ETH zürich

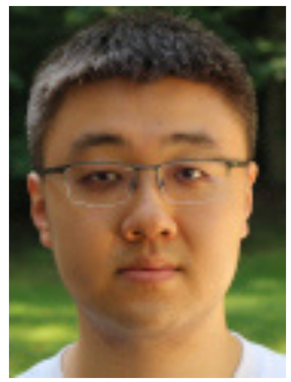
Funded by the European Union

European Research Council
Established by the European Commission

Geometry of Riemann surfaces through the lens of Brownian Motion

Yilin Wang (ETH)

Joint with Yuhao Xue
(IHES)



IMSI
University of Chicago 07. 2026

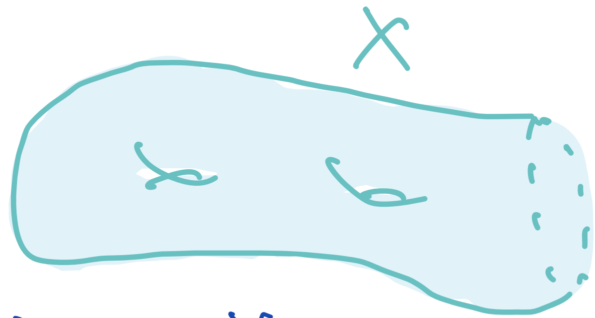
Overview

- Uniformization theorem
- Brownian motion on Riemannian surfaces
- Warm-up : Harmonic measure
- Identity on length spectra
- Determinant of Laplacian

Riemannian surface (X, g)

Assumption:

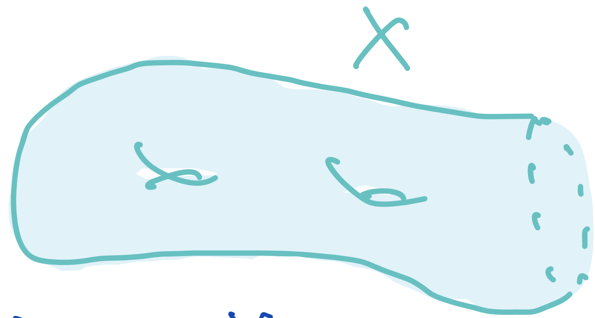
- X without boundary
  (∂X is not included in X)
- g a smooth Riemannian metric on X



$\rightsquigarrow$ complex structure on X (Riemann surface)

$g \sim e^{2\sigma} g$ equivalence class of metrics

Riemannian surface (X, g)



Assumption:

- X without boundary
  (∂X is not included in X)
- g a smooth Riemannian metric on X

$\rightsquigarrow$ Complex structure on X (Riemann surface)

$g \sim e^{2\sigma} g$ equivalence class of metrics

Remark:

→ X may have punctures
→ X does not need to be geometrically finite
(∞ genus, ∞ punctures allowed)

Uniformization theorem (Riemann surface)

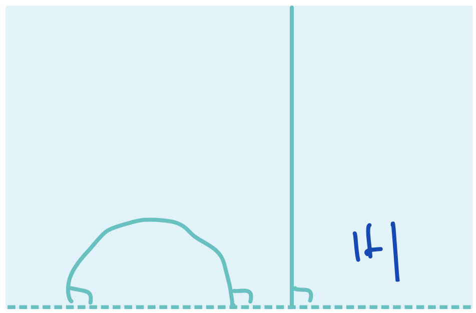
$$\begin{array}{ccc|c|c}
\tilde{X} & = & \hat{\mathbb{C}} & \mathbb{C} & \mathbb{H}^2 \\
{\scriptstyle\text{universal cover}}\Big\downarrow & & & & \\
X & = & \hat{\mathbb{C}} & \mathbb{C},\ \mathbb{C}\setminus\{0\},\ \mathbb{C}/\Lambda = T & \text{everything else}
\end{array}$$

Uniformization theorem (metric formulation)

$\widetilde{X}$	$=$	$\hat{\mathbb{C}}$	$\mathbb{C}$	$\mathbb{H}$
universal cover $\downarrow$	complete metric with curvature	1	0	-1
X	$=$	$\hat{\mathbb{C}}$	$\mathbb{C}$, $\mathbb{C}\setminus\{0\}$, $\mathbb{C}/\Lambda = T$	everything else
(complete)		round metric	flat metric	hyperbolic metric

Complete metric examples

$\mathbb{H} = \{ z = x + iy \mid y > 0 \}$



geodesics

$g = dx^2 + dy^2$ not complete

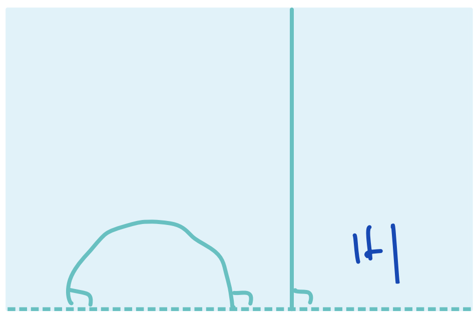
$g_{\mathbb{H}} = \dfrac{dx^2 + dy^2}{y^2}$ complete

curvature $= -1$

hyperbolic plane

Complete metric examples

$\mathbb{H} = \{ z = x + iy \mid y > 0 \}$



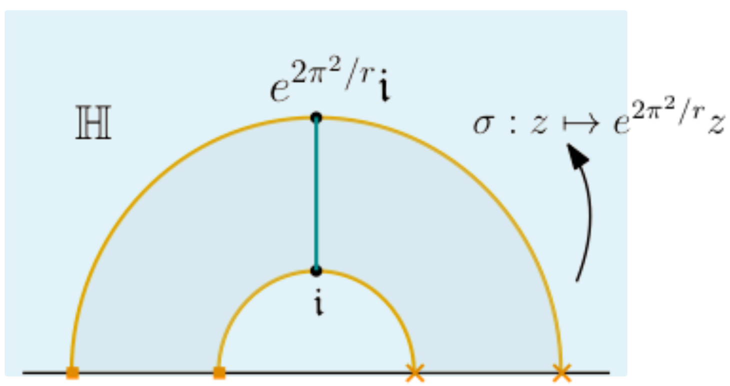
geodesics

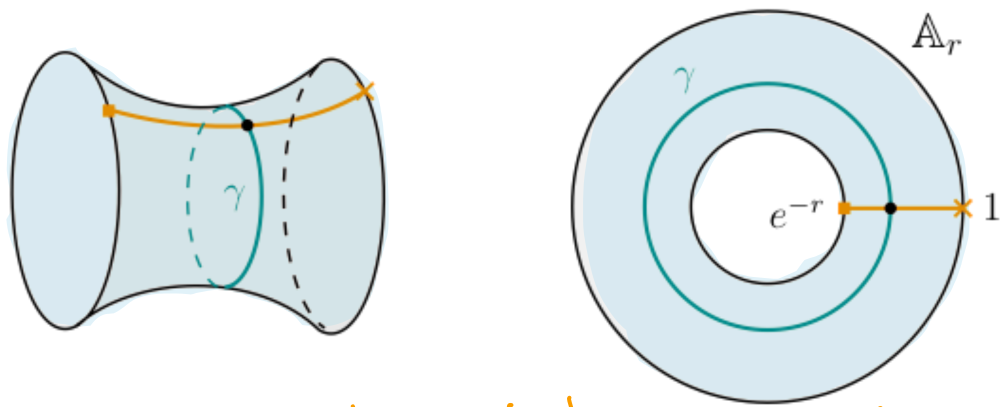
$g = dx^2 + dy^2$ not complete

$g_{\mathbb{H}} = \dfrac{dx^2 + dy^2}{y^2}$ complete
curvature $= -1$

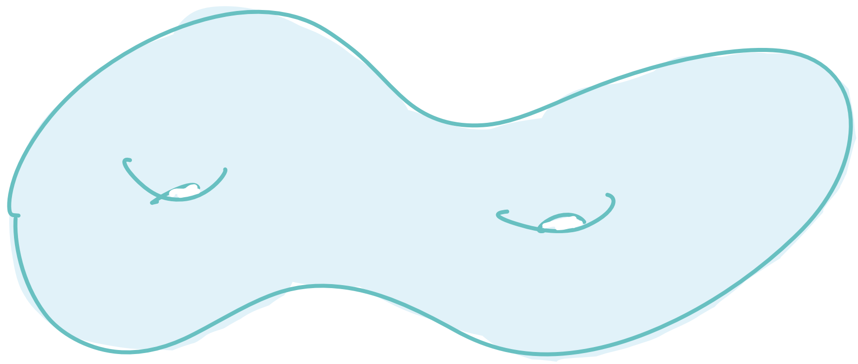
hyperbolic plane

$A_r = \langle z \mapsto e^{2\pi^2/r} z \rangle \backslash \mathbb{H}$



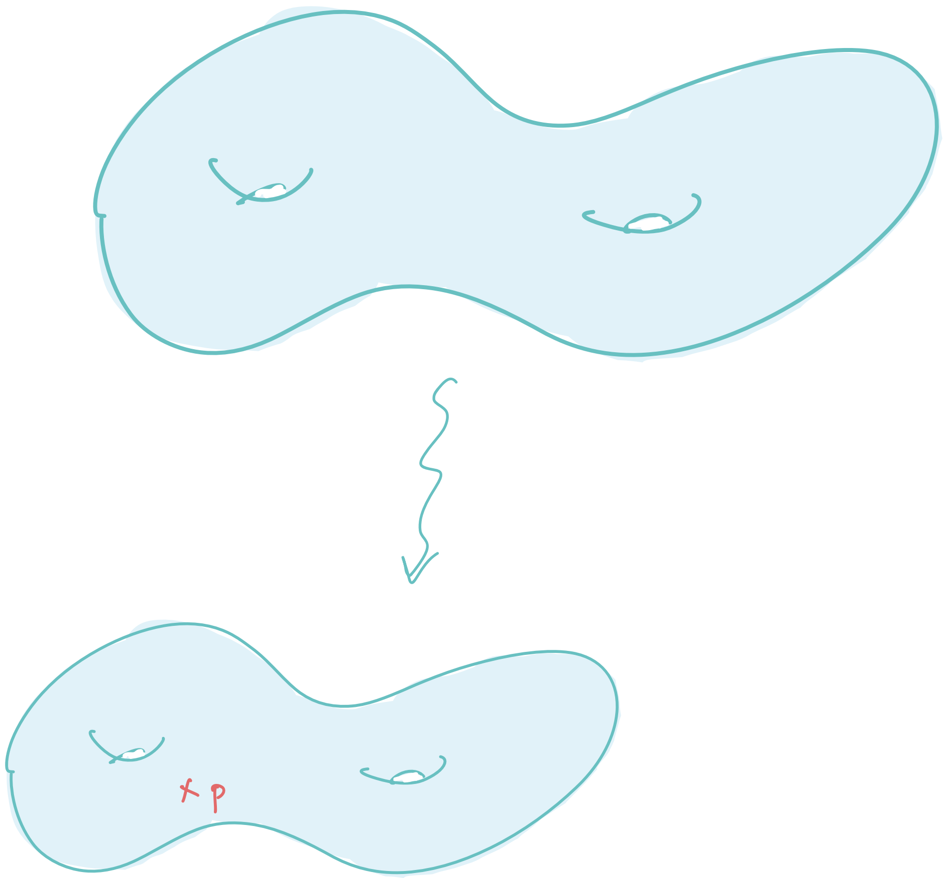


hyperbolic annulus



(Σ, g)

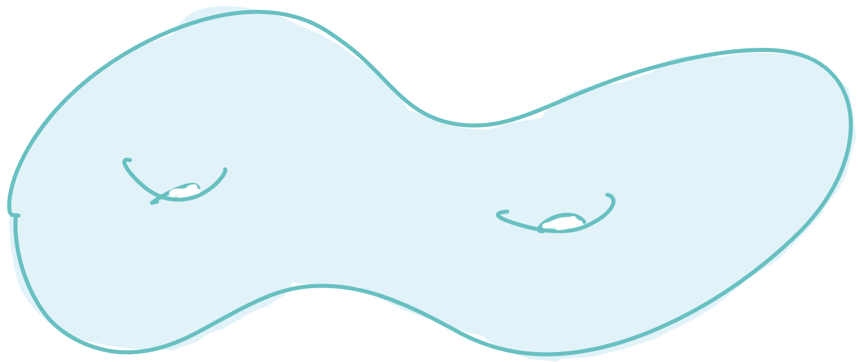
hyperbolic surface



(Σ, g)

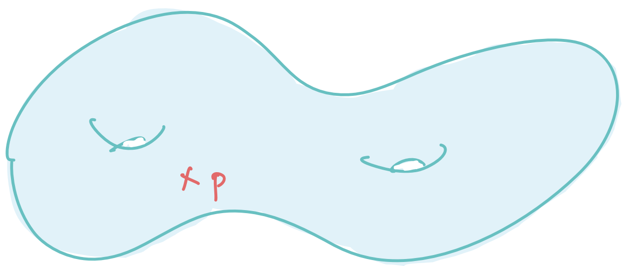
hyperbolic surface

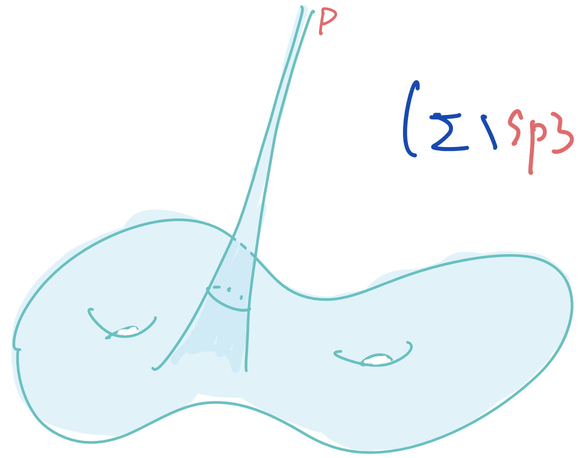
$(\Sigma \setminus \{p\}, g)$ not complete



