

Reversibility of Loewner energy via local reversals

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Random Explorations Workshop

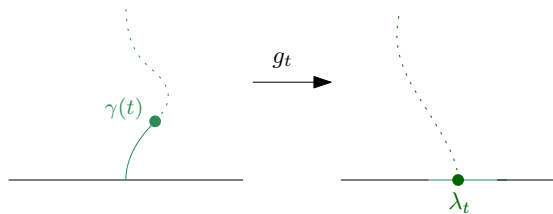
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- Reversibility of Loewner energy
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- Comparison with SLE reversibility proofs



Chordal Loewner chains

- γ simple chord (crosscut) in $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ from 0 to ∞



$$g_t : \mathbb{H} \setminus \gamma((0, t]) \rightarrow \mathbb{H} \text{ with } g_t(z) = z + o(1) \text{ as } z \rightarrow \infty$$

- Capacity parameterization: γ can be continuously parameterized by $t \in (0, \infty)$ such that

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \text{ as } z \rightarrow \infty$$

- Carathéodory's theorem $\Rightarrow g_t$ extends continuously to $\gamma(t)$.

Driving function of γ is $t \mapsto \lambda_t := g_t(\gamma(t))$

γ can be recovered from $(\lambda_t)_{t \in [0, \infty))}$ through the **Loewner equation**

$$z \notin \gamma((0, t]) \Leftrightarrow \text{ODE (in } t) \partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t}, \quad g_0(z) = z \text{ can be solved on } [0, t]$$

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When does a continuous function $(\lambda_t)_{t \in [0, \infty)}$ encode a simple chord in $\mathbb{H}$ from 0 to ∞ ?

- $\lambda \in C^{1/2}$ with $\|\lambda\|_{C^{1/2}} < 4$

by Marshall–Rohde and Lind

- ▶ K -**quasichord** with $K = K(\|\lambda\|_{C^{1/2}})$
- ▶ Quantitative bound for K by Schoug–Shekhar–Viklund

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When does a continuous function $(\lambda_t)_{t \in [0, \infty)}$ encode a simple chord in $\mathbb{H}$ from 0 to ∞ ?

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 - ▶ K -**quasichord** with $K = K(\|\lambda\|_{C^{1/2}})$
 - ▶ Quantitative bound for K by Schoug–Shekhar–Viklund
- $\lambda_t = \sqrt{\kappa} B_t$ with $\kappa \in (0, 4]$ and B_t standard Brownian motion (a.s.)
by Rohde–Schramm
 - ▶ Gives *Schramm–Loewner evolution* (**SLE $_{\kappa}$**)

Loewner energy

First studied by Yilin Wang (also Friz–Shekhar)

$$I(\gamma) := \frac{1}{2} \int_0^\infty \left| \frac{d\lambda_t}{dt} \right|^2 dt$$

if λ is absolutely continuous in t , and $I(\gamma) = \infty$ otherwise.

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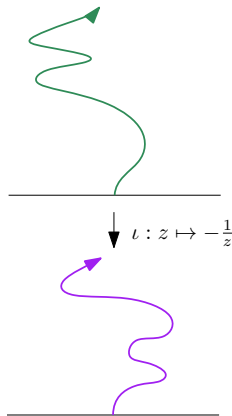
if λ is absolutely continuous in t , and $I(\gamma) = \infty$ otherwise.

Theorem (Loewner energy reversibility; Wang '16; S. '24)

Let $\iota(z) = -\frac{1}{z}$. For any crosscut γ in $\mathbb{H}$ from 0 to ∞ ,

$$I(\gamma) = I(\iota(\gamma)).$$

Remark: $I(\gamma) = 0$ if and only if $\gamma = i\mathbb{R}_+$. Only case that is easy to see.

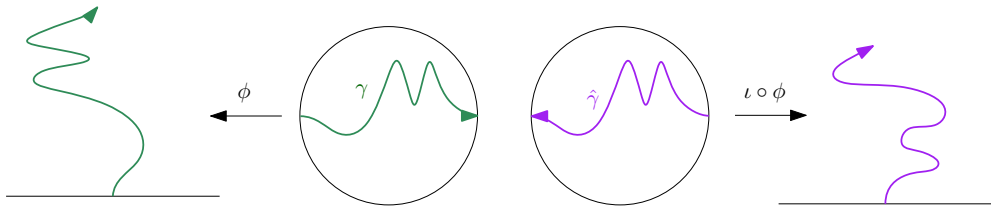


Given an arbitrary **simply connected** domain D and an **oriented simple** chord (crosscut) γ crossing from prime end a to prime end b , we define

$$I(\gamma) := I(\phi(\gamma))$$

where $\phi : D \rightarrow \mathbb{H}$ is a conformal map such that $\phi(a) = 0$ and $\phi(b) = \infty$.

- Defined unambiguously b/c Loewner energy invariant under dilation
($I(\gamma) = I(c\gamma)$ for $c > 0$)



Going forward, we denote the orientation reversal of γ as $\hat{\gamma}$.

Basic properties of Loewner energy:

- $I(\gamma) = 0$ if and only if γ is a **hyperbolic geodesic** ($i\mathbb{R}_+$ on $\mathbb{H}$)
- Define the energy of the initial part of a simple chord γ going $0 \rightarrow \mathbb{H} \rightarrow \infty$ as

$$I(\gamma([0, T])) = \int_0^T |\dot{\lambda}_t|^2 dt$$

and to be conformally invariant on other domains. Satisfies **additivity**.

$$I\left(\left[\begin{array}{c} \text{box with a green curve segment} \\ \text{green 'x' on the right boundary} \end{array}\right]\right) + I\left(\left[\begin{array}{c} \text{