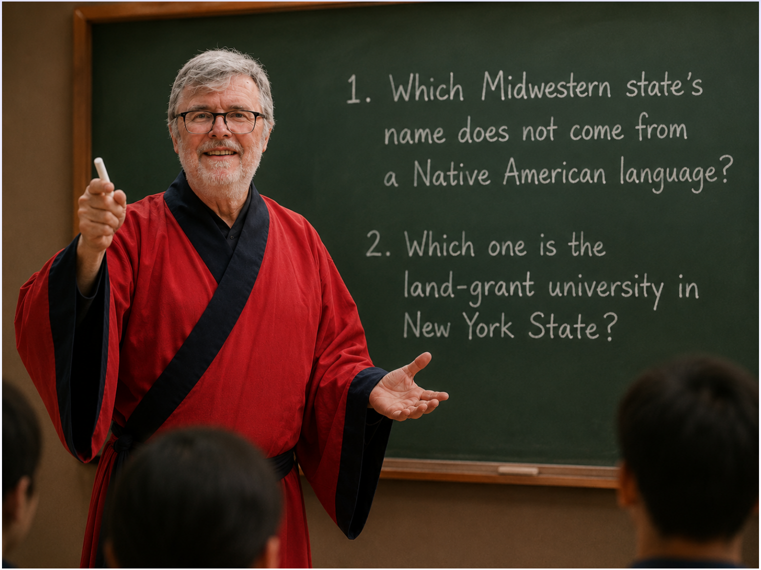
Langevin Dynamics for non-intersecting curves

Xuan Wu

University of Illinois Urbana-Champaign

Random Explorations
IMSI


1. Which Midwestern state's name does not come from a Native American language?
2. Which one is the land-grant university in New York State?

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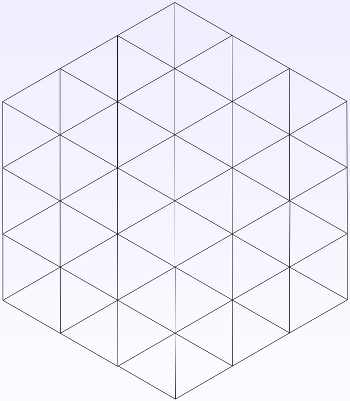
Lozenge Tilings



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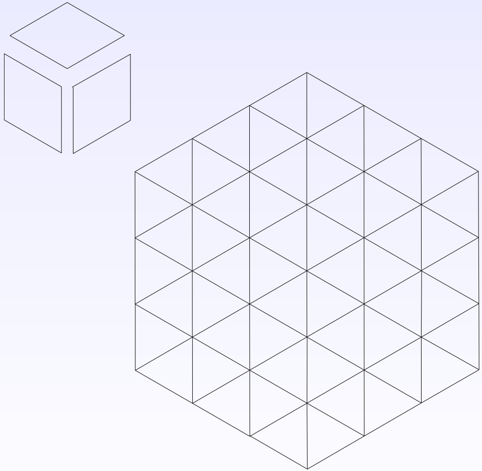

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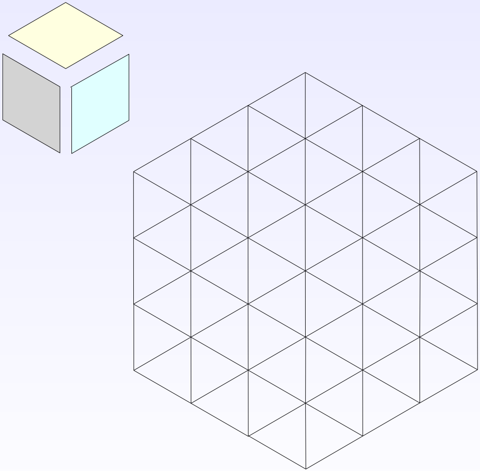

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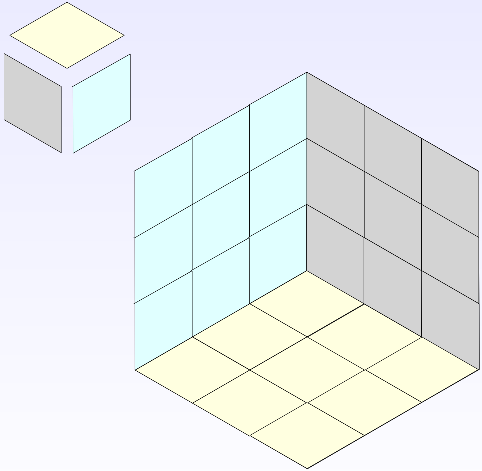