(Σ, g)

hyperbolic surface





$(\Sigma \setminus \{p\}, \tilde{g})$

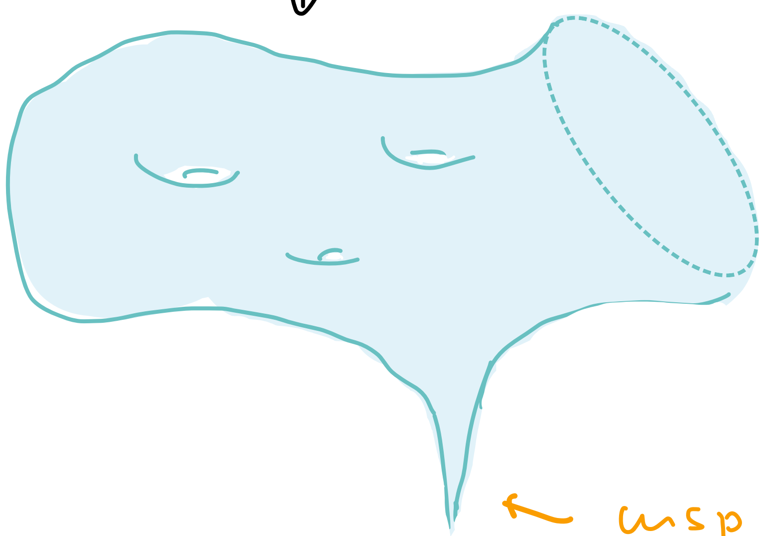
$(\Sigma \setminus \{p\}, g)$ not complete $\rightsquigarrow$ $\exists!\, \tilde{g}$ complete hyperbolic and conformal to g

Complete hyperbolic surface

(X, g_0) has curvature $\equiv -1$

$\Gamma \triangleleft PSL(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H})$

discrete torsion free subgroup



$\Gamma \backslash \mathbb{H} = X$, $\Gamma \simeq \pi_1(X)$

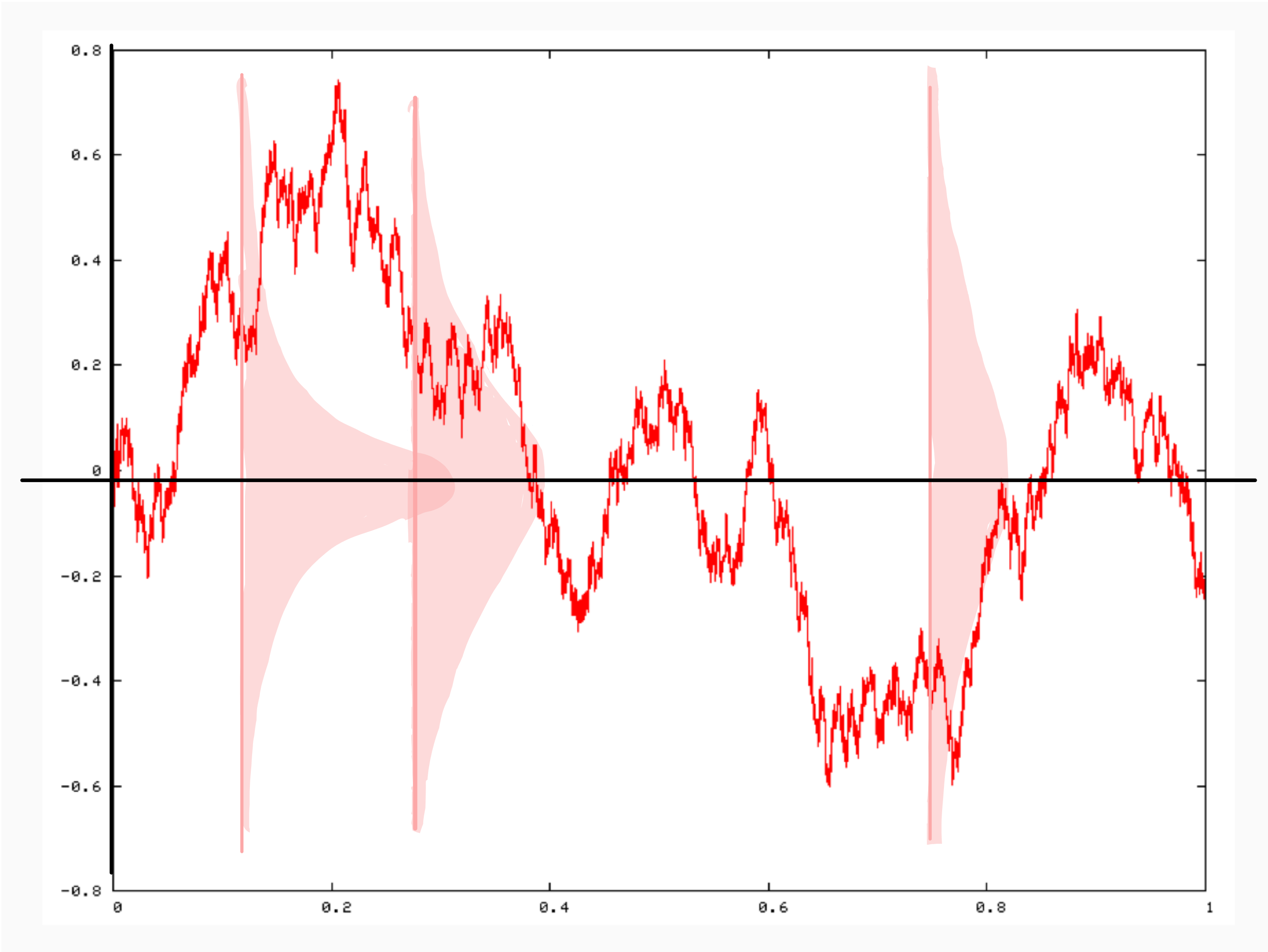
funnel

both ∞-ly far

cusp

Standard Brownian Motion in $\mathbb{R}$

B_t



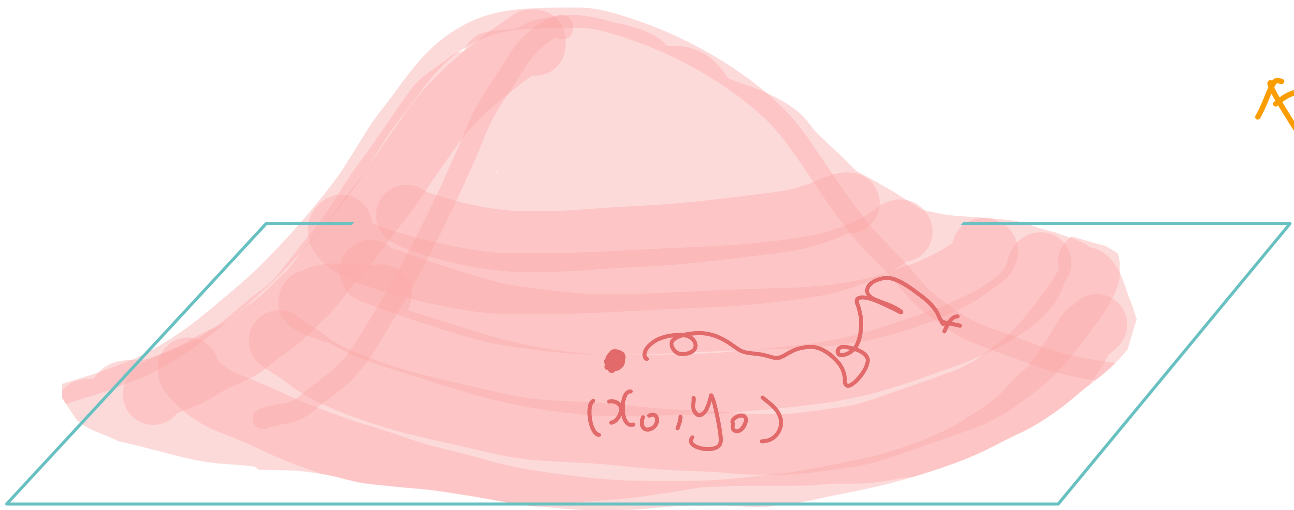
$$\mathbb{P}_0(B_t \in dx) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t}\right) = P_t(0,x)$$

Standard Brownian Motion in $\mathbb{R}^2$

$B_t = (B_t^{(1)}, B_t^{(2)})$

independent

1-D BM



$$\mathbb{P}_{(x_0,y_0)}(B_t \in (dx, dy)) = \underset{\substack{\uparrow \\ e^{t\Delta}}}{P_t}((x_0,y_0),(x,y))\,dx\,dy$$

$$\Delta = \partial_{xx} + \partial_{yy}$$

Brownian Motion on (X, g)

= the diffusion generated by

$\Delta_g = \mathrm{div}_g \, \mathrm{grad}_g$ Laplace-Beltrami operator

Brownian Motion on (X, g)

= the diffusion generated by

$\Delta_g = \mathrm{div}_g \, \mathrm{grad}_g$ Laplace-Beltrami operator

- In $\mathbb{R}^2$, $g = e^{2\sigma}(dx^2 + dy^2) \Rightarrow \Delta_g = e^{-2\sigma}(\partial_{xx} + \partial_{yy})$

$\Rightarrow$ B is a time-changed standard Brownian motion

$\Rightarrow$ The trajectory is conformally invariant

Brownian Motion on (X, g)

= the diffusion generated by

$\Delta_g = \mathrm{div}_g \, \mathrm{grad}_g$ Laplace-Beltrami operator

- In $\mathbb{R}^2$, $g = e^{2\sigma}(dx^2 + dy^2) \Rightarrow \Delta_g = e^{-2\sigma}(\partial_{xx} + \partial_{yy})$

$\Rightarrow$ B is a time-changed standard Brownian motion

$\Rightarrow$ The trajectory is conformally invariant

$$\mathbb{P}_z(B_t \in dw) = \underset{\substack{\uparrow \\ e^{t\Delta_{\mathbb{H}}}}}{P_t^{\mathbb{H}}}(z, w)\, dw$$

Brownian Motion on (X, g)

= the diffusion generated by

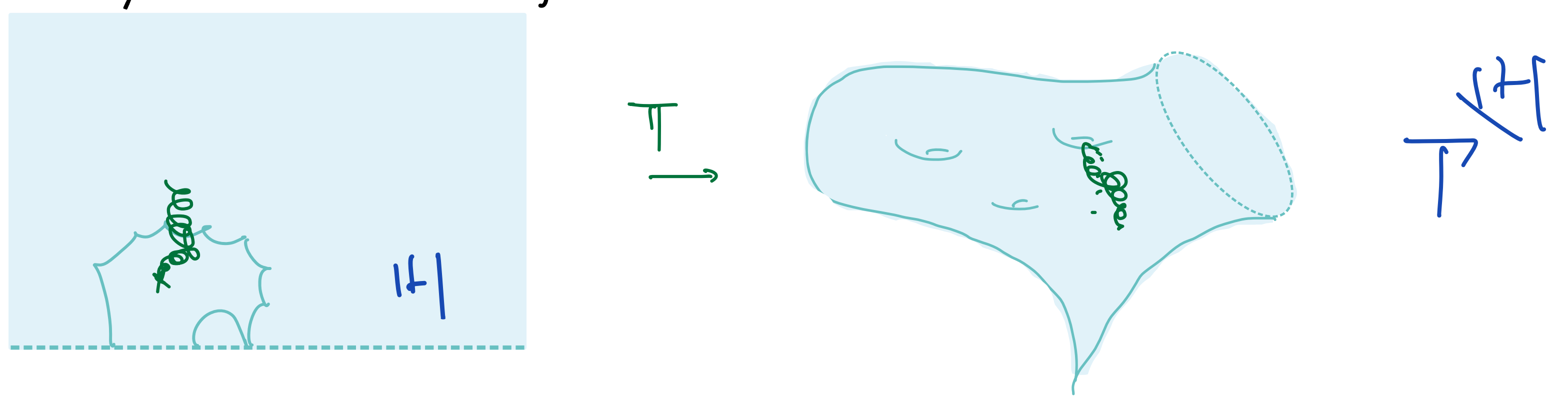
$\Delta_g = \mathrm{div}_g \, \mathrm{grad}_g$ Laplace-Beltrami operator

- In $\mathbb{R}^2$, $g = e^{2\sigma}(dx^2 + dy^2) \Rightarrow \Delta_g = e^{-2\sigma}(\partial_{xx} + \partial_{yy})$

$\Rightarrow$ B is a time-changed standard Brownian motion

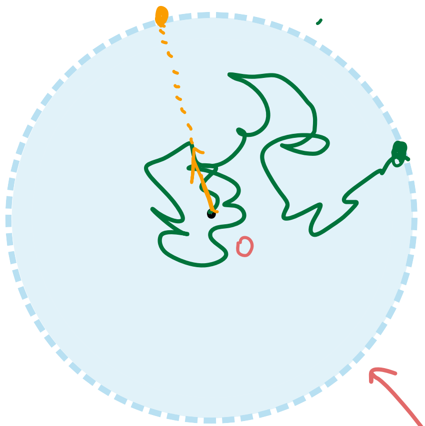
$\Rightarrow$ The trajectory is conformally invariant

