip-regularity hypothesis (3) of Viklund. Crossing bounds are far easier — and already known for the models of interest.

Proof of the Bridge (sketch)

A bound η on η_{tip} fails iff there is a nested (δ, η)-*bottleneck*: a fjord of depth $\geq \eta$, mouth $\leq \delta$.

Sketch.

1. **Cut** γ into N arcs of diameter $\leq \eta/4$; by Aizenman–Burchard, $N \leq M(\gamma, \eta/8)$ is stochastically bounded.

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3. **Union bound:** $E(\delta, \eta) \subset \bigcup_{j<k} E_{j,k}$, and H1 gives $\mathbb{P}(E_{j,k}) \leq K(\delta/\eta)^{\Delta}$. With $m = \eta^{-2(1+\epsilon)}$,

$$\mathbb{P}(E(\delta, \eta)) \leq C_2 \Big[\eta^{\epsilon \Delta k - 2(1+\epsilon)} + \eta^{-4(1+\epsilon)} (\delta/\eta)^{\Delta} \Big].$$

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4. **Optimize:** $\delta = c\,\eta^{1+\tilde{\epsilon}}$, $\tilde{\epsilon} \in (0, 4(1+\epsilon)/\Delta)$, gives $\mathbb{P} \leq c\,\eta^{\alpha}$. □

Two Supporting Inputs

Aizenman–Burchard

Hypothesis H1 (power-law bound on multiple annulus crossings) yields a stochastically bounded $M(\gamma, l) \le \tilde{C}(\gamma)\, l^{-2\beta}$ and Hölder-($< \frac{1}{2}$) tightness. Kemppainen–Smirnov: *KS* $\Rightarrow$ *H1*. This legitimizes "N stochastically bounded."

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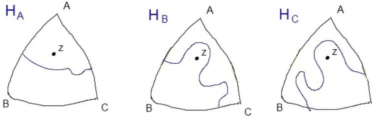
Convergence of discrete domains

If ψ is α-Hölder, then $|\psi(z) - \psi^n(z)| < Cn^{-\beta}$ *up to the boundary*: Beurling pins ∂D^n within $\sqrt{C_2/n}$ of $\partial\mathbb{D}$, and Marchenko's lemma converts this into $|\rho^n(z) - z| \leq n^{-\gamma}$ uniformly. This upgrades parameter-plane convergence to convergence in D.

Roadmap

The Cardy–Smirnov Observable

Smirnov (2000): combinatorial description of the discrete approximation of the mapping $H = H_D$ to the standard equilateral triangle $\mathbb{T}$ with the vertices $\{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$, a complexification of an earlier *Cardy observable*



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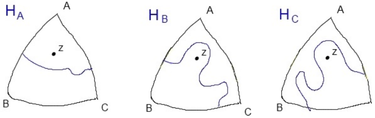

On the conformal triangle, the crossing-probability functions H_A^n, H_B^n, H_C^n assemble into the *Cardy-Smirnov (CS) observable*

$$H^n = H_A^n + \tau\, H_B^n + \tau^2 H_C^n, \qquad \tau = e^{2\pi i/3}.$$

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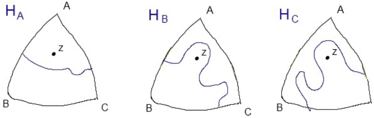

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Theorem (Smirnov)

H^n is a martingale observable for the exploration process: crossing probabilities conditioned on $\gamma^n[0,t]$ equal those in the slit domain $D \setminus \gamma^n[0,t]$. Its limit when $n \to \infty$ is the conformal map H onto the equilateral triangle T sending $(A, B, C) \to (1, \tau, \tau^2)$.

Almost-Holomorphicity

H^n is not discrete holomorphic, but *almost*:

Definition ((σ, ρ)-holomorphic)

$0 < \sigma, \rho \leq 1$ with, for large n:

1. **Hölder up to ∂D^n:** bulk and boundary moduli $|Q^n(z) - Q^n(w)| \leq c\,(|z-w|/\epsilon)^\sigma$;
2. **almost-closed contours:** for every lattice contour Λ, $\left|\oint_\Lambda Q^n\,dz\right| \leq c\,|\Lambda|\,n^{-\rho}$.

Proposition (Smirnov)

The CS observable H^n is (σ, ρ)-holomorphic for some $\sigma, \rho > 0$.

Clause (1) "saves" the construction near the rough boundary; clause (2) is Smirnov's near-holomorphicity made quantitative.

Rate for the Observable, via the Cauchy Transform

Theorem (B.–Chayes–Lei)

For a domain with prime ends A, B, C, there is a universal $s > 0$ with

$$|H^n - H| \leq n^{-s} \qquad (n \geq n_0(R)).$$

where R is the corresponding conformal radius.

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- Use *Harris systems* to control the boundary.
- Use the classical Marchenko Lemma to estimate links $H^n(z) \sim H^n_\square \sim H^n_\diamond \cdots \sim H(z)$.