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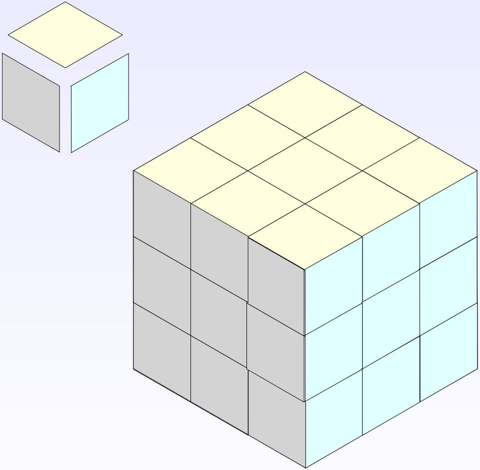

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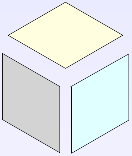

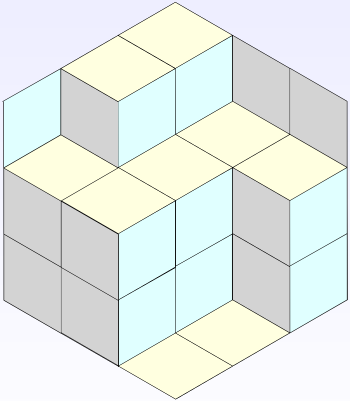

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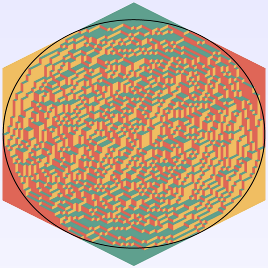

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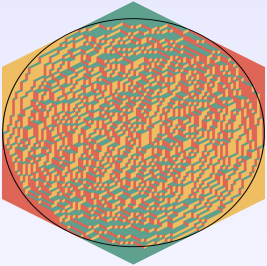
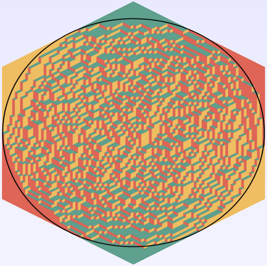


Limit Shape:

$$\frac{1}{N} h_N(Nx) \longrightarrow h^*(x)$$

Lozenge Tilings



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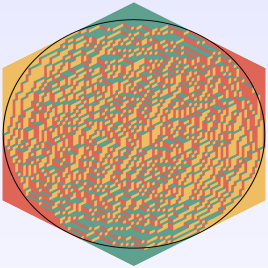

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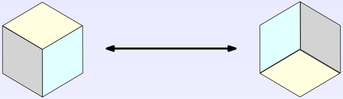
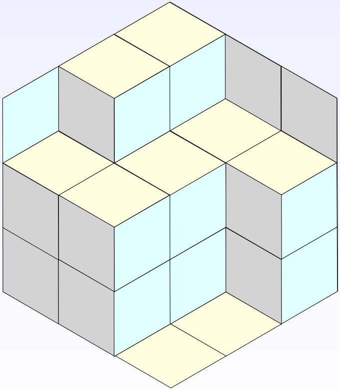
Local Limit:

$$\text{Law near } Nx \longrightarrow \nu(\nabla h^*(x))$$

Glauber Dynamics





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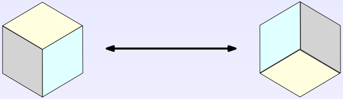

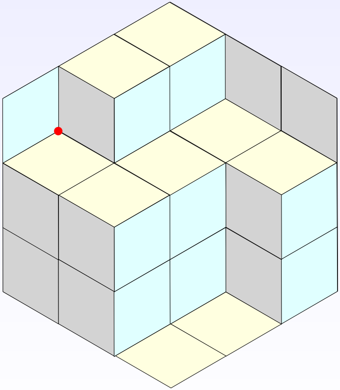

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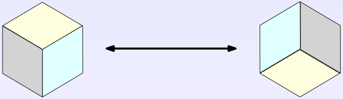

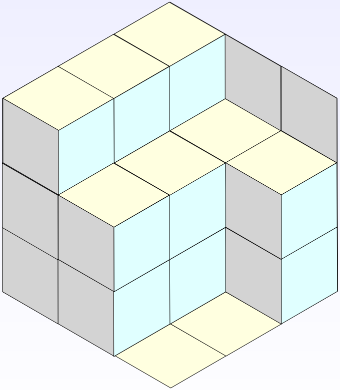

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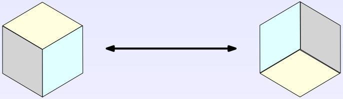