Project to hyperbolic surfaces



Goal : explore the conformal invariance of BM

Warm-up Harmonic measure



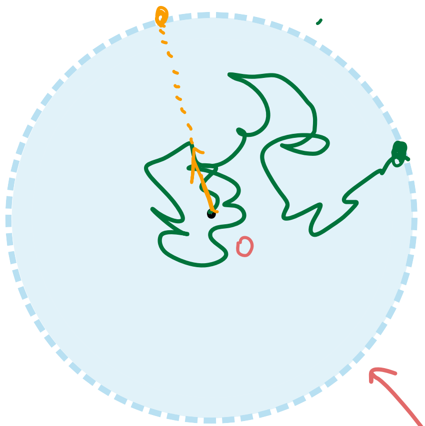
$W_0(d\theta) = \frac{d\theta}{2\pi}$

Starting from 0

- pick a direction uniformly follow the geodesic
- Brownian motion

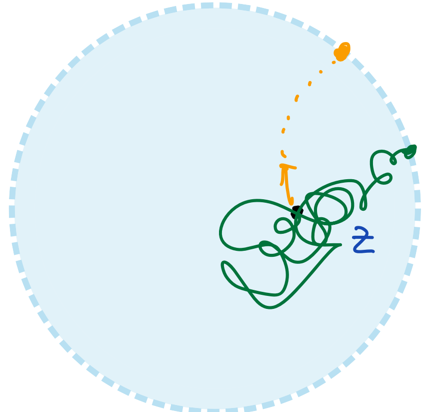
hitting distributions for both BM & geodesic

Warm-up   Harmonic measure



$\omega_o(d\theta) = \frac{d\theta}{2\pi}$

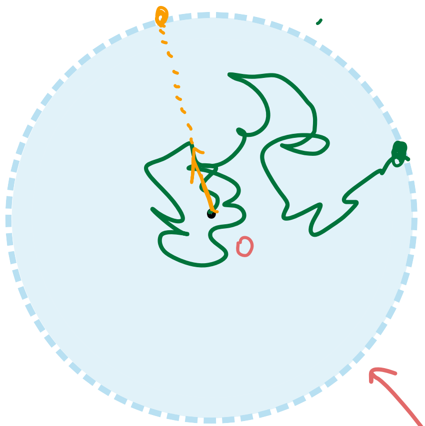
$\xrightarrow[A(o)\,=\,z]{A \text{ isometry of } \mathbb{H}}$



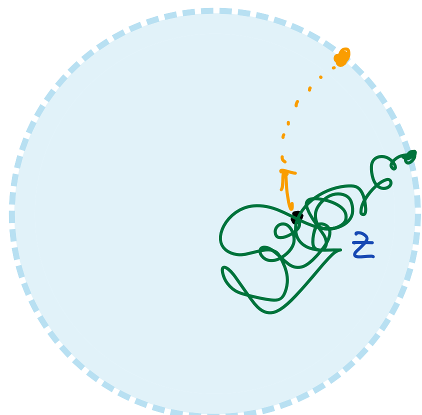
$\omega_z(d\theta) = A_*(\omega_o)$

hitting distributions for both BM & geodesic

Warm-up Harmonic measure



$$\xrightarrow[A(0) \,=\, z]{A \text{ isometry of } \mathbb{H}}$$



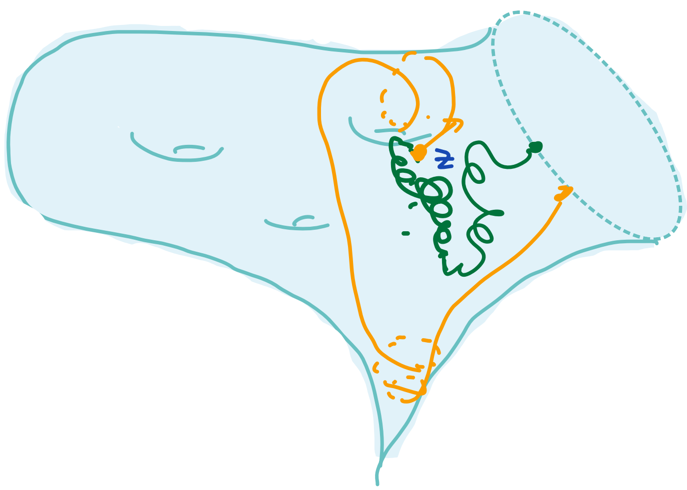
$W_0(d\theta) = \frac{d\theta}{2\pi}$

$W_z(d\theta) = A_*(W_0)$

hitting distributions for both BM & geodesic

Assume $\Sigma = \mathbb{H}/\Gamma$

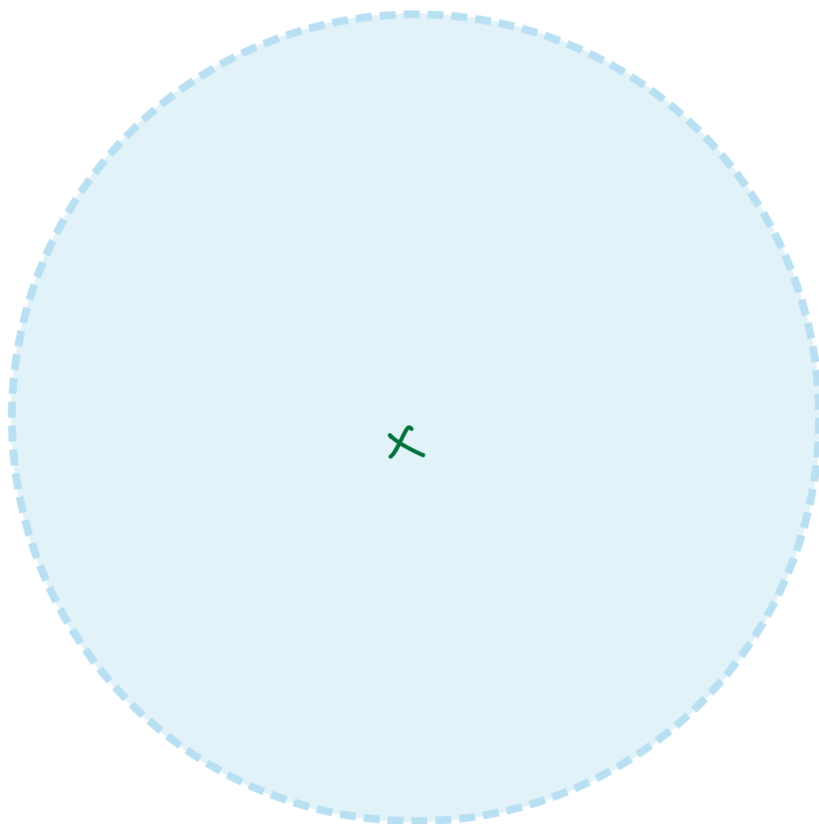
& Brownian motion is transient



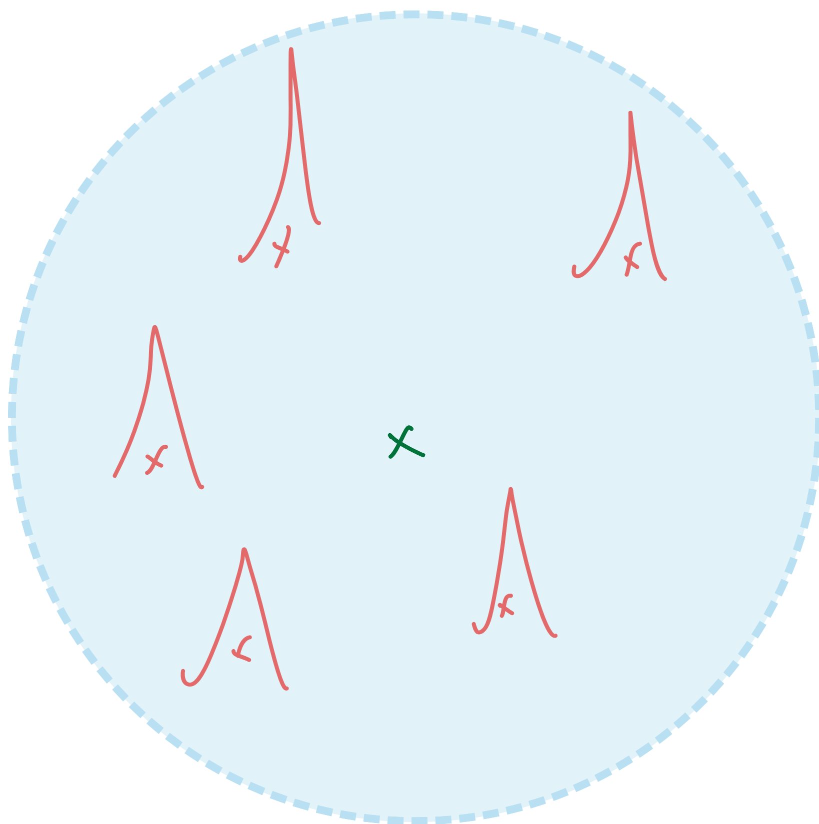
$\partial_\infty \Sigma$

Hitting distributions on $\partial_\infty \Sigma$ $\begin{cases} \text{for uniformly chosen geodesic} \\ \text{for Brownian motion} \end{cases}$

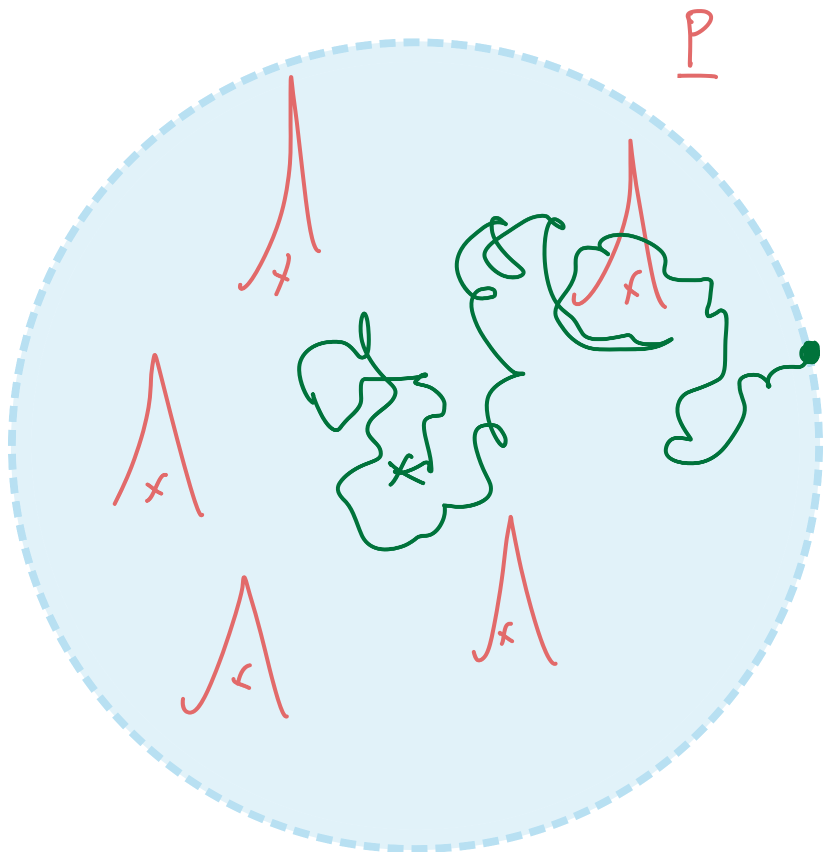
are equal



Hitting distributions on $\partial_\infty \Sigma$ $\begin{cases} \text{for uniformly chosen geodesic} \\ \text{for Brownian motion} \end{cases}$ are equal



Hitting distributions on $\partial_\infty \Sigma$ $\begin{cases} \text{for uniformly chosen geodesic} \\ \text{for Brownian motion} \end{cases}$ are equal

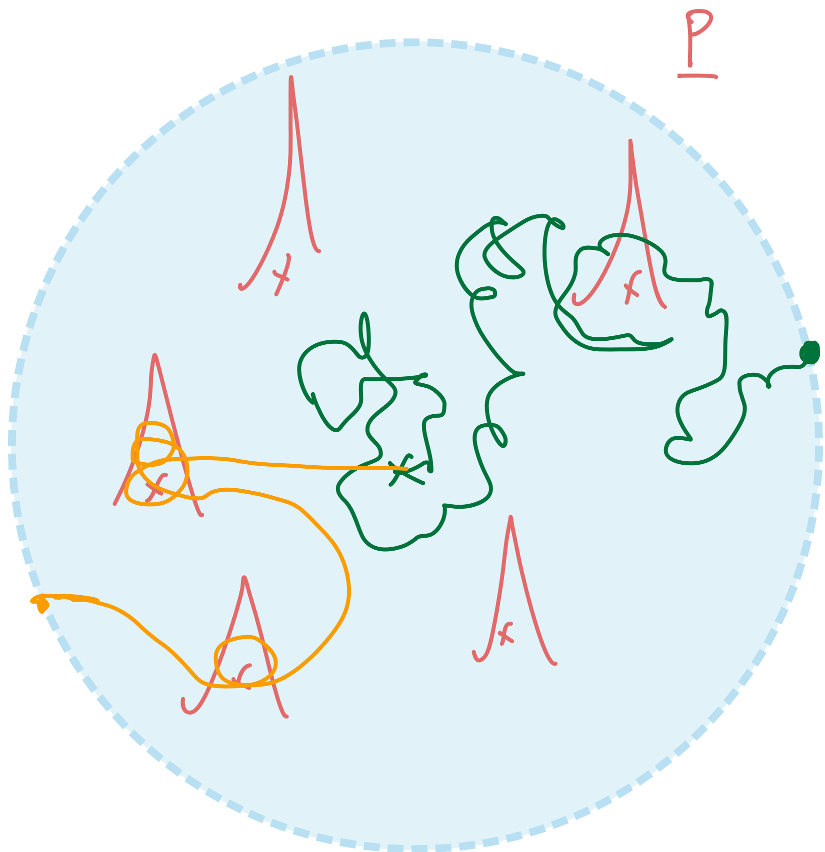


$$W_o(d\theta) = \frac{d\theta}{2\pi}$$

as BM has 0 probability hitting P

& conformally invariant

Hitting distributions on $\partial_\infty \Sigma$ $\begin{cases} \text{for uniformly chosen geodesic} \\ \text{for Brownian motion} \end{cases}$ are equal



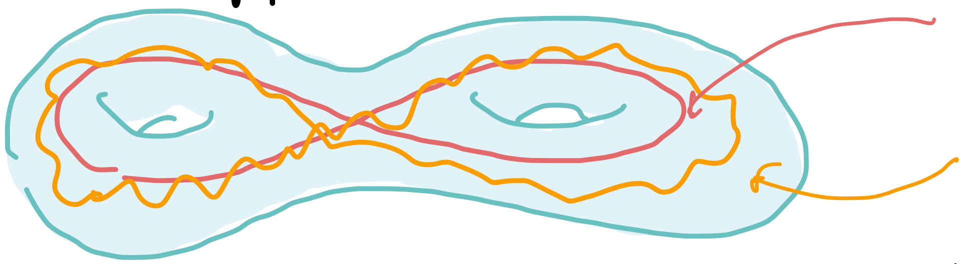
$$W_o(d\theta) = \frac{d\theta}{2\pi}$$

So is for geodesic!

