Circle packing and Riemann uniformization of random planar maps in an ergodic scale-free environment

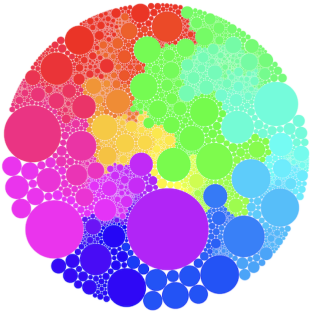
Pu Yu

New York University

Joint work with Nina Holden
Random Explorations Conference, Chicago
July 8, 2026

Embedding of planar maps

- Given a planar map, how should we draw it on the plane?
- Some canonical embeddings: Tutte embedding, Cardy embedding, Smith embedding, etc.;
- This talk: circle packing and Riemann uniformization embedding.



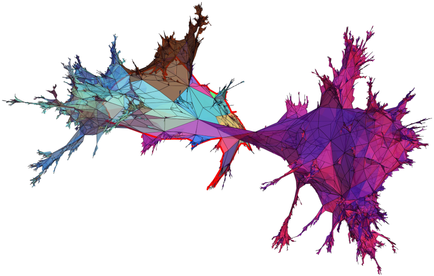


Figure: Simulation by Jason Miller and Jérémie Bettinelli.

For a planar map $G = (V, E)$, each of its vertex v is represented by a disk D_v, with all the disks non-overlapping, and $(v, v') \in E$ if and only if D_v and $D_{v'}$ are tangent.

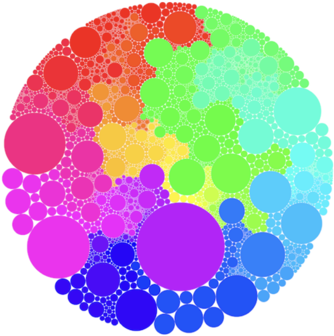
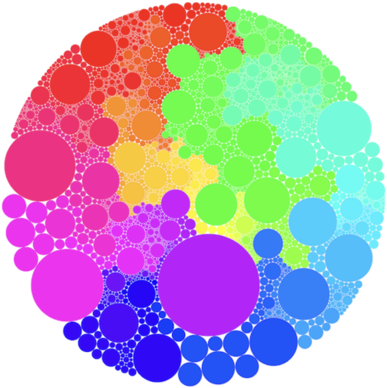


Figure: Circle packing of a triangulation from uniform spanning tree weighted planar maps. Simulation by Jason Miller.

The Koebe-Andreev-Thurston Theorem

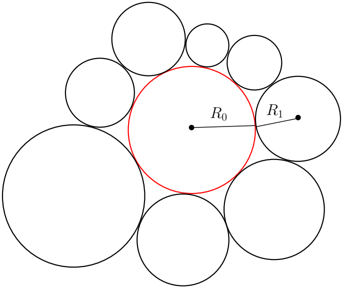
- Circle packing for planar maps always exists.
- If the planar map G is a disk triangulation, then up to Möbius transforms, there exists a unique circle packing for G in the unit disk $\mathbb{D}$ such that all the disks for boundary vertices are tangent to $\partial\mathbb{D}$.



Circle packing for infinite triangulations

Lemma (Ring Lemma, Rodin-Sullivan '87)

Consider a flower with d petals. Then there exists $c > 0$ such that $R_1/R_0 \geq e^{-cd}$.



Existence of circle packing for infinite triangulations follows from the Ring Lemma, and uniqueness is proved by Schramm ('91). Dichotomy: the carrier is either the unit disk (hyperbolic) or the whole plane (parabolic).

Circle packing for infinite triangulations

(He-Schramm) If the SRW on the map G is recurrent, then G is circle packing parabolic; if SRW on G is transient and G has bounded degree, then G is circle packing hyperbolic.