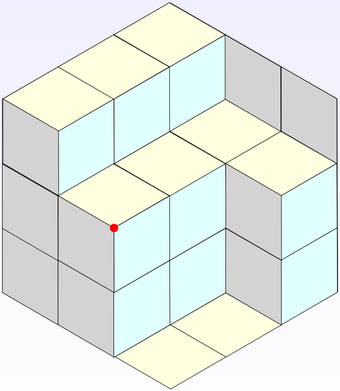

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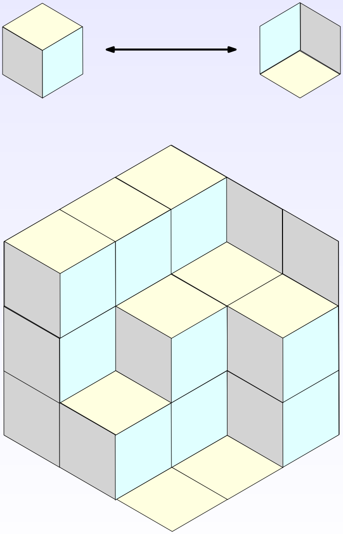

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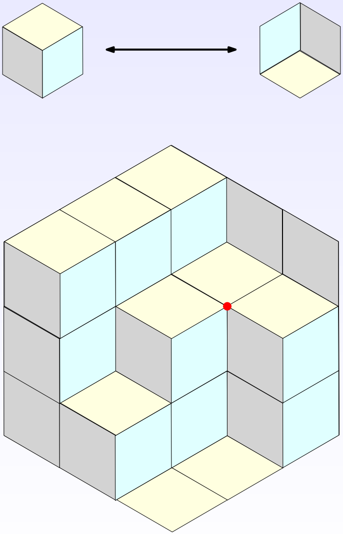

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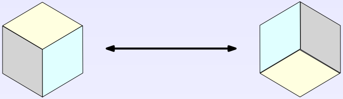

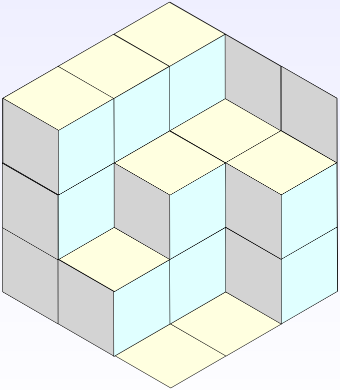

Glauber Dynamics

Conjecture

$$\frac{1}{N} h_N(N^2 t, Nx) \longrightarrow h^*(t, x)$$

$h^*(t, x)$ solves

$$\partial_t h^* = -\mu(\nabla h^*) \frac{\delta \mathcal{F}[h^*]}{\delta h^*}$$

Glauber Dynamics

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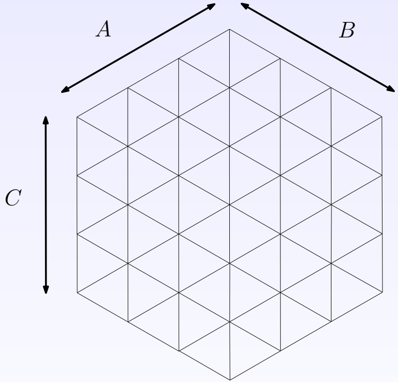
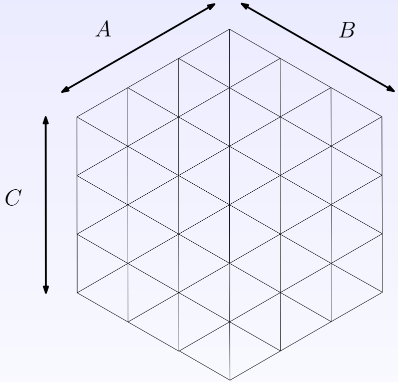
Long Hexagon



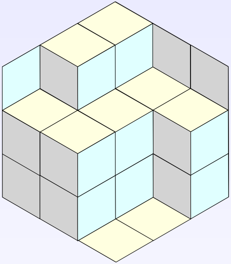
regular hexagon: $A = B = C \to \infty$

Long Hexagon



regular hexagon: $A = B = C \to \infty$
long hexagon: $A = B \to \infty$, $C = d$ fixed

Lozenge Tilings as Non-Intersecting Random Walks



Lozenge tilings $\Longleftrightarrow$ non-intersecting RW

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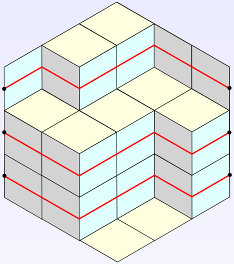

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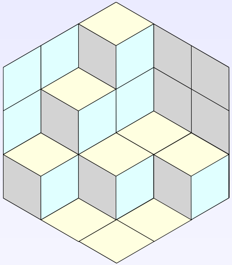

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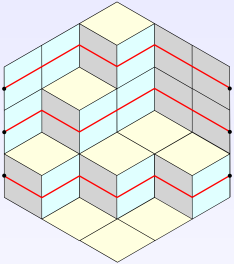