Not obvious without BM

Length spectra

Free homotopy classes of closed curves



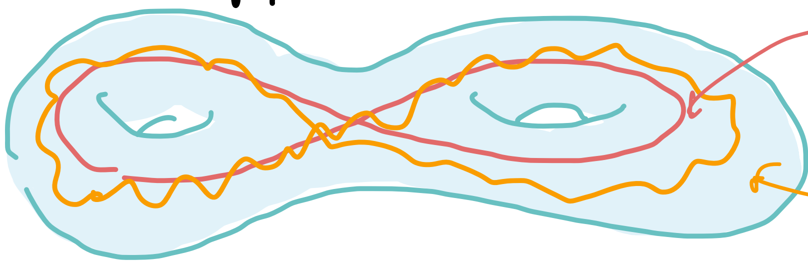
freely homotopic

Facts • If X is hyperbolic, in $\forall$ free homotopy class of essential curves

not homotopic to a point

$\exists!$ hyperbolic geodesic γ

Free homotopy classes of closed curves



freely homotopic

Facts • If X is hyperbolic, in $\forall$ free homotopy class of essential curves

not homotopic to a point

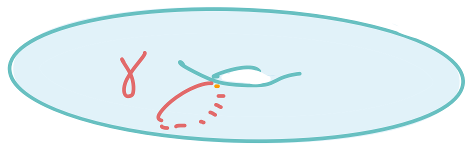
$\exists!$ hyperbolic geodesic γ

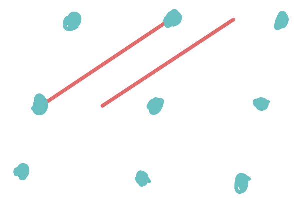
$\ell_X(\gamma) \leftarrow$ length of geodesic

$\downarrow$

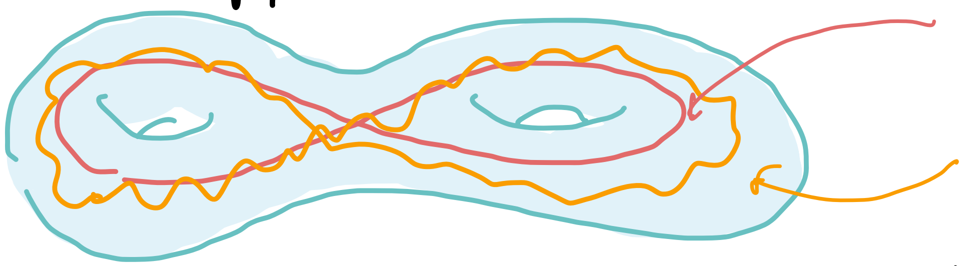
$\ell_T(\gamma)$

• If T is a flat torus $\mathbb{C}/\Lambda$





Free homotopy classes of closed curves



freely homotopic

Facts • If X is hyperbolic, in $\forall$ free homotopy class of essential curves

not homotopic to a point

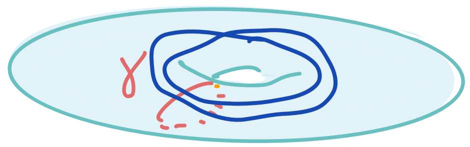
$\exists !$ hyperbolic geodesic γ

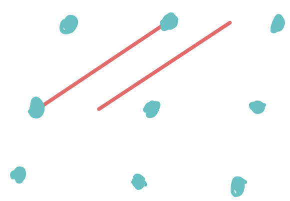
$\ell_X(\gamma) \leftarrow$ length of geodesic

$\downarrow$

$\ell_T(\gamma)$

• If T is a flat torus $\mathbb{C}/\Lambda$





• Otherwise, all γ are homotopic to a point

• $m_X(\gamma)$ = iteration # of γ $\quad$ $m_X(\gamma) = 1 \iff \gamma$ primitive

Length spectrum of hyperbolic surface

$= \{$ lengths of primitive geodesics w. multiplicity $\}$

$\{ \ell_1 \leq \ell_2 \leq \ell_3 \quad \cdots \}$

Length spectrum of hyperbolic surface

$= \{$ lengths of primitive geodesics w. multiplicity $\}$

(For compact hyperbolic surfaces)

- [Huber] [Selberg] $\xleftrightarrow{1:1}$ spectrum of Δ

Length spectrum of hyperbolic surface

= { lengths of primitive geodesics w. multiplicity }

(For compact hyperbolic surfaces)

- [Huber] [Selberg] $\xleftrightarrow{1:1}$ spectrum of Δ
- (Prime geodesic theorem) counting of primitive geodesics [Huber] [Randol] [Margulis] ... of length $\leq L$ $\overset{L\to\infty}{\sim} e^L/L$

Length spectrum of hyperbolic surface

= { lengths of primitive geodesics w. multiplicity }

(For compact hyperbolic surfaces)

- [Huber] [Selberg] $\xleftrightarrow{1:1}$ spectrum of Δ
- (Prime geodesic theorem) counting of primitive geodesics
  [Huber] [Randol] [Margulis] ... of length $\leq L$ $\overset{L\to\infty}{\sim} e^L/L$
- Simple length spectrum [Mirzakhani] $\overset{L\to\infty}{\sim} c(X)\, L^{6g-6}$

Length spectrum of hyperbolic surface

= { lengths of primitive geodesics w. multiplicity }

(For compact hyperbolic surfaces)

- [Huber] [Selberg] $\xleftrightarrow{1:1}$ spectrum of Δ
- (Prime geodesic theorem) counting of primitive geodesics of length $\leq L$ $\overset{L\to\infty}{\sim} e^L/L$
  [Huber] [Randol] [Margulis] ...
- Simple length spectrum [Mirzakhani] $\overset{L\to\infty}{\sim} c(X)\, L^{6g-6}$
- Multiplicity ?

Length spectrum of hyperbolic surface

= { lengths of primitive geodesics w. multiplicity }

(For compact hyperbolic surfaces)

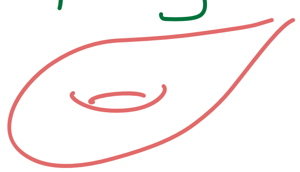
- [Huber] [Selberg] $\xleftrightarrow{1:1}$ spectrum of Δ
- (Prime geodesic theorem) counting of primitive geodesics of length $\leq L$ $\underset{L\to\infty}{\sim} e^L/L$
  [Huber] [Randol] [Margulis] ...
- Simple length spectrum [Mirzakhani] $\underset{L\to\infty}{\sim} c(X)\, L^{6g-6}$
- Multiplicity [Randol] is arbitrarily high on any hyperbolic surface

Length spectrum of hyperbolic surface

= { lengths of primitive geodesics w. multiplicity }

(For compact hyperbolic surfaces)

- [Huber] [Selberg] $\xleftrightarrow{1\,:\,1}$ spectrum of Δ
- (Prime geodesic theorem) counting of primitive geodesics of length $\le L$ $\overset{L\to\infty}{\sim} e^L/L$
  [Huber] [Randol] [Margulis] ...
- Simple length spectrum [Mirzakhani] $\overset{L\to\infty}{\sim} c(X)\, L^{6g-6}$
- Multiplicity [Randol] is arbitrarily high on any hyperbolic surface
- Simple geodesic length multiplicity? Many conjectures → number theory
  [Schmutz-Schaller]



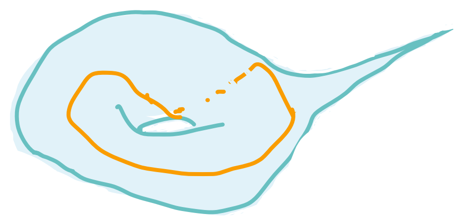
- Many identities about lengths spectrum

Thm (McShane's identity Invent. Math. 98)

$\forall$ $X = T \setminus \{p_1\}$ once punctured torus.

$$\sum_{\gamma} \frac{1}{e^{\ell_X(\gamma)} + 1} = 1$$

γ oriented simple closed geodesic



- Many identities about lengths spectrum

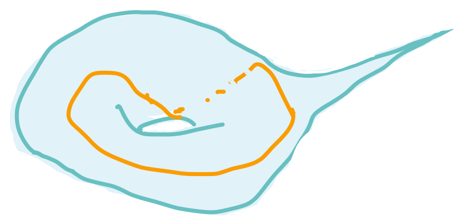
⇝ all identities on 1 surface

Thm (McShane's identity Invent. Math. 98)

$\forall$ $X = T \setminus \{p_1\}$ once punctured torus.

$$\sum_{\gamma} \frac{1}{e^{\ell_X(\gamma)} + 1} = 1$$

γ oriented simple closed geodesic



[Mirzakhani] Invent. 07 — Generalized McShane's identity ⇒ Computation of the Weil-Petersson volume of moduli spaces.

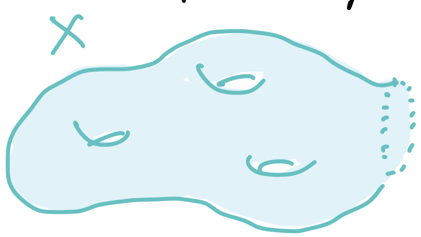
[Basmajian] AJM 93 — Relation length of geodesic boundary ⇔ lengths of orthogeodesics

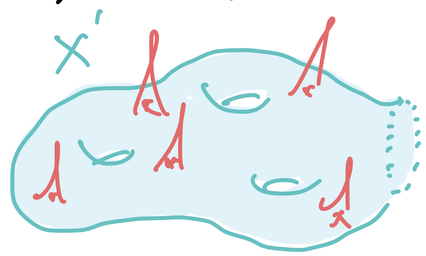
[Bridgeman] G & T 11 — Identities on orthogeodesics by decomposing T^1M

Also [Bridgeman – Kahn] [Calegari] [Parlier] ...

Main theorem 1 (W. - Xue)

Let X be complete hyperbolic surface, $P = \{p_1, \cdots\} \subset X$



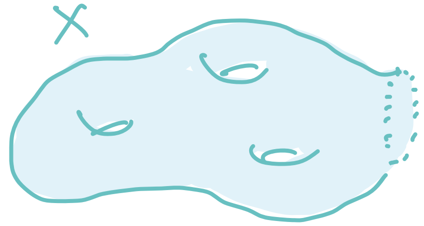


be a closed, countable subset of X, X' complete hyperbolic conformal to $X \setminus P$.

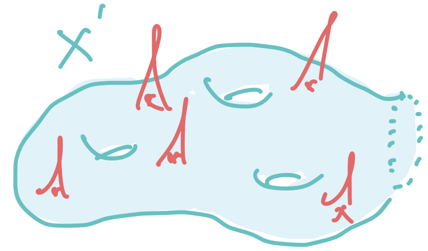
Main theorem 1 (W. – Xue)

Let X be complete hyperbolic surface, $P = \{p_1, \cdots\} \subset X$

X



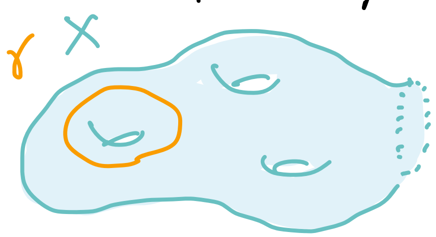
X'

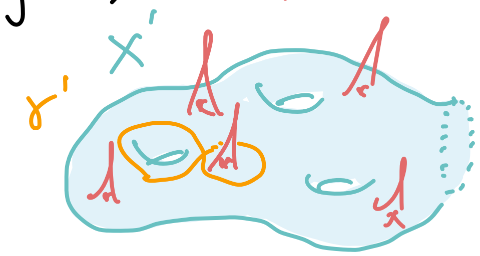


be a closed, countable subset of X, X' complete hyperbolic conformal to $X \setminus P$.