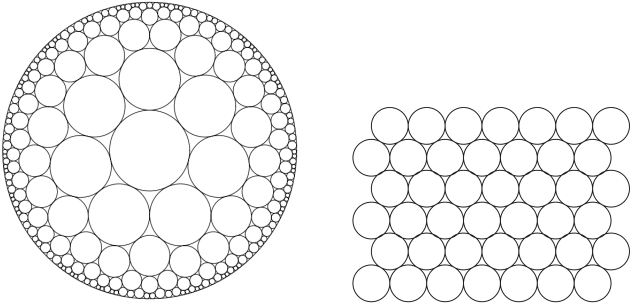
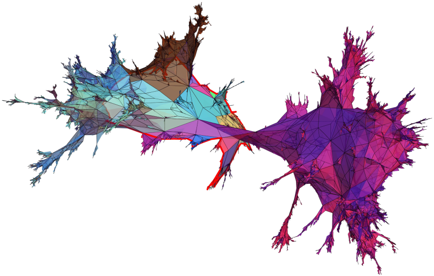


Figure: **Left:** A circle packing for the 7-regular lattice in the unit disk. Figure by James T. Gill and Steffen Rohde. **Right:** A circle packing for the triangular lattice, and the carrier is the whole plane.

The Riemann uniformization embedding

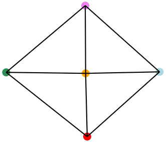
- Collapse all the faces of degree 1 and 2.
- View each face of the planar map as a unit side length regular polygon and glue them together according to the planar map.
- This defines a Riemann surface, where the chart near a point in the interior of an edge or a face is taken to be the identity map, and near a vertex v the chart is $z \mapsto z^\lambda$.
- For planar maps with whole-plane topology, the surface M could be conformally equivalent to $\mathbb{C}$ (parabolic) or $\mathbb{D}$ (hyperbolic).



Cell configurations

A cell configuration $(\mathcal{H}, \mathcal{M}, \mathfrak{c})$ consists of the following objects.

- A locally finite collection $\mathcal{H}$ of compact connected subsets of $\mathbb{C}$ ("cells") with non-empty interiors whose union is all of $\mathbb{C}$ and such that the intersection of any two elements of $\mathcal{H}$ has zero Lebesgue measure.
- A planar map $\mathcal{M}$, which we call the *associated map* of $\mathcal{H}$, with vertices $\{v_H : H \in \mathcal{H}\}$. We require that if v_H and $v_{H'}$ are connected by an edge of $\mathcal{M}$ then $H \cap H' \neq \emptyset$. (Slightly different from Gwynne-Miller-Sheffield as they worked on graphs).
- A function $\mathfrak{c} = \mathfrak{c}_{\mathcal{H}}$ ("conductance") on the edges $\mathcal{M}$ with values in $(0, \infty)$.



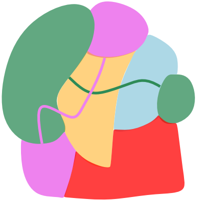


Figure: An illustration of five cells in a cell configuration $\mathcal{H}$. The cells in $\mathcal{H}$ may not be simply connected and could cross from each other.

Gwynne-Miller-Sheffield Invariance principle

Conditions on the random cell system $\mathcal{H}$:

- Translation invariance modulo scaling: There exists domains $U_j \uparrow \mathbb{C}$ and $C_j > 0$ (possibly random), such that if one samples z_j from Lebesgue measure on U_j, then $C_j(\mathcal{H} - z_j)$ converges to $\mathcal{H}$ in law.
- Ergodicity modulo scaling: If F is a function on cell systems such that for any C, z, $F(C(\mathcal{H} - z)) = F(\mathcal{H})$, then F is a.s. constant.
- Line connectivity: For every $\varepsilon > 0$, almost surely for large enough r, for each vertical/horizontal line segment $\ell \subseteq B(0; r)$ of length εr, if H, H' intersect ℓ, then one can find $H \sim H_1 \sim ... \sim H_j \sim H'$ such that all these cells intersects ℓ.
- Moment bounds: $\mathbb{E}\left(\frac{\mathrm{diam}(H_0)^2}{\mathrm{area}(H_0)}(\pi(H_0) + \pi^*(H_0))\right) < \infty$, where H_0 is the cell containing 0, $\pi(H_0) = \sum_{H \sim H_0} \mathfrak{c}(H, H_0)$ and $\pi^*(H_0) = \sum_{H \sim H_0} \mathfrak{c}(H, H_0)^{-1}$.

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- Ergodicity modulo scaling: If F is a function on cell systems such that for any C, z, $F(C(\mathcal{H} - z)) = F(\mathcal{H})$, then F is constant.
- Line connectivity: For each vertical/horizontal line segment ℓ, if H, H' intersect ℓ, then one can find $H \sim H_1 \sim ... \sim H_j \sim H'$ such that all these cells intersects ℓ.
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Gwynne-Miller-Sheffield Invariance principle: Assuming these conditions, there exists a deterministic matrix Σ such that the random walk on the random cell system converges to Brownian motion (modulo time parameterization) with convariance matrix Σ.