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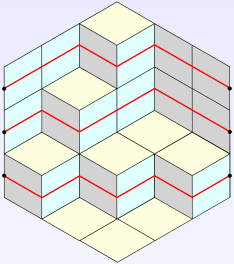

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Glauber dynamics

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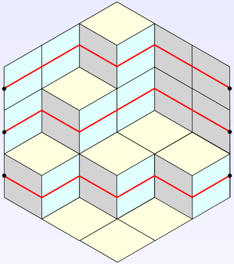

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Glauber dynamics   Glauber dynamics

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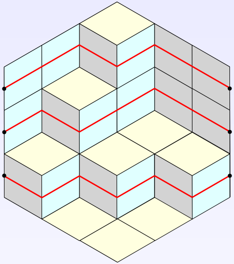

non-intersecting RW

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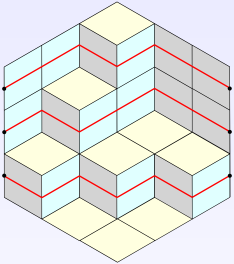
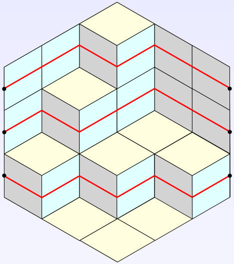


non-intersecting RW $\longrightarrow$ Dyson Brownian Bridge

Glauber dynamics

Lozenge Tilings as Non-Intersecting Random Walks



$$\begin{tikzcd}
\text{non-intersecting RW} \arrow[r] \arrow[loop below] & \text{Dyson Brownian Bridge} \arrow[loop below] \\
\text{Glauber dynamics} \arrow[r] & \text{some dynamics } (*)
\end{tikzcd}$$

Dynamics on non-intersecting curves

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The dynamics $(*)$:

Dynamics on non-intersecting curves

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Dynamics on non-intersecting curves

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The dynamics $(*)$:

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$\boxed{\text{Langevin Dynamics}}$

Langevin Dynamics

Gibbs measure $\Longrightarrow$ reversible Markov process

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Example

Gibbs measure: $Z^{-1}\exp(-H(z))dz$

Markov process: $dX_t = -\frac{1}{2}\nabla H(X_t)dt + dB_t$

Langevin Dynamics

Gibbs measure $\Longrightarrow$ reversible Markov process

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Markov process: $dX_t = -\dfrac{1}{2}\nabla H(X_t)dt + dB_t$

$H(z) = \frac{1}{2}z^2 \Longrightarrow$ Ornstein–Uhlenbeck process

Langevin Dynamics

$$\text{Gibbs measure} \Longrightarrow \text{reversible Markov process}$$

Example

Gibbs measure: Brownian motion/bridge

Langevin Dynamics

Gibbs measure $\Longrightarrow$ reversible Markov process

Example

Gibbs measure: Brownian motion/bridge

$$Z^{-1} \exp\left(- \int |\partial_x u(x)|^2 dx \right) du$$

Langevin Dynamics

Gibbs measure $\Longrightarrow$ reversible Markov process

Example

Gibbs measure: Brownian motion/bridge

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Markov process: stochastic heat equation

$$\partial_t u = \frac{1}{2}\partial_{xx} u + \xi$$

Langevin Dynamics for Dyson Brownian motion/bridge

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Brownian motion/bridge conditioned in $\Delta = \{y \,:\, y_1 \geq y_2 \geq \cdots \geq y_d\}$

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Markov process:

$$\partial_t u = \frac{1}{2} \partial_{xx} u - \frac{1}{2} \nabla V(u) + \xi$$

Reflected Stochastic Heat Equation

Definition

Fix $d \in \mathbb{N}$. A random function $u : [0, \infty) \times [0, 1] \to \mathbb{R}^d$ solves the reflected stochastic heat equation (RSHE) provided

$$\partial_t u = \frac{1}{2}\partial_{xx} u - \frac{1}{2}\nabla V(u) + \xi,$$

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∇V is ill-defined!

Reflected Stochastic Heat Equation

Definition

Fix $d \in \mathbb{N}$. A random function $u : [0,\infty) \times [0,1] \to \mathbb{R}^d$ and a vector-valued Radon measure η solve the RSHE provided

(1) $\partial_t u = \frac{1}{2}\partial_{xx} u + \eta + \xi.$

(2) $u(t,x) \in \Delta$ for all (t,x).