Main theorem 1 (W. - Xue)

Let X be complete hyperbolic surface, $P = \{p_1, \cdots\} \subset X$





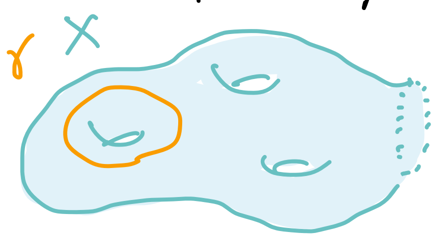
be a closed, countable subset of X, X' complete hyperbolic conformal to $X \setminus P$.

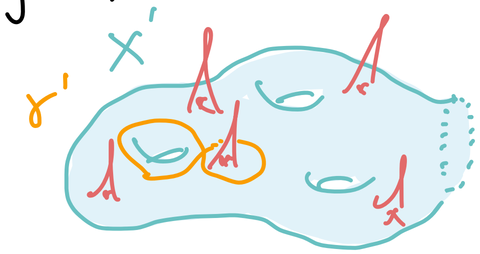
Let γ be a closed geodesic in X,

$$\frac{1}{m_X(\gamma)} \frac{1}{e^{\ell_X(\gamma)} - 1} = \sum_{\gamma' \simeq_X \gamma} \frac{1}{m_{X'}(\gamma')} \frac{1}{e^{\ell_{X'}(\gamma')} - 1}.$$

Main theorem 1 (W. - Xue)

Let X be complete hyperbolic surface, $P=\{p_1,\cdots\}\subset X$





be a closed, countable subset of X, X' complete hyperbolic conformal to $X\setminus P$.

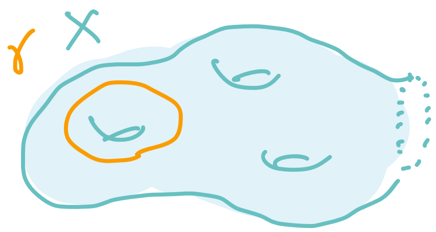
Let γ be a closed geodesic in X,

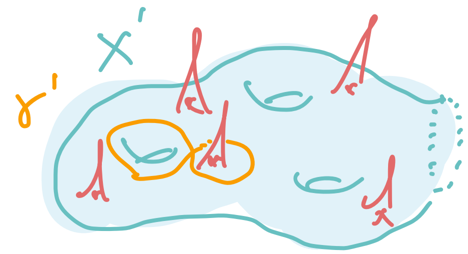
$$\frac{1}{m_X(\gamma)}\frac{1}{e^{\ell_X(\gamma)}-1}=\sum_{\gamma'\sqsupset_X\gamma}\frac{1}{m_{X'}(\gamma')}\frac{1}{e^{\ell_{X'}(\gamma')}-1}.$$

If T is a flat torus, $X'=T\setminus P$,

$$\frac{\mathrm{Area}(T)}{\pi\,\ell_T(\gamma)^2}=\sum_{\gamma'\sqsupset_X\gamma}\frac{1}{m_{X'}(\gamma')}\frac{1}{e^{\ell_{X'}(\gamma')}-1}.$$

Comments





$$\frac{\mathrm{Area}(T)}{\pi\, \ell_T(\gamma)^2} \quad \text{or} \quad \cancel{\frac{1}{m_X(\gamma)}}\, \frac{1}{e^{\ell_X(\gamma)}-1} = \sum_{\gamma' \sqsupset_X \gamma} \cancel{\frac{1}{m_{X'}(\gamma')}}\, \frac{1}{e^{\ell_{X'}(\gamma')}-1} .$$

...

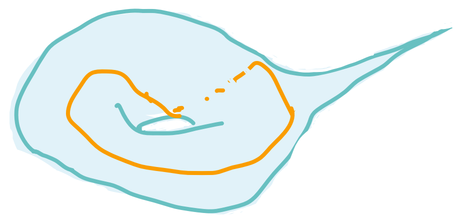
Remark: If γ is primitive $(m_X(\gamma) = 1)$, then $m_{X'}(\gamma') = 1$.
since $m_{X'}(\gamma') \mid m_X(\gamma)$.

Thm (McShane's Identity Invent. Math. 98)

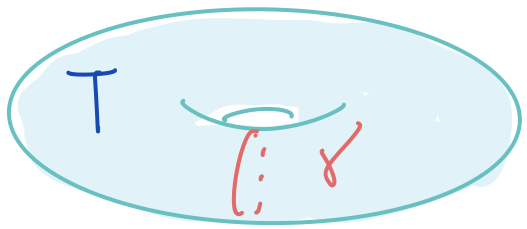
$\forall$ $X = T \setminus \{p\}$ once punctured torus

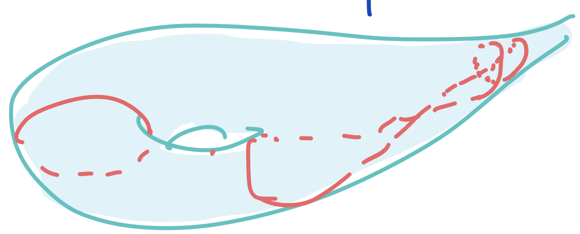
$$\sum_{\gamma} \frac{1}{e^{\ell_X(\gamma)}+1} = 1$$

γ oriented simple closed geodesic



Compare to our identity

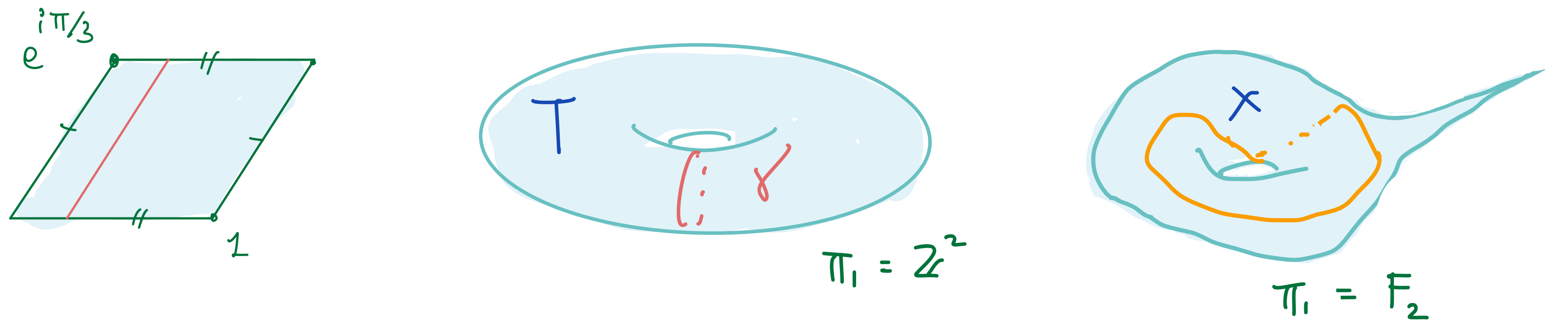




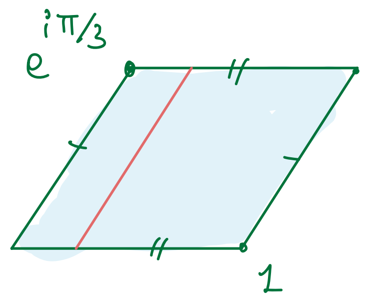
$$\frac{\text{Area}(T)}{\pi \,\text{Length}(\gamma)^2} = \sum_{\gamma' \sim_T \gamma} \frac{1}{e^{\ell_X(\gamma')}-1}$$

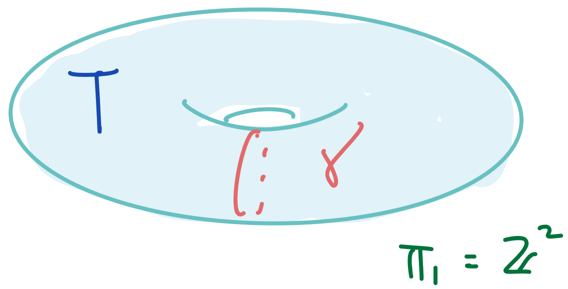
$\leftarrow$ Non simple

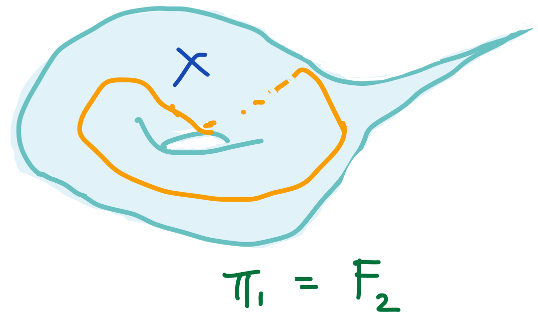
Simulations (Mingkun Lin) for modular torus



Simulations (Mingkun Lin) for modular torus





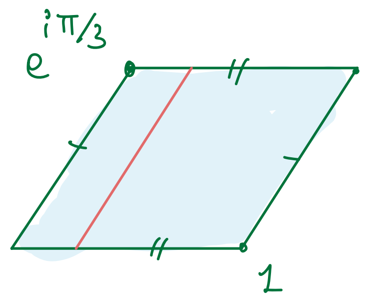


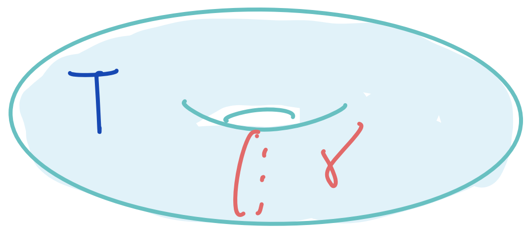
McShane: $$\sum_{\gamma} \frac{1}{e^{\ell_X(\gamma)}+1} = 1$$

γ oriented simple closed geodesic

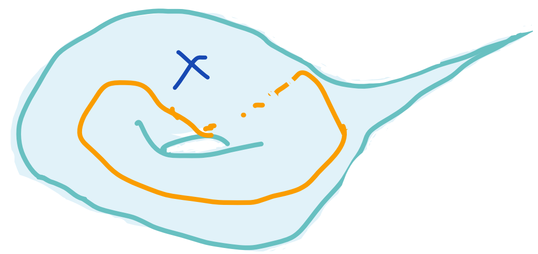
convergence 99% $\simeq$ 40 terms

Simulations (Mingkun Lin) for modular torus





$\pi_1 = \mathbb{Z}^2$



$\pi_1 = F_2$

McShane: $\displaystyle\sum_{\gamma} \frac{1}{e^{\ell_X(\gamma)} + 1} = 1$

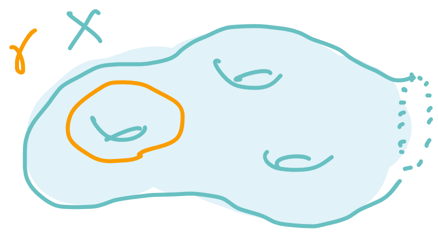
γ oriented simple closed geodesic

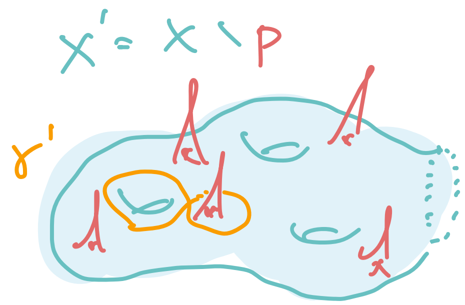
convergence 99% $\simeq$ 40 terms

W.-Xue: $\displaystyle\frac{\text{Area}(T)}{\pi \, \text{Length}(\gamma)^2} = \sum_{\gamma' \underset{T}{\sim} \gamma} \frac{1}{e^{\ell_X(\gamma')} - 1}$ ⇐ Non simple

convergence 75% 80,000 terms
85% 50,000,000 terms

Main result:



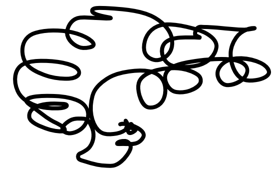


$$\frac{\mathrm{Area}(T)}{\pi\, \ell_T(\gamma)^2} \quad \text{or} \quad \frac{1}{m_X(\gamma)}\,\frac{1}{e^{\ell_X(\gamma)}-1} = \sum_{\gamma' \sqsupseteq_X \gamma} \frac{1}{m_{X'}(\gamma')}\,\frac{1}{e^{\ell_{X'}(\gamma')}-1}.$$

Proof idea:

- → Use Brownian motion on (X, g)
- → Conformal invariance of 2D BM
- → Probability of 2D BM hitting $\{x\}$ is 0

Brownian loop measure



- ∞, σ-finite measure $\mu_{X,g}$ on Brownian type loops on (X,g)
- Introduced by Lawler-Werner to study SLE in $\mathbb{C}$
- Used to construct family of simple random curves
  SLE, CLE

  [Lawler-Schramm-Werner] [Sheffield-Werner]

- "Total mass = $\log\det \Delta_{X,g}$"
  ↪ Partition function of GFF
  ↪ conformal anomaly of CFT

  [LeJan] [Dubédat] [Ang-Park-Pfeffer-Sheffield]

Definition & Properties

Brownian Bridge on (X,g) of total mass $P_t(x,x)$

(heat kernel with Dirichlet ∂-condition)

$$\mu_{X,g} := \int_0^\infty \frac{1}{t} \int_X \mathbb{W}^t_{x \to x} \, d\mathrm{vol}_g(x) \, dt$$

measure on loops in (X,g)

Definition & Properties

Brownian Bridge on (X,g) of total mass $P_t(x,x)$

(heat kernel with Dirichlet ∂-condition)

$$\mu_{X,g} := \int_0^\infty \frac{1}{t} \int_X \mathbb{W}^t_{x \to x} \, d\mathrm{vol}_g(x) \, dt$$