Theorem (Holden-Y. '26)

Consider a random cell configuration $\mathcal{H}$ whose associated map is a simple infinite plane triangulation. Assume $\mathcal{H}$ satisfies the conditions above and a moment bound

$$\mathbb{E}\left[\frac{\mathrm{diam}(H_0)^2}{\mathrm{area}(H_0)}\deg(H_0)^4\right] < \infty. \tag{1}$$

Then $\mathcal{H}$ is a.s. circle packing parabolic, and there exists a circle packing $\mathcal{P}$ for $\mathcal{H}$ and a deterministic 2×2 matrix $\mathbf{A}$ with $\det(\mathbf{A}) = \pm 1$ such that the following is true. For $H \in \mathcal{H}$, letting $c(H) = \frac{1}{\mathrm{area}(H)} \int_H z\, dz$ be the Euclidean center of H, and letting $o(H)$ be the center of the disk for H in $\mathcal{P}$, almost surely,

$$\lim_{r \to \infty} \frac{1}{r} \max_{H \in \mathcal{H}(B(0;r))} |\mathbf{A}c(H) - o(H)| = 0. \tag{2}$$

Finally, if $\mathcal{H} \stackrel{d}{=} e^{i\theta}\mathcal{H}$ for some $\theta \in (0, 2\pi)\setminus\{\pi\}$, then $\mathbf{A}$ can be taken to be the identity matrix or reflection matrix.

Main Results

Theorem (Holden-Y. '26)

Consider a random cell configuration $\mathcal{H}$ whose associated map is an infinite plane triangulation. Assume $\mathcal{H}$ satisfies the conditions above and a moment bound

$$\mathbb{E}\left[\frac{\mathrm{diam}(H_0)^2}{\mathrm{area}(H_0)}\deg(H_0)^4\right] < \infty. \tag{3}$$

Then the Riemann surface $M(\mathcal{H})$ is a.s. parabolic, and there exists a conformal map $\varphi : M(\mathcal{H}) \to \mathbb{C}$ and a deterministic 2×2 matrix $\mathbf{A}$ with $\det(\mathbf{A}) = \pm 1$ such that the following is true. For $H \in \mathcal{H}$, letting $c(H) = \frac{1}{\mathrm{area}(H)} \int_H z\,dz$ be the Euclidean center of H, and letting v_H be the vertex for H on $M(\mathcal{H})$, almost surely,

$$\lim_{r\to\infty} \frac{1}{r} \max_{H \in \mathcal{H}(B(0;r))} |\mathbf{A}c(H) - \varphi(v_H)| = 0. \tag{4}$$

Finally, if $\mathcal{H} \overset{d}{=} e^{i\theta}\mathcal{H}$ for some $\theta \in (0, 2\pi)\setminus\{\pi\}$, then $\mathbf{A}$ can be taken to be the identity matrix or reflection matrix.

Main Results

We write $\mathcal{H}_0^{\text{face}}$ for the union of cells incident to the same face as H_0, and write d_0^f for the maximum degree of faces incident to H_0.

Theorem (Holden-Y. '26)

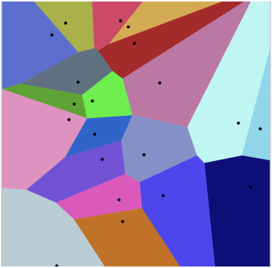
Consider a random cell configuration $\mathcal{H}$ whose associated map has plane topology. Assume $\mathcal{H}$ satisfies the conditions above and a moment bound

$$\mathbb{E}\Big[\max_{H \in \mathcal{H}_0^{\text{face}}} \Big(\frac{\text{diam}(H)^2}{\text{area}(H)} \big(d_0^f + \deg(H)\big)^6\Big)\Big] < \infty. \tag{5}$$

Then the same conclusions hold for the Riemann uniformization embedding for $\mathcal{H}$.