(3) For any random function v with $v(t,x) \in \Delta$ for all (t,x),

$$\langle v - u, \eta \rangle \geq 0.$$

Reflected Stochastic Heat Equation

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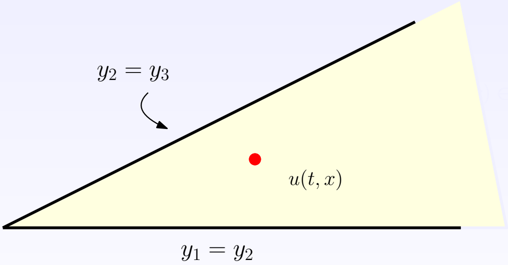

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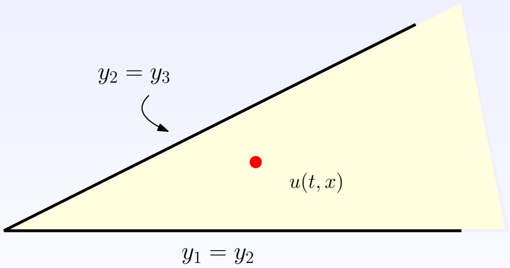

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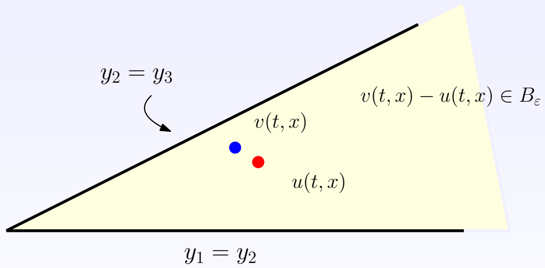

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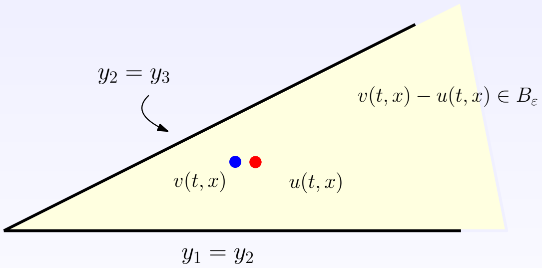

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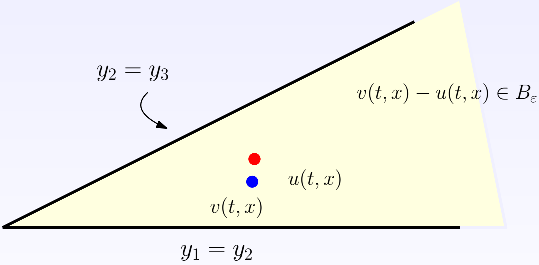

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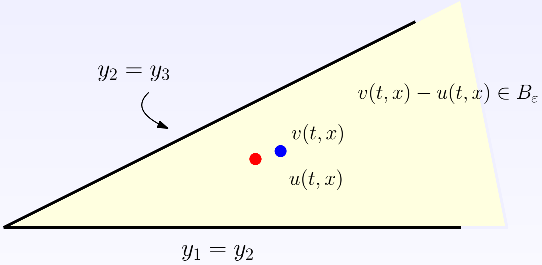

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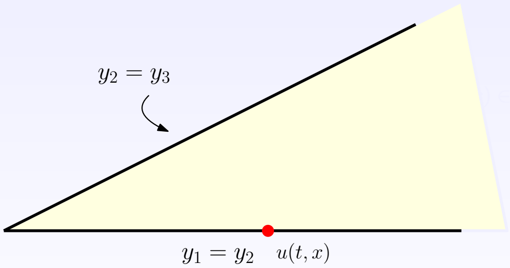

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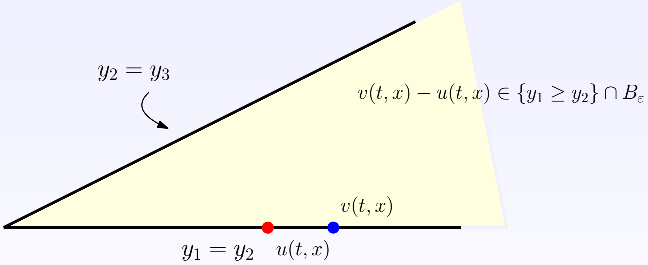

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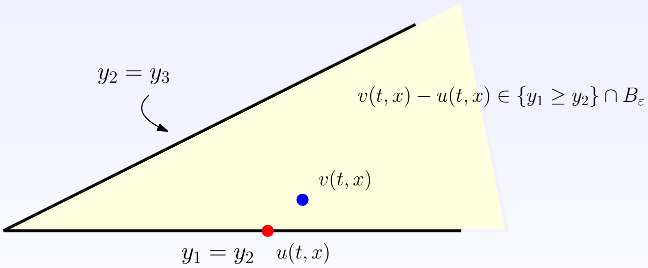

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Reflected Stochastic Heat Equation