- Infinite measure on loops in X
- Restriction: If $X' \subset X$, $\mu_{X',g} = \mu_{X,g} \mathbb{1}_{\{\cdot \subset X'\}}$.
- Conformal invariance: $\forall \sigma \in C^\infty(X, \mathbb{R})$

$$\mu_{X,g} = \mu_{X, e^{2\sigma} g}$$

(forgetting the parametrization but keeping the orientation)

Definition & Properties

Brownian Bridge on (X,g) of total mass $P_t(x,x)$

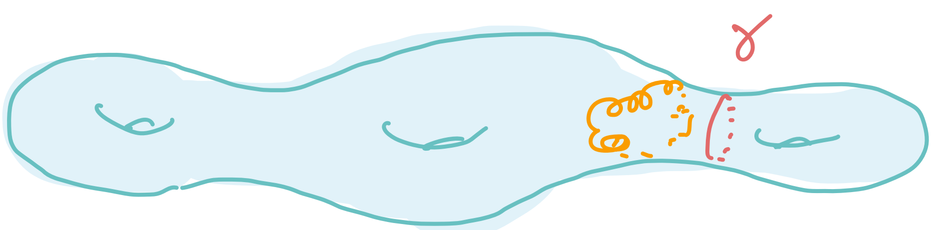
(heat kernel with Dirichlet ∂-condition)

$$\mu_{X,g} := \int_0^\infty \frac{1}{t} \int_X \mathbb{W}^t_{x \to x} \, d\mathrm{vol}_g(x) \, dt$$

- Infinite measure on loops in X
- Restriction: If $X' \subset X$, $\mu_{X',g} = \mu_{X,g} \mathbb{1}_{\{\cdot \subset X'\}}$.
- Conformal invariance: $\forall \sigma \in C^\infty(X, \mathbb{R})$

$$\mu_{X,g} = \mu_{X, e^{2\sigma} g}$$

(forgetting the parametrization but keeping the orientation)

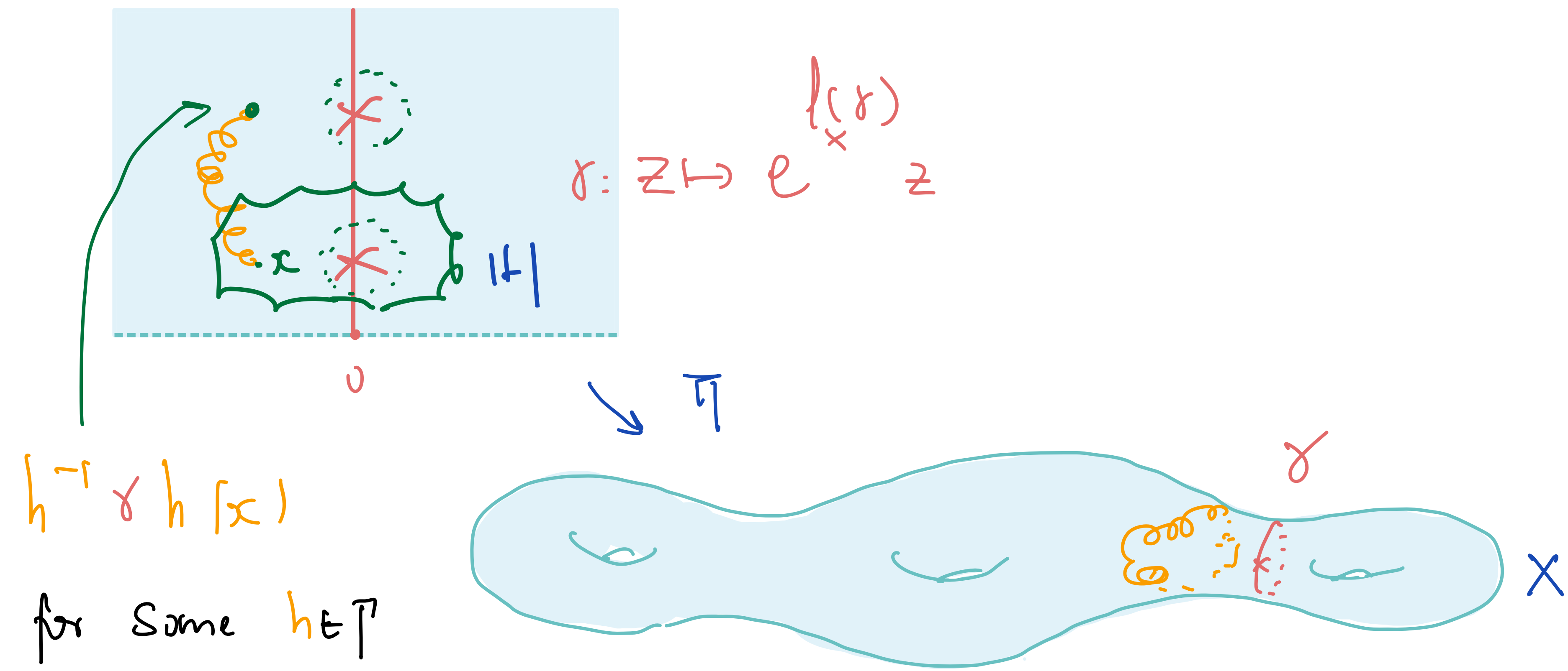


loops in a non-trivial free homotopic class $\mu_{X,g}([\gamma]) < \infty$ for compact surface [LeJan]

Computing $\mu_{X,g}([\gamma])$

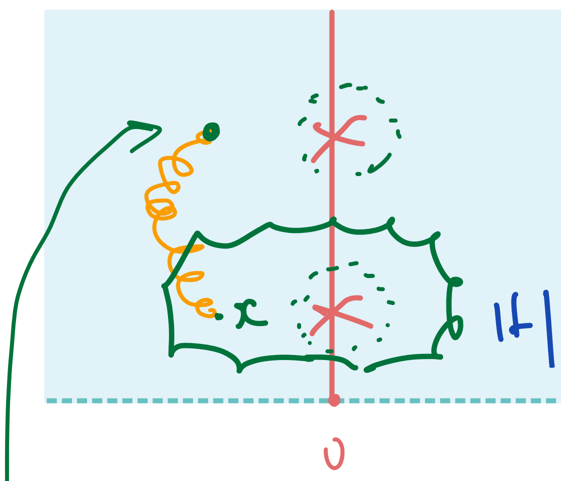
first assume γ primitive

- Conformal invariance $\Rightarrow$ Take $g = g_0$ complete constant curvature metric.



Computing $\mu_{x,g}([\gamma])$

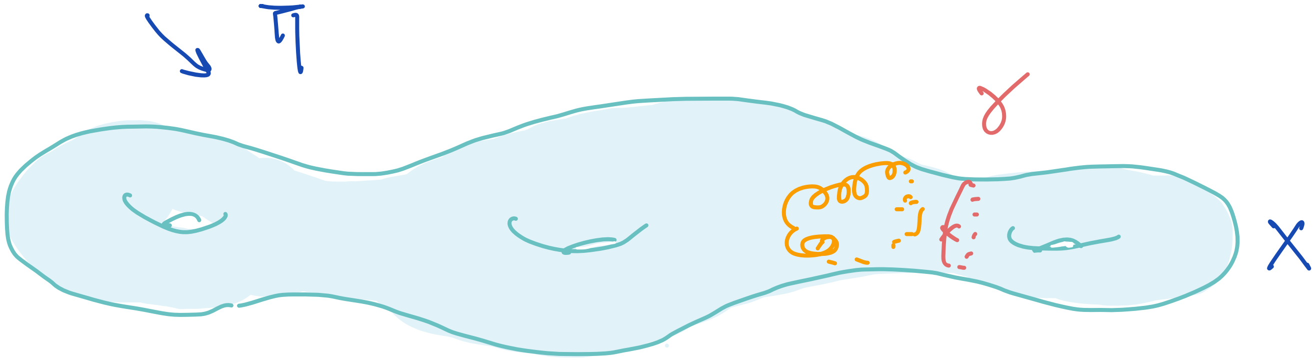
- Conformal invariance $\Rightarrow$ Take $g = g_0$ complete constant curvature metric.



$\gamma : z \mapsto e^{\ell_x(\gamma)} z$

$\underbrace{h^{-1}\gamma h}_{\gamma'}(x)$

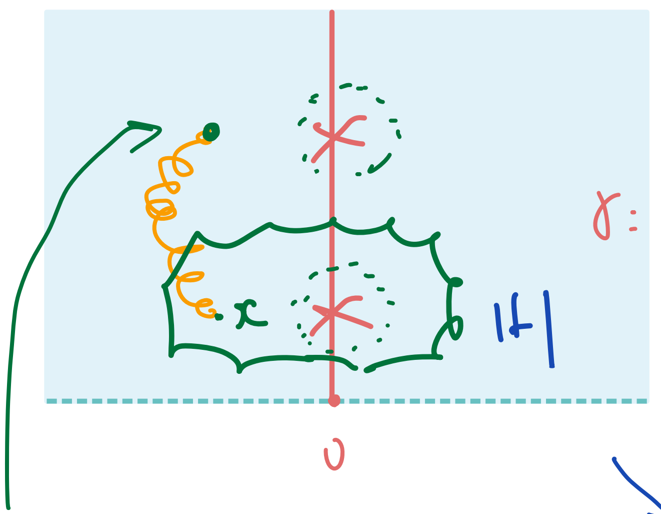
for some $h \in \Gamma$



$$\mu_x([\gamma]) = \int_0^\infty \frac{1}{t} \int_F \sum_{\gamma' \in \text{conjugacy class of } \gamma \text{ in } \Gamma} \underbrace{P_t^{\mathbb{H}}(x, \gamma' x)}_{P_t^{\mathbb{H}}(h(x),\, \gamma h(x))} \, d\mathrm{vol}_x \, dt$$

Computing $\mu_{x,g}([\gamma])$

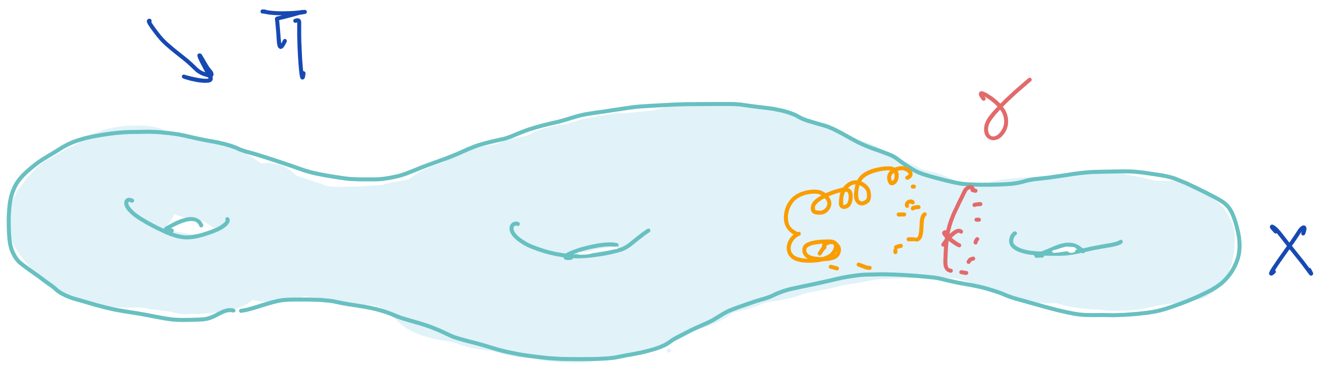
• Conformal invariance $\Rightarrow$ Take $g = g_0$ complete constant curvature metric.



$\gamma : z \mapsto e^{\ell_x(\gamma)} z$

$\underbrace{h^{-1}\gamma h}_{\gamma'}(x)$

for some $h \in \Gamma$

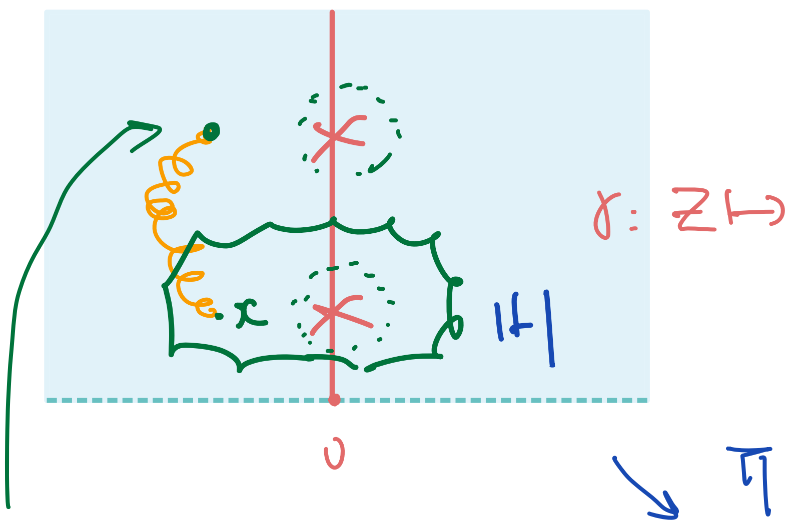


$$\mu_x([\gamma]) = \int_0^\infty \frac{1}{t} \int_{\cup h(F)} P_t^{\mathbb{H}}(x, \gamma x)\, d\mathrm{vol}_x\, dt$$

if γ primitive $\quad h^{-1}\gamma h = \tilde{h}^{-1}\gamma\tilde{h} \iff h = \gamma\tilde{h}$

Computing $\mu_{x,g}([\gamma])$

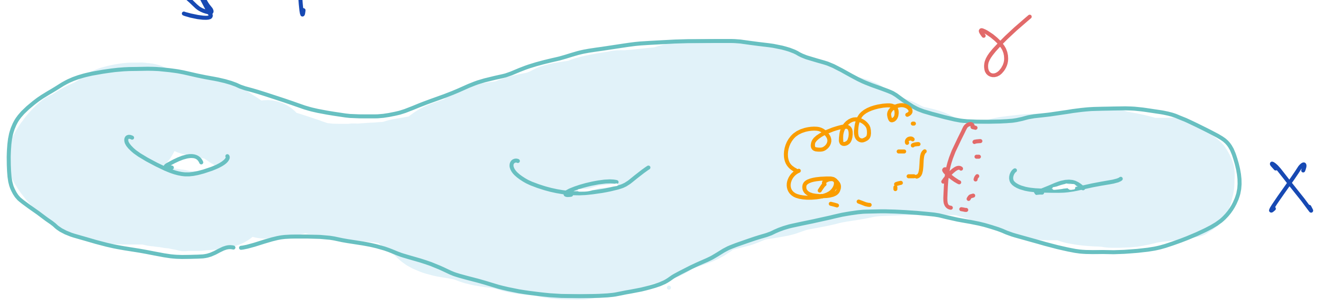
- Conformal invariance $\Rightarrow$ Take $g = g_0$ complete constant curvature metric.