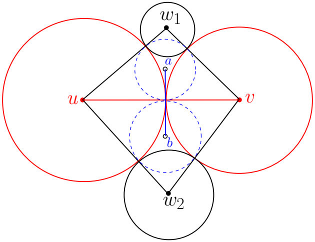
An example

- Consider a Poisson point process Λ of rate λ on $\mathbb{C}$.
- The Voronoi tessellation associated with Λ induces a cell configuration $\mathcal{H}$, where the cells $\{H_z : z \in \Lambda\}$ are the Voronoi cells, i.e., points in $\mathbb{C}$ such that $z \in \Lambda$ is the closest element in Λ.
- $\mathcal{H}$ satisfies all the conditions as well as our moment bound.
- By our main results, the circle packing/Riemann uniformization embedding for $\mathcal{H}$ would be close to $\mathcal{H}$ on a large scale.
- A finite volume version of this result in the setting of circle packing was proven by Ivrii-Marković '23.
- Our main results can also be applied to other natrual planar maps from e.g., percolations.



Random walk on circle packing

Given a circle packing, the Dubejko conductance of the edge uv is defined to be $\mathfrak{c}(u,v) = \frac{|o_a - o_b|}{|o_u - o_v|} = \frac{\sqrt{r_u r_v}}{r_u + r_v} \cdot \left(\sqrt{\frac{r_{w_1}}{r_{w_1} + r_u + r_v}} + \sqrt{\frac{r_{w_2}}{r_{w_2} + r_u + r_v}}\right)$.



Under this conductance, the random walk on centers of the disks becomes a martingale.

Ingredients on circle packing

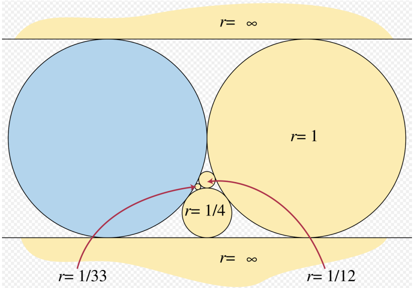
- Bou-Rabee and Gwynne: for a collection of circle packings, if the maximal diameter of the disks converge to 0, then random walk with Dubejko conductance converges to Brownian motion (modulo time parameterization).
- Gurel-Gurevich, Jerison, and Nachmias: a combinatorial criterion of macroscopic circles via vertex extremal length.
- We can also attempt to use the Gwynne-Miller-Sheffield invariance principle, where we work with the walk on $\mathcal{H}$ with the Dubejko conductance.

Proof idea for circle packing

- Claim: If we have two planar embeddings of the same infinite plane triangulation, such that the same random walk on the two embeddings both converge to Brownian motion modulo time parameterization, then the two embeddings are close on a large scale.
- Let us assume that under two embeddings, we scale by $n >> 1$ such that two vertices v_0 and v_1 are both embedded at 0 and 1, and consider a vertex v embedded within $B(0; 10)$ under the first embedding.
- By convergence of RW to Brownian motion, the random walk from v_0 has uniform positive probability to wrap around v by the time it wraps around v_1.
- Then v must be embedded in $B(0; K)$ for some $K > 0$, otherwise the random walk from v_0 will have overwhelming probability to wrap around v_1 before wrapping around v.
- Using this type of wrapping around argument, one can further prove some kind of tightness for the homeomorphism of $\mathbb{C}$ induced by the two embeddings.
- Any limiting homeomorphism of $\mathbb{C}$ sends traces of BM to traces of BM, and has to be identity or a reflection.

Lemma (Rodin-Sullivan '87)

Consider a flower with d petals. Then there exists $c > 0$ such that $R_1/R_0 \geq e^{-cd}$.



Note: This bound is sharp, and the optimum bound is $1/(F_{2d-3} - 1)$ where $\{F_n\}$ is the Fibonacci sequence. Figure from Wikipedia.

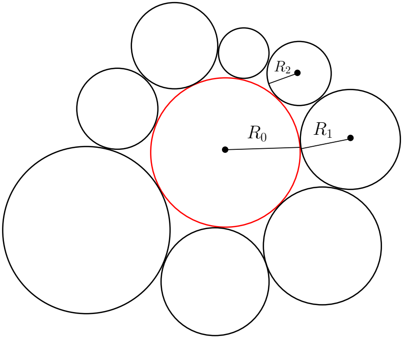
Proof idea for circle packing (Cont'd)

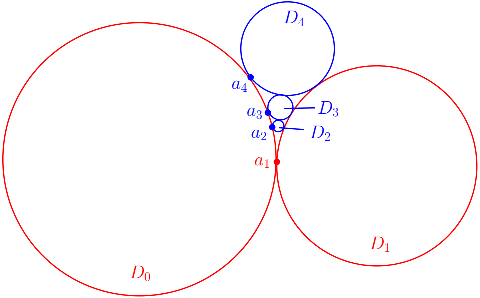
- Using the argument above, it remains to prove the convergence of the random walk on $\mathcal{H}$ with Dubejko conductance.
- The circle packing parabolicity for $\mathcal{H}$ follows from the recurrence of SRW on $\mathcal{H}$ by He-Schramm.
- Claim: if triangulations T^n converges to infinite plane triangulation T in e.g., Benjamini-Schramm topology, then the Dubejko conductance of T^n converges to the Dubejko conductance of T. This gives the translation invariance modulo scaling with the Dubejko conductance.
- To see this claim, first we have tightness on the radii ratio thanks to the Ring Lemma. If the radii ratio for T^n converges to something other than the radii ratio for T, then we obtain a second circle packing for T, contradicting with the uniqueness.
- It remains to work on the moment bound. We will use the following three-circle variant of the Ring Lemma.

A three-circle variant of the Ring Lemma

Lemma (Holden-Y. '26)

Consider a flower with d petals. Then $R_2 \geq 0.01 \min\{R_0, R_1\} d^{-2}$.





A three-circle variant of the Ring Lemma: proof

- Assume on the contrary $R_2 < 0.01 \min\{R_0, R_1\} d^{-2}$. Let us try to put in the disks $D_3, \ldots, D_d$. Assume for a moment that $D_3, \ldots, D_d$ are all tangent to D_1.
- By Descartes' Theorem,

$$\frac{1}{R_3} = \frac{1}{R_2} + \frac{1}{R_0} + \frac{1}{R_1} - 2\sqrt{(\frac{1}{R_0} + \frac{1}{R_1})\frac{1}{R_2} + \frac{1}{R_0 R_1}}. \tag{6}$$

Since $R_2 < 0.01 \min\{R_0, R_1\} d^{-2}$,

$$\frac{1}{R_3} \geq \frac{1}{R_2} + \frac{1}{R_0} + \frac{1}{R_1} - 2\sqrt{2(\frac{1}{R_0} + \frac{1}{R_1})\frac{1}{R_2}} \geq \left(\sqrt{\frac{1}{R_2}} - 2\sqrt{\frac{1}{R_0} + \frac{1}{R_1}}\right)^2.$$

- By an iterative argument, $\sqrt{\frac{1}{R_{j+1}}} \geq \sqrt{\frac{1}{R_j}} - 2\sqrt{\frac{1}{R_0} + \frac{1}{R_1}}$, and one can further get $R_j < \min\{R_0, R_1\} d^{-2}/36$ for $j = 2, \ldots, d$. Then the arcs between the tangent points between D_j, D_0 and the tangent points between D_{j+1} and D_0 for $j = 1, \ldots, d-1$ cannot cover 25% of ∂D_0 so we cannot have a flower.
- By a Möbius transform sending the tangent point between D_0 and D_1 to ∞, the case where $D_3, \ldots, D_d$ are all tangent to D_1 is the best we can get, so in any case no flower if $R_2 < 0.01 \min\{R_0, R_1\} d^{-2}$.

Proof idea for circle packing (Cont'd)

- If one directly use the optimal bound from the Ring Lemma for the moment bound in GMS, then a certain exponential tail on $\deg(H_0)$ would be required.
- By our three-circle variant of the Ring Lemma, each case where we encounter a huge disk tangent to a tiny disk, on the inverse conductance side, instead of paying an $\exp(c \deg(H))$ price to move from the huge disk and the tiny disk, we use a bypass to move at most $\deg(H_0)$ steps, where for each step the price is at most $\deg(H)^{-2}$.
- This eventually allows us to reduce from the exponential bound to the 4th moment bound on degree as in the statement.

Area/length arguments

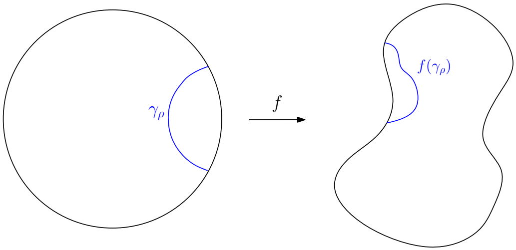
- Let D be a Jordan domain and $f : \mathbb{D} \to D$ be conformal. Let $\gamma_\rho = \partial B(1;\rho) \cap \mathbb{D}$.
- Then

$$\mathrm{len}(f(\gamma_\rho))^2 = \left(\int_{\gamma_\rho} |f'(z)||dz| \right)^2 \leq \int_{\gamma_\rho} |f'(z)|^2 |dz| \cdot \int_{\gamma_\rho} |dz| \leq 2\pi\rho \int_{\gamma_\rho} |f'(z)|^2 |dz|.$$

- This implies that

$$\int_0^\infty \frac{\mathrm{len}(f(\gamma_\rho))^2}{\rho} d\rho \leq 2\pi \int_{\mathbb{D}} |f'(z)|^2 dz = 2\pi \cdot \mathrm{area}(D).$$

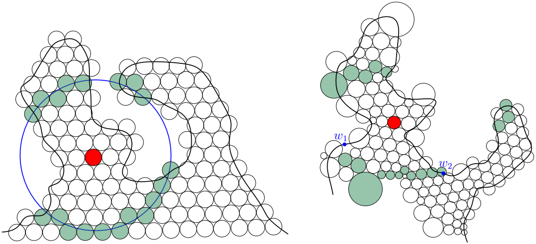
- This is sometimes known as Wolff's estimate, and can be applied to prove, e.g., f can be continuously extended to $\overline{\mathbb{D}}$.



Area/length arguments for circle packing

- Let $\mathcal{P}, \widetilde{\mathcal{P}}$ be isomorphic circle packings of T and $z_0 \in \mathbb{C}$. Let $r_1 > r_0 > 0$. For each vertex v, we write $[a_v, b_v]$ for the range of ρ where $C_{\mathrm{sp}}(z_0;\rho) \cap D_v \neq \emptyset$.
- Let $L(\rho)$ be the sum of the diameter of disks in $\widetilde{\mathcal{P}}$ that intersect $C_{\mathrm{sp}}(z_0;\rho)$. Then

$$\begin{aligned}\int_{r_0}^{r_1} \frac{L(\rho)}{\rho} d\rho &= \sum_{v\in V*} \mathrm{diam}_{\mathrm{sp}}(\widetilde{D}_v) \int_{a_v}^{b_v} \frac{1}{\rho} d\rho \\ &\leq \sum_{v\in V*} \mathrm{diam}_{\mathrm{sp}}(\widetilde{D}_v) \frac{b_v - a_v}{a_v} \leq \Big(\sum_{v\in V*} \mathrm{diam}_{\mathrm{sp}}(\widetilde{D}_v)^2 \Big)^{1/2} \Big(\sum_{v\in V*} \frac{(b_v-a_v)^2}{a_v^2} \Big)^{1/2}.\end{aligned} \tag{7}$$

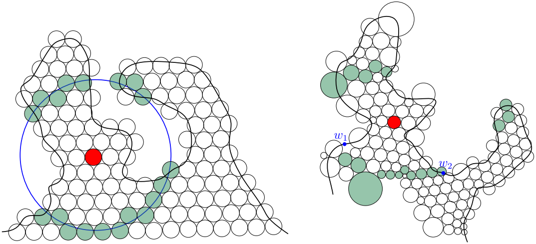


Area/length arguments for circle packing

- The sum $\sum \mathrm{diam}_{\mathrm{sp}}(\widetilde{D}_v)^2$ is bounded, and if the maximal diameter of disks is less than $0.01r_0$, then the sum

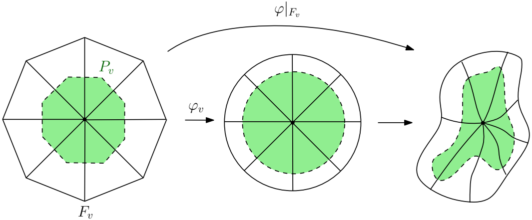
$$\sum_{v \in V^*} \frac{(b_v - a_v)^2}{a_v^2} \leq 2\pi \log(r_1/r_0).$$

- In particular, $\inf\{\rho \in [r_0, r_1] : L(\rho)\} \leq C \log(r_1/r_0)^{-1/2}$.
- This is one of the arguments used in He-Schramm's proof on convergence of circle packing maps to conformal maps, which gives some kind of regularity estimate for circle packing maps.



Area/length arguments for cell configurations

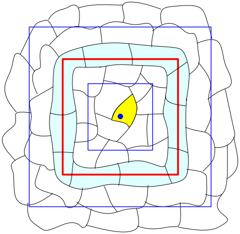
- Let us consider the cell configuration $\mathcal{H}$ and its surface $M(\mathcal{H})$. At this moment we work on triangulations.
- For each vertex v, consider the set P_v, which we call semi-flowers, having closest distance to the vertex v on $M(\mathcal{H})$. Let F_v be the union of the equilateral triangles incident to v.
- Let $\varphi_v : F_v \to \mathbb{D}$ be a conformal map sending v to 0, which is roughly $z \mapsto z^{6/\deg(v)}$. Then $\varphi_v(P_v)$ is approximately $B(0; 2^{-6/\deg(v)})$.
- By Koebe distortion theorem, one can prove that for any arbitrary conformal map $\varphi : M(\mathcal{H}) \to \mathbb{C}$, the ratio of out radius and in radius for $\varphi(P_v)$ is no greater than $c\deg(v)^2$.

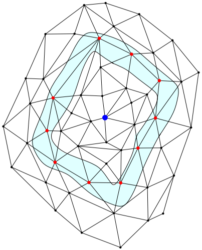


Area/length arguments for cell configurations

- Now we repeat the area/length argument for circle packing. Let $r_1 > r_0 > 0$. For each cell H, we write $[a_H, b_H]$ for the range of ρ where $\partial B(z_0; \rho) \cap H \neq \emptyset$.
- Let $L(\rho) = \sum_{H: H \cap \partial B(z_0;\rho) \neq \emptyset} \mathrm{diam}(\varphi(P_H))$. Then

$$\begin{aligned}\int_{r_0}^{r_1} \frac{L(\rho)}{\rho} d\rho &= \sum_{H \in \mathcal{H}^*} \mathrm{diam}(\varphi(P_H)) \int_{a_v}^{b_v} \frac{1}{\rho} d\rho \leq \sum_{H \in \mathcal{H}^*} \mathrm{diam}(\varphi(P_H)) \frac{b_H - a_H}{a_H} \\ &\leq \Big(\sum_{H \in \mathcal{H}^*} \mathrm{diam}_{\mathrm{sp}}(\varphi(P_H))^2 \mathrm{deg}(H)^{-4} \Big)^{1/2} \Big(\sum_{H \in \mathcal{H}^*} \frac{(b_H - a_H)^2}{a_H^2} \mathrm{deg}(H)^4 \Big)^{1/2}.\end{aligned} \tag{8}$$





Area/length arguments for cell configurations

- Using that the ratio for our radius and in radius for $\varphi(P_H)$ is bounded by $c\deg(H)^2$, the sum
$$\sum_{H\in\mathcal{H}^*} \mathrm{diam}_{\mathrm{sp}}(\varphi(P_H))^2 \deg(H)^{-4}$$
can be bounded by the area of a number of disks with disjoint interiors;
- On a large scale, the sum
$$\sum_{H\in\mathcal{H}^*} \frac{(b_H - a_H)^2}{a_H^2} \deg(H)^4$$
can be treated by the ergodic averaging of $\mathcal{H}$ as in the GMS paper.
- This gives regularity estimates of $M(\mathcal{H})$, e.g., $M(\mathcal{H})$ is a.s. parabolic, the map ϕ_0 between $M(\mathcal{H})$ and the a priori embedding $\mathcal{H}$ is continuous, and has at most polynomial growth.

Proof of uniformization convergence for triangulations

- Following similar treatment as GMS, by working with the continuous Dirichlet energy, the map ϕ_0 will be close to a harmonic map $\phi_\infty : M(\mathcal{H}) \to \mathbb{C}$ on a large scale.
- Consider a conformal map $\varphi : M(\mathcal{H}) \to \mathbb{C}$. Then $\phi_\infty \circ \varphi^{-1}$ is a harmonic function on $\mathbb{C}$.
- By our regularity estimates on ϕ_0, $\phi_\infty \circ \varphi^{-1}$ has polynomial growth and has to be an actual polynomial.
- If this polynomial has degree at least 2, then two cells could be far away on $M(\mathcal{H})$ and close on $\mathcal{H}$, which violates the continuity of the map ϕ_0.
- Thus $\phi_\infty \circ \varphi^{-1}$ has to be a random linear function, and by ergodicity arguments we can express $\phi_\infty \circ \varphi^{-1}$ as a composition of a deterministic linear map and a random rotation/scaling.