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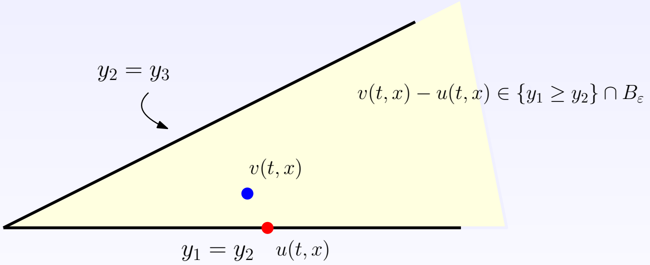

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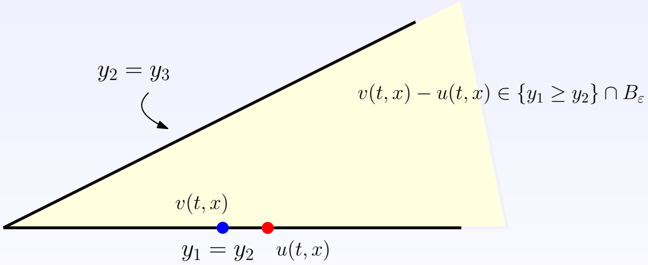

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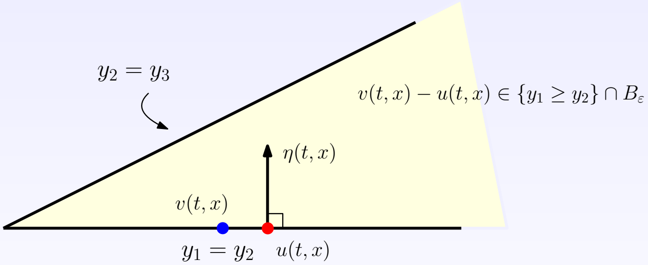

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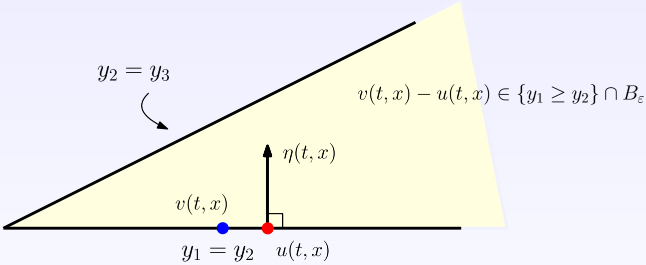

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Definition

Fix $d \in \mathbb{N}$. A random function $u : [0,\infty) \times [0,1] \to \mathbb{R}^d$ and a vector-valued Radon measure η solve the RSHE provided

(1) $\partial_t u = \frac{1}{2}\partial_{xx} u + \eta + \xi$.

(2) $u(t,x) \in \Delta$ for all (t,x).

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Main Results

Theorem 1 (Hung-Tsai-W. 26+)

Given a (random) continuous function $\varphi : [0,1] \to \mathbb{R}^d$ *with* $\varphi(x) \in \Delta$, *there exists a unique solution* $u(t,x)$ *to the RSHE with* $u(0,x) = \varphi(x)$.

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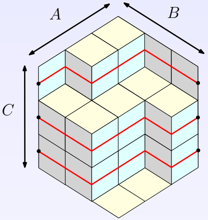
Theorem 2 (Hung-Tsai-W. 26+)

Under the long hexagon limit, the stationary Glauber dynamics of Lozenge tiling/non-intersecting RW converges to the RSHE.

Recap



Glauber dynamics

reflected stochastic heat equation

$\Big\downarrow d \to \infty$

????

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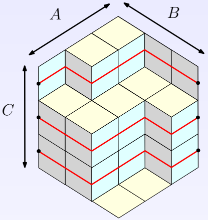

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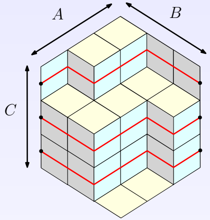

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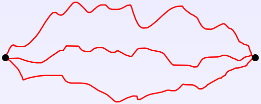

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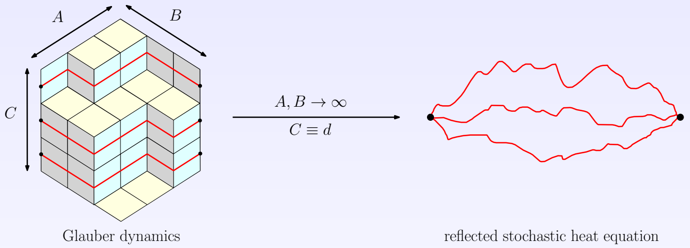
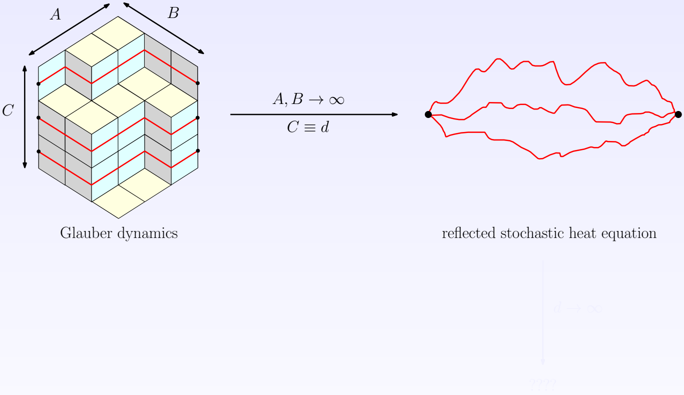