$\gamma : z \mapsto e^{l_x(\gamma)} z$

$\underbrace{h^{-1}\gamma h}_{\gamma'}(x)$

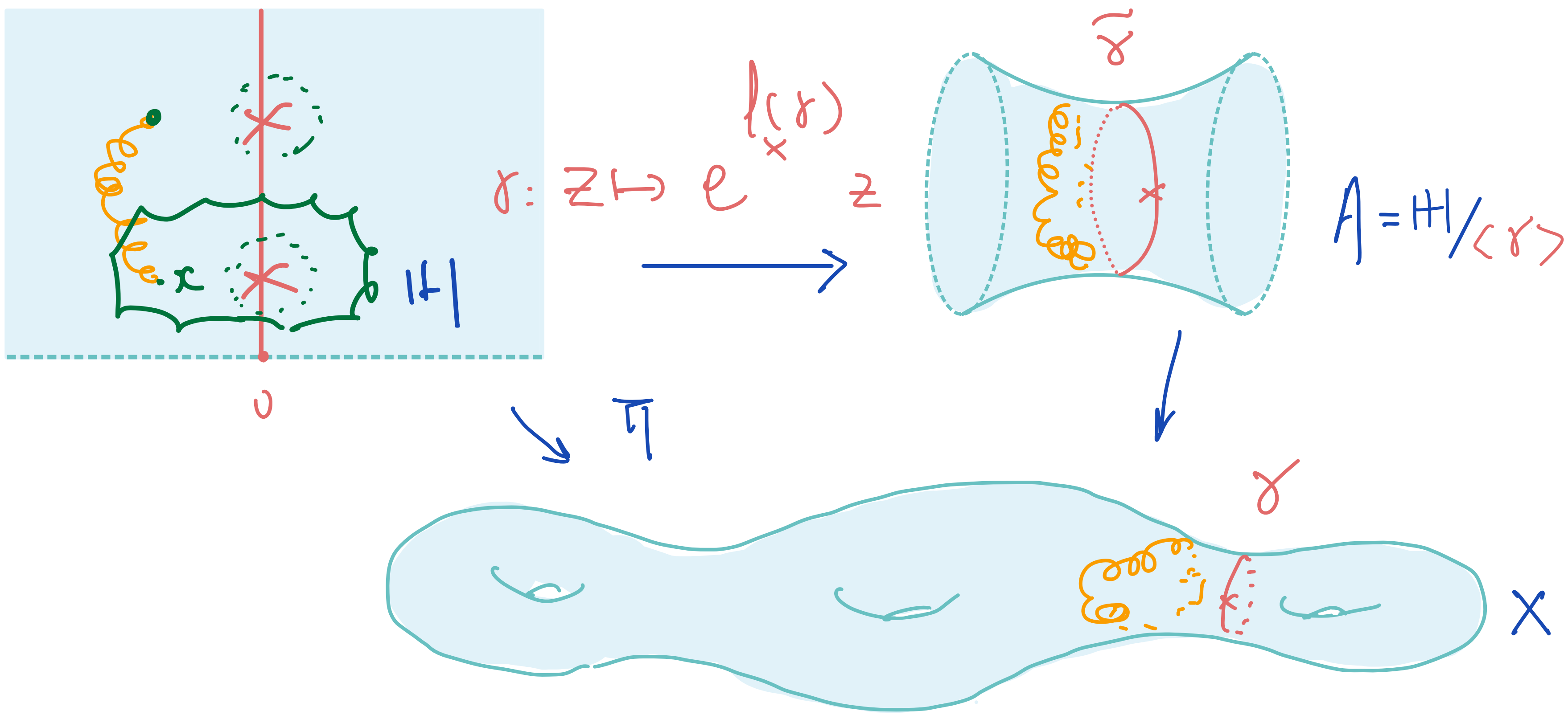
for some $h \in \Gamma$



$$\mu_x([\gamma]) = \int_0^\infty \frac{1}{t} \int_{\mathbb{H}/\langle\gamma\rangle} P_t^{\mathbb{H}}(x, \gamma x)\, d\mathrm{vol}\, x\, dt$$

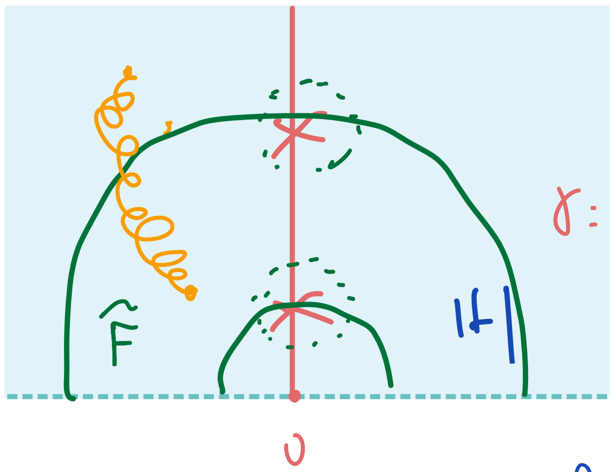
Computing $\mu_{X,g}([\gamma])$

- Conformal invariance $\Rightarrow$ Take $g = g_0$ complete constant curvature metric.

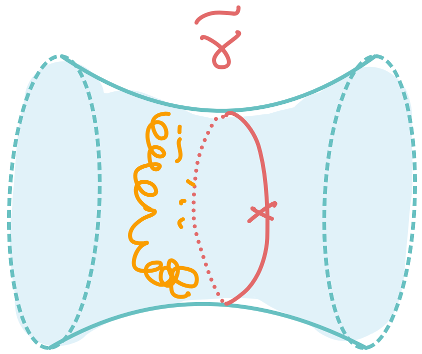


In other words: $\mu_X([\gamma]) = \mu_A([\tilde{\gamma}])$.

Step 2: Compute $\mu_A([\tilde{\gamma}])$



$\gamma : z \mapsto e^{\ell_X(\gamma)} z$



$A = \mathbb{H} / \langle \gamma \rangle$

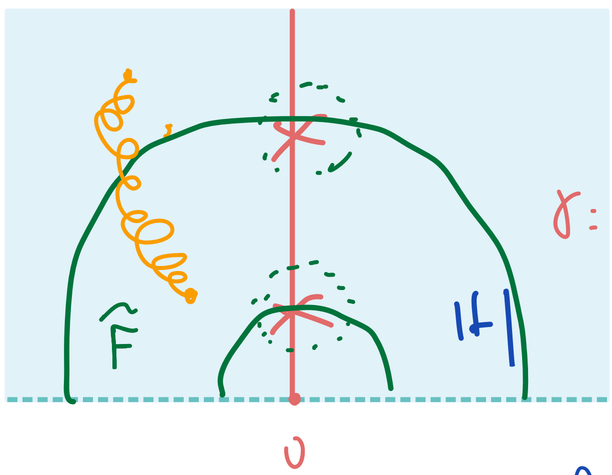
$$\mu_A([\tilde{\gamma}]) = \int_0^\infty \frac{1}{t} \int_{x \in \tilde{F}} p_t^{\mathbb{H}}(x, e^{\ell} x) \, d\mathrm{vol}_{\mathbb{H}}(x) \, dt$$

$$= \frac{1}{e^{\ell_A(\tilde{\gamma})} - 1}$$

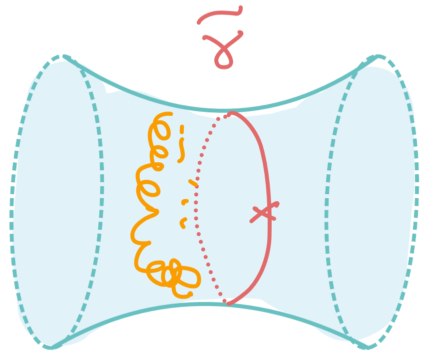
$$\Rightarrow \quad \mu_X([\gamma]) = \frac{1}{e^{\ell_X(\gamma)} - 1}$$

$\boxtimes$

Step 2: Compute $\mu_A([\tilde{\gamma}])$



$\gamma : z \mapsto e^{l_x(\gamma)} z$



$A = \mathbb{H}/\langle \gamma \rangle$

$$\mu_A([\tilde{\gamma}]) = \int_0^\infty \frac{1}{t} \int_{x \in \tilde{F}} p_t^{\mathbb{H}}(x, e^{l} x)\, d\mathrm{vol}_{\mathbb{H}}(x)\, dt$$

$$= \frac{1}{e^{l_A(\tilde{\gamma})} - 1}$$

$$\Rightarrow \mu_x([\gamma]) = \frac{1}{e^{l_x(\gamma)} - 1}$$

when γ is primitive

$\boxtimes$

More generally $\mu_x([\gamma]) = \dfrac{1}{m(\gamma)} \dfrac{1}{e^{l_x(\gamma)} - 1}$

Proof of main theorem 1

$X' = X \setminus p$

$$\frac{\mathrm{Area}(T)}{\pi\, \ell_T(\gamma)^2} \text{ or } \frac{1}{m_X(\gamma)} \frac{1}{e^{\ell_X(\gamma)}-1} = \sum_{\gamma' \sqsupseteq_X \gamma} \frac{1}{m_{X'}(\gamma')} \frac{1}{e^{\ell_{X'}(\gamma')}-1} .$$

$$\mu_X([\gamma]) = \sum_{\gamma' \sqsupseteq_X \gamma} \mu_{X'}([\gamma'])$$

p is polar for BM

μ_X is conformally invariant

Total mass of Brownian loops

$\to$ Determinant of Δ

"Total mass of Brownian loop on (X, g)

$= -\log \det_{\zeta} \Delta_g$ "

[Ang, Park, Pfeffer, Sheffield] also [Le Jan] [Dubédat]

Theorem 4.10 (See [1, Thm. 1.3]). *For a closed Riemannian surface (X, g), the total mass of Brownian loops with quadratic variation greater than 4t and less than 4T is given by*

$$\frac{\mathrm{Vol}_g(X)}{4\pi t} - \log \det{}_\zeta \Delta_g - \frac{\chi(X)}{6}(\log t + \gamma) + \log T + \gamma + O(t) + O(e^{-\alpha T})$$

as $t \to 0$ and $T \to \infty$, where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant, $\chi(X) = 2 - 2\,\mathrm{genus}(X)$ is the Euler characteristic of X, and $\alpha > 0$.

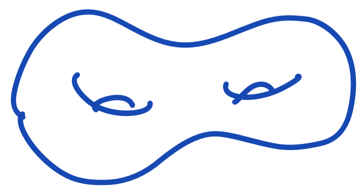
[Ang, Park, Pfeffer, Sheffield] also [Le Jan] [Dubédat]

Theorem 4.10 (See [1, Thm. 1.3]). *For a closed Riemannian surface (X, g), the total mass of Brownian loops with quadratic variation greater than 4t and less than 4T is given by*

$$\frac{\mathrm{Vol}_g(X)}{4\pi t} - \log\det{}_\zeta \Delta_g - \frac{\chi(X)}{6}(\log t + \gamma) + \log T + \gamma + O(t) + O(e^{-\alpha T})$$

as $t \to 0$ and $T \to \infty$, where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant, $\chi(X) = 2 - 2\,\mathrm{genus}(X)$ is the Euler characteristic of X, and $\alpha > 0$.

Does hyperbolic geometry help?



First consider closed hyperbolic surface with $<\infty$ genus.

Normalization by homotopy classes

$\mathcal{G}(X) = \{\text{geodesics of } X\}$. $\mathcal{P}(X) := \{\text{primitive geodesics of } X\}$

Prime geodesic theorem (Huber, Randol)

$$\underbrace{N_X(L)}_{\#\{\gamma \in \mathcal{P}(X) \mid \ell_X(\gamma) \le L\}} = \underbrace{\mathrm{Li}(\mathrm{e}^L)}_{\simeq\, e^L/L} + \sum_{0<\lambda_j\le\frac{1}{4}} \mathrm{Li}(\mathrm{e}^{s_j L}) + O_X(\mathrm{e}^{\frac{3}{4}L}/L)$$

where $\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$

$s_j = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_j}$

Normalization by homotopy classes

$\mathcal{G}(X) = \{$ geodesics of $X\}$. $\mathcal{P}(X) := \{$primitive geodesics of $X\}$

Prime geodesic theorem (Huber, Randol)

$$\underbrace{N_X(L)}_{\substack{\shortparallel \\ \#\{\gamma \in \mathcal{P}(X) \mid \ell_X(\gamma) \leq L\}}} = \underbrace{\mathrm{Li}(e^L)}_{\simeq e^L/L} + \sum_{0<\lambda_j\leq\frac{1}{4}} \mathrm{Li}(e^{s_j L}) + O_X(e^{\frac{3}{4}L}/L)$$

where $\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$

$s_j = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_j}$

Main Theorem 2 (W.-Xue)

$$-\log \det{}_\zeta \Delta = -\mathrm{Area}(X)\, E + C + \sum_{\gamma \in \mathcal{G}(X) \setminus \mathcal{P}(X)} \mu_X([\gamma])$$

contribution from non-primitive cu

$$+ \int_{L=0}^{\infty} \frac{1}{e^L - 1}\, d(N_X(L) - \mathrm{Li}(e^L))$$

renormalization

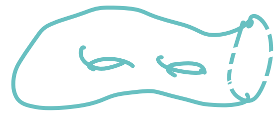
contribution from primitive curves

C, E are universal constants.