Power-Law Convergence of Percolation to SLE_6

The CS observable is a non-degenerate BPZ solution, an almost martingale, (σ, ρ)-holomorphic with the rate above, and the family satisfies KS. Hence percolation is 6-admissible, and the Main Theorem applies.

Theorem (B.–Richards)

There is $u > 0$ such that, for the percolation exploration process γ^n and its image $\tilde{\gamma}^n$ in $(\mathbb{H}, 0, \infty)$, there is a coupling with $\sqrt{6}\, B$ with

$$\mathbb{P}\left(\sup_{t \in [0, T_\rho]} d_*(\tilde{\gamma}^n(t), \tilde{\gamma}(t)) > n^{-u} \right) < n^{-u}.$$

If D is α-Hölder, then in D the rate is n^{-v}, $v = v(\alpha)$.

Roadmap

The Harmonic Explorer Observable

The HE observable is

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The only obstruction to continuum harmonicity is a controlled discretization defect: on a δ-isoradial graph

$$|\Delta^\delta H_D(v)| = O\big((\delta/d_u)^{2\beta}\, d_u^{-1}\big)\,, \qquad \beta > \tfrac{1}{2}, \quad d_u = \operatorname{dist}(u, \partial D),$$

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The next theorem converts this defect into power-law closeness $|H_D - h| = O(\delta^{\alpha(\eta)})$, the harmonic analogue of the Cauchy-transform / Harris-system argument for percolation.

Approximating continuous Dirichlet solution.

Theorem (B-Chelkak-Richards)

Let $D \subset \mathbb{C}$ be a bounded simply connected discrete domain drawn on a δ-scaled isoradial graph. Assume that H_D satisfies the Laplacian-defect bound of the previous slide in (the bulk of) D, and that $|H_D| \leq 1$ in D.
Let h be the harmonic continuation of the function H_D from the boundary. Then for any $\eta \in (0,1)$ there exists an exponent $\alpha(\eta) > 0$ such that, provided that δ is small enough (depending only on η), the following estimate holds:

$$|H_D(u) - h(u)| \; = \; O(\delta^{\alpha(\eta)}) \quad \text{uniformly in} \, \{u \in D \,|\, d_u > \delta^\eta\}\,,$$

where the implicit constant in the O-estimate depends only on the diameter of D.

Convergence for Harmonic Explorer

Theorem (B-Chelkak-Richards)

There exists $u > 0$ such that for any domain D with two boundary points A, B and any sequence of "good" n-lattice approximations (by the rescalings of any isoradial lattice) (D^n, A^n, B^n), there is a coupling of the Harmonic Exploration process γ^n in D^n from A^n to B^n and the SLE_4-curve γ in D from A to B such that, in the parameter plane,

$$\mathbb{P}\left[\mathrm{dist}_{sup}\left(\varphi^n(\gamma^n), \varphi(\gamma)\right) > n^{-u}\right] < n^{-u}.$$

Moreover, if D is an α-Hölder domain, then in the domain D itself:

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The same result holds for Ising and FK-Ising.

Conclusion and Open Problems

The framework.

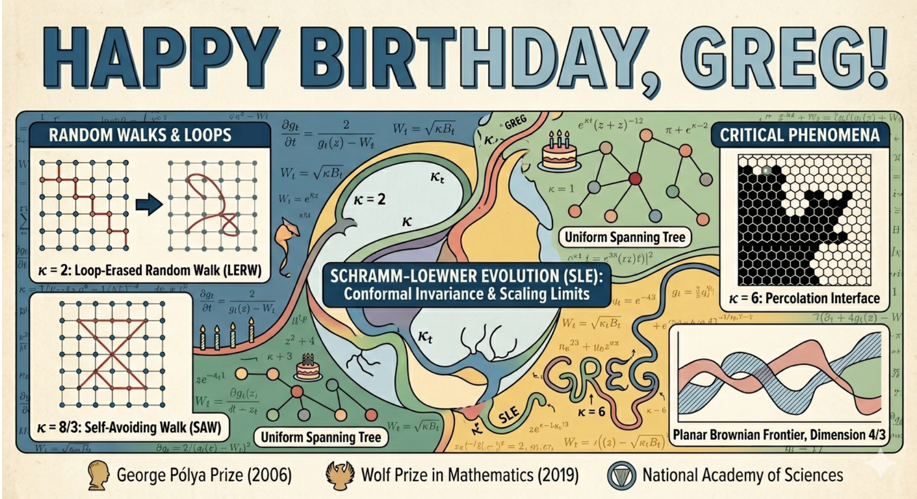
Observable convergence rate n^{-s} + KS condition $\Longrightarrow$ interface rate $n^{-u(s,\,\textsc{ks})}$.

The tip-modulus input supplied automatically by KS, with an α-Hölder upgrade to the physical domain.

Open problems.

1. **Beyond Hölder.** Relax domain regularity via Garban–Rohde–Schramm.
2. **Sharp exponents.** Our u is explicit but far from optimal. What is the true rate?
3. **More models.** End-to-end Percolation Exploration, $O(n)$ loop models.

Happy Birthday, Greg!



Happy Birthday, Greg! ver.2