Theorem 1: well-posedness of RSHE

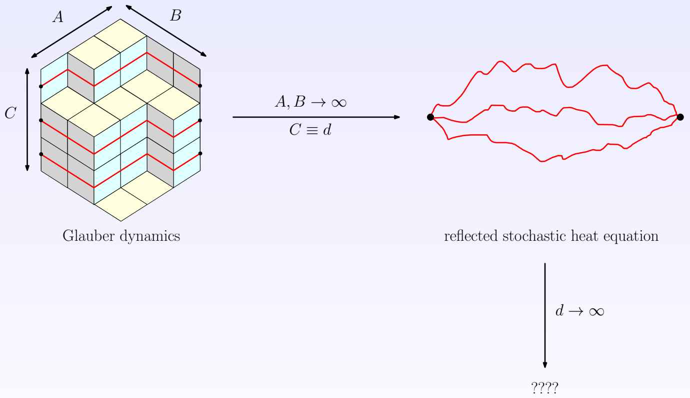
Recap



Theorem 1: well-posedness of RSHE

Theorem 2: stationary Glauber dynamics converges to RSHE

Recap



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Theorem 2: stationary Glauber dynamics converges to RSHE

Conjectural Hydrodynamic Limit

For fixed $d \in \mathbb{N}$, let $u_1(t,x), u_2(t,x), \dots u_d(t,x)$ be the solution to RSHE. Define

$$U^{(d)}(t,x,y) = \frac{1}{\sqrt{d}} u_{dy}(t,x)$$

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Conjecture

$$\lim_{d\to\infty} U^{(d)}(t,x,y) \to U^*(t,x,y)$$

$U^*(t,x,y)$ *solves the equation*

$$\partial_t U^* = \partial_{xx} U^* + \pi^2 (\partial_y U^*)^{-4} \partial_{yy} U^*$$

Conjectural Hydrodynamic Limit

$$\partial_t U^* = \partial_{xx} U^* + \pi^2 (\partial_y U^*)^{-4} \partial_{yy} U^*$$

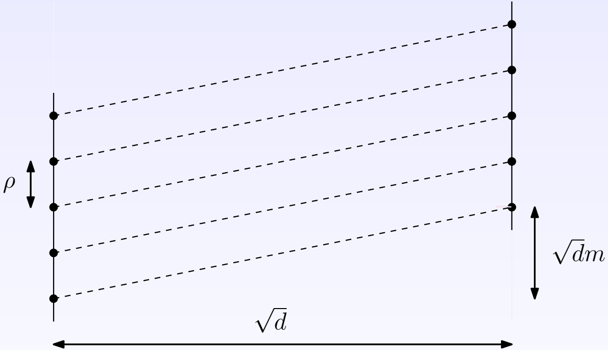
The RHS is the gradient of

$$\mathcal{F}(U) = \int \underbrace{\frac{1}{2}|\partial_x U|^2 + \frac{\pi^2}{6}|\partial_y U|^{-2}}_{\text{surface tension } \sigma} \, dxdy$$

$$\sigma(m, \rho) = \frac{1}{2} m^2 + \frac{\pi^2}{6} \rho^{-2}$$

Surface Tension

Let $\partial_x U = m$ and $\partial_y U = \rho$.



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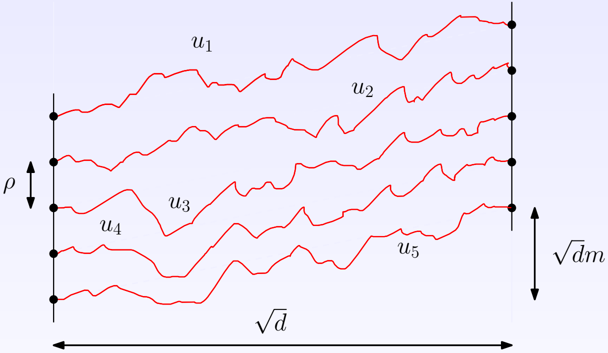

$\mathbb{P}$: law of independent Brownian bridges

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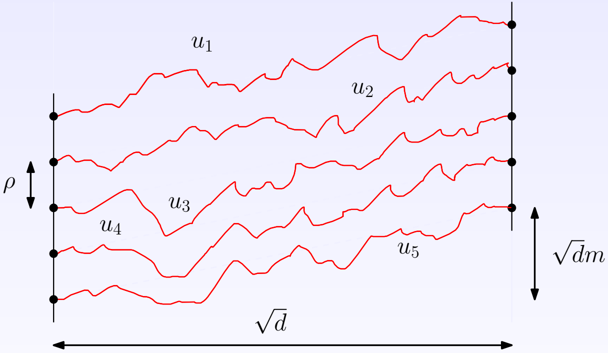

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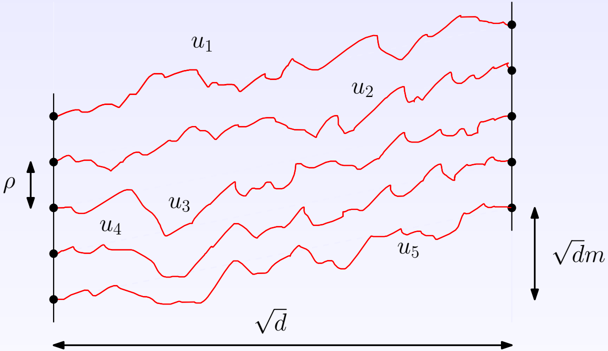

$$Z_{m,\rho,d} = \mathbb{P}\Big[(u_1, u_2, \ldots, u_d) \in \Delta\Big]$$

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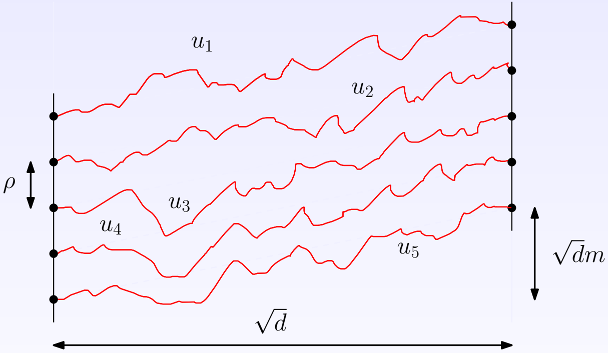

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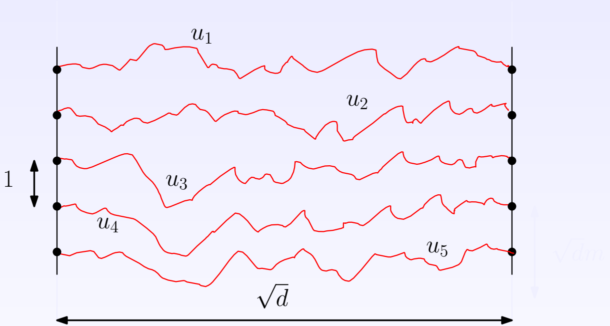
Surface Tension

Let $\partial_x U = 0$ and $\partial_y U = 1$.



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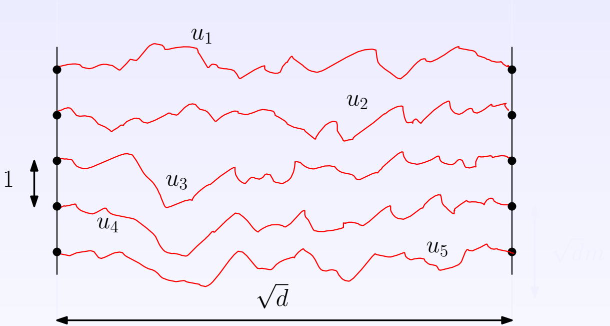

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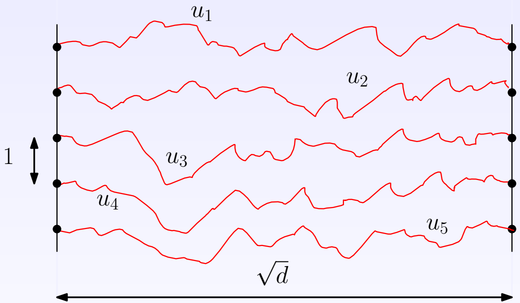

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$$\begin{aligned} Z_{0,1,d} &= \det\left[\frac{1}{(2\pi\sqrt{d})^{1/2}}\exp\left(-\frac{(j-i)^2}{2\sqrt{d}}\right)\right]_{1\le i,j\le d} \\ &= (2\pi\sqrt{d})^{-d/2}\exp\left(-6^{-1}\sqrt{d}(d+1)(2d+1)\right)\det\left[e^{d^{-1/2}ij}\right]_{1\le i,j\le d} \\ &= (2\pi\sqrt{d})^{-d/2}\prod_{\ell=1}^{d-1}(1-e^{-d^{-1/2}\ell})^{d-\ell} \end{aligned}$$

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Happy Birthday Greg!