General planar map setting

- For general planar maps, we can turn it into a triangulation by adding a vertex within each face and connect this vertex with origin vertices on the face.
- One can take the circle packing for this new triangulation to generate a circle packing for the original planar map, and our result can be extended to this particular version of circle packing for general random planar map.
- There are also many other ways to add constraints to get uniqueness for circle packing for general maps. It would be interesting to look into these cases as well.

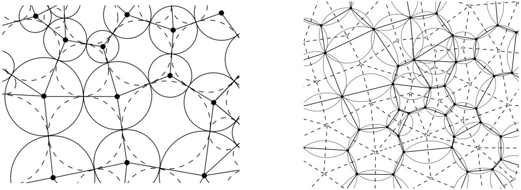
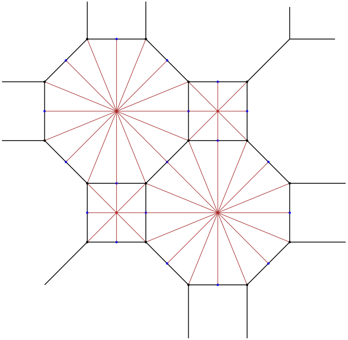
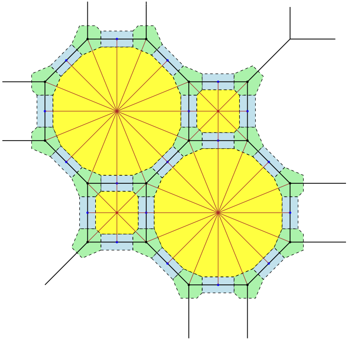


Figure: Figure by Boris A. Springborn.

General planar map setting

- For the setting of uniformization of general maps, we again first turn it to a triangulation by adding vertices within each face and edge.
- The same argument for triangulations can be adapted if we choose the semi-flowers for each vertex more carefully, so that the geometry of the semi-flowers under conformal maps can be controlled in terms of $\deg(v)^2$.





Convergence of random planar map to LQG

LQG is conjectured/proved to be scaling limit of random planar maps. But what is the exact statement?

- Convergence under peanosphere topology. There are certain random planar maps that can be encoded by a random walk $\mathcal{Z} = (\mathcal{L}, \mathcal{R})$, and the random walk converges to the Brownian motion in mating-of-trees. Examples include UST-weighted maps ($\sqrt{2}$-LQG), UIPT ($\sqrt{8/3}$-LQG), bipolar orientations ($\sqrt{4/3}$-LQG) and Schnyder woods (1-LQG).
- Convergence in the sense of metric space. View random planar map as a random metric space, and the convergence is in Gromov-Hausdorff topology. Le Gall and Miermont proved that uniform quadranguations converges to the Brownian map, which is the same as $\sqrt{8/3}$-LQG by Miller and Sheffield.

Convergence of random planar map to LQG (Cont'd)

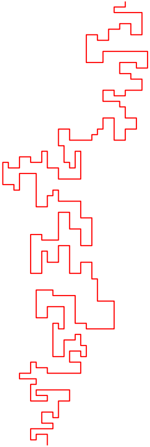
LQG is conjectured/proved to be scaling limit of random planar maps. But what is the exact statement?

- Convergence under embeddings. One embed the random planar maps using certain embeddings, which induces a random counting measure on vertices. One then looks into the convergence of this random measure.
- Examples: Convergence of UIPT under Cardy embedding to $\sqrt{8/3}$-LQG (Holden-Sun), convergence of mated-CRT map under Tutte embedding (Gwynne-Miller-Sheffield) and under Smith embedding (Bertacco-Gwynne-Sheffield).

Convergence of random planar map to LQG (Cont'd)

In forthcoming works by us, our main results will be applied to prove that:

- The mated-CRT map converges to LQG under circle packing/Riemann uniformization embedding;
- Consider the spanning tree weighted map M, its associated triangulation T and path $\mathfrak{tr}$ in disk/sphere/whole plane topology, which can be generated by a walk $\mathcal{Z}$ on $\mathbb{Z}^2$ via Mullin bijection.
- Then one can couple $\mathcal{Z}$ with a 2D Brownian motion Z, which generates a space-filling SLE_8 decorated LQG surface (D, η, h) such that under circle packing of T/ Riemann uniformization embedding of M, the counting measure on vertices converges weakly to the Liouville measure μ_h, while the trace $\mathfrak{tr}$ converges to η under curve topology.



Happy birthday, Greg!