∞ area surfaces

If X is geometrically finite and ∞ area

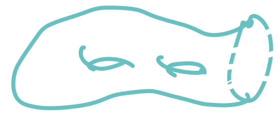
(at least 1 funnel)



∞ area surfaces

If X is geometrically finite and ∞ area

(at least 1 funnel)



Prime geodesic theorem (Huber)

$\delta \in [0,1)$

$$N_x(L) \sim \frac{e^{\delta L}}{\delta L} \quad \text{as } L \to \infty$$

$= \mathrm{Hdim}(\Lambda(\Gamma))$

$\Rightarrow \quad \sum_{[\gamma]} \mu_x([\gamma]) < \infty$. No renormalization needed.

∞ area surfaces

If X is geometrically finite and ∞ area

(at least 1 funnel)

Prime geodesic theorem (Huber)

$$N_x(L) \sim \frac{e^{\delta L}}{\delta L} \quad \text{as } L \to \infty$$

$\delta \in [0,1)$

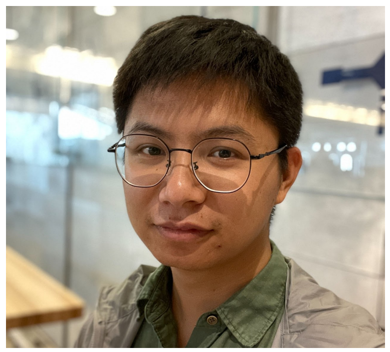
$\shortparallel$

$\mathrm{Hdim}(\Lambda(\Gamma))$

$$\Rightarrow \quad \sum_{[\gamma]} \mu_x([\gamma]) < \infty.$$

No renormalization needed.

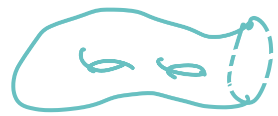
Thm (Roman Lemonde - Jian Wang)



∞ area surfaces

If X is geometrically finite and ∞ area

(at least 1 funnel)



Prime geodesic theorem (Huber)

$\delta \in [0,1)$
$\parallel$
$\mathrm{Hdim}(\Lambda(\Gamma))$

$$N_X(L) \sim \frac{e^{\delta L}}{\delta L} \quad \text{as } L \to \infty$$

$$\Rightarrow \quad \sum_{[\gamma]} \mu_X([\gamma]) < \infty .$$

No renormalization needed.

Thm (Roman Lemonde - Jian Wang) $\sum \cdots = -\log \underbrace{Z_X(1)}$

Selberg zeta function

More generally, k = killing rate

$$\sum_{[\gamma]} \mu_X^k([\gamma]) = -\log Z_X\left(\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + k}\right)$$

if $\frac{1}{2} + \sqrt{\frac{1}{4} + k} > \delta$.

$$Z_X(s) := \prod_{\gamma \in \mathcal{P}_X} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell_\gamma}\right)$$

Conclusion

Mass of Brownian loop in a homotopy class $[\gamma]$ on a surface (X, g) is expressed as a function of $\ell_X(\gamma)$ length of the geodesic for the complete metric g_0 of constant curvature and conformal to g.

Conclusion

Mass of Brownian loop in a homotopy class $[\gamma]$ on a surface (X, g) is expressed as a function of $\ell_X(\gamma)$ length of the geodesic for the complete metric g_0 of constant curvature and conformal to g.

$\rightarrow$ Tool to study the length spectrum of Riemann surface

$\Rightarrow$ Identity on length spectra of X & $X \setminus p$

$\Rightarrow$ Another renormalization of total mass of BL

$\longleftrightarrow$ $\log \det \Delta$ & Selberg Zeta function.


Happy Birthday Greg!!!

Key: $\zeta_\Delta(s) = \sum_{i\geq 1} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\mathrm{Tr}(e^{t\Delta}) - 1 \right) dt$

Selberg trace formula

$$\sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j} = \mathrm{Area}(X) \frac{\mathrm{e}^{-t/4}}{(4\pi t)^{3/2}} \int_0^\infty \frac{r\mathrm{e}^{-r^2/(4t)}}{\sinh(r/2)} \mathrm{d}r + \underbrace{\sum_{\gamma \in \mathcal{P}(X)} \sum_{m=1}^{\infty} \frac{\mathrm{e}^{-t/4}}{(4\pi t)^{1/2}} \frac{\ell(\gamma)}{2\sinh(m\ell(\gamma)/2)} \mathrm{e}^{-\frac{(m\ell(\gamma))^2}{4t}}}_{S_X(t) = \sum_{\gamma \in \mathcal{G}(X)} \int_X W^t_{x\to x}([\gamma])\, d\mathrm{vol}_x}.$$

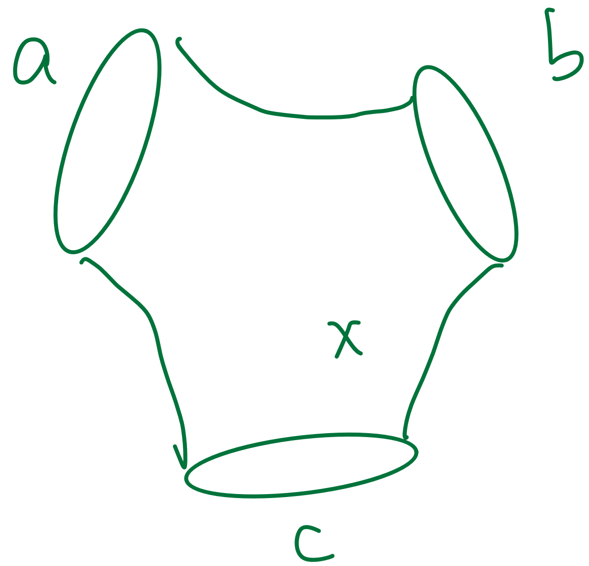
$$-\log \det{}_\zeta \Delta = -\mathrm{Area}(X)E - \gamma + \int_0^1 \frac{S_X(t)}{t}\,\mathrm{d}t + \int_1^\infty \frac{S_X(t) - 1}{t}\,\mathrm{d}t$$

Separate $\gamma \in \mathcal{P}(X)$ and $\gamma \in \mathcal{G}(X) \setminus \mathcal{P}(X)$

+ calculation ...

Remark: BLM is defined for any smooth metric but $\mu_x([\gamma])$ is a function of $\ell_x(\gamma)$ of complete constant curvature metric.

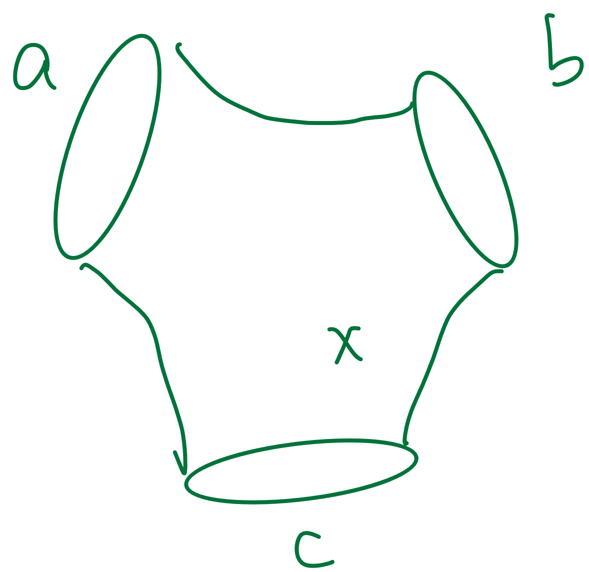
Example:



geodesic boundary (a, b, c)

Remark: BLM is defined for any smooth metric but $\mu_x([\gamma])$ is a function of $\ell_x(\gamma)$ of complete constant curvature metric.

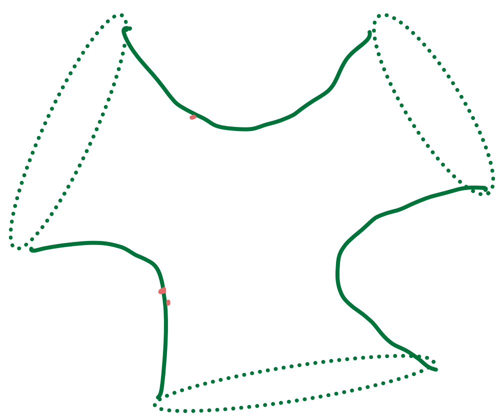
Example:



geodesic boundary (a, b, c)

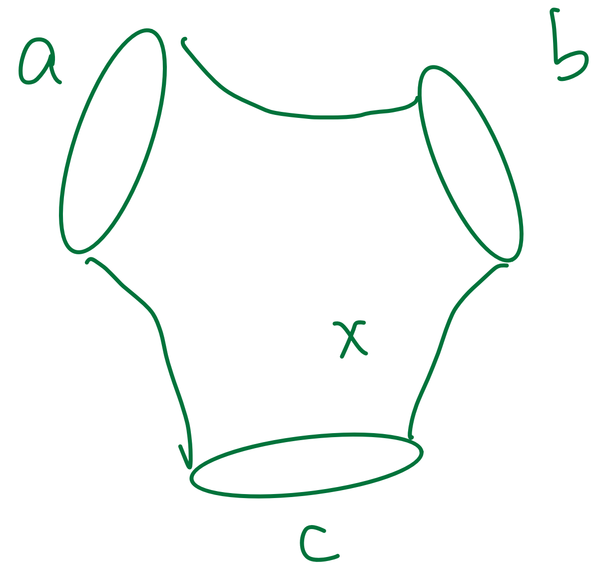
$\updownarrow$

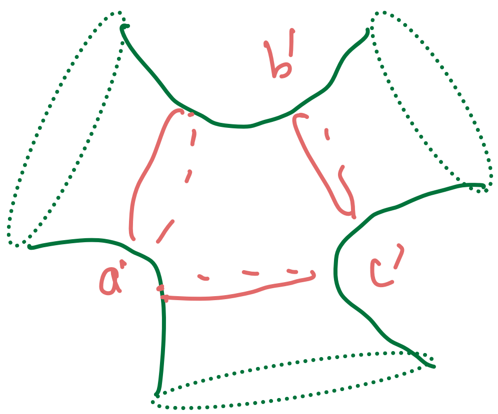
complete surface



Remark: BLM is defined for any smooth metric but $\mu_x([\gamma])$ is a function of $\ell_x(\gamma)$ of complete constant curvature metric.

Example:





geodesic boundary (a, b, c)

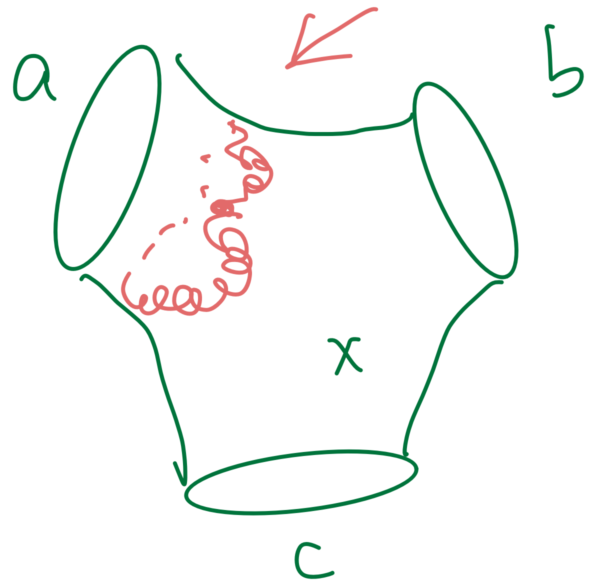
$\updownarrow$ $\qquad$ $\updownarrow$??

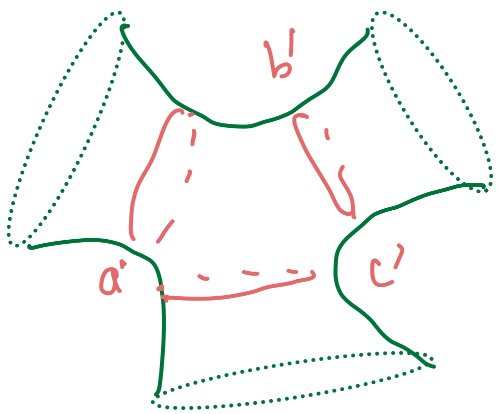
complete surface (a', b', c')

Remark: BLM is defined for any smooth metric but $\mu_x([\gamma])$ is a function of $\ell_x(\gamma)$ of complete constant curvature metric.

Example:

$$\frac{1}{e^{a'}-1} = \mu_x([a])$$





geodesic boundary (a, b, c)

$\updownarrow$ $\quad$ $\updownarrow$??

complete surface (a', b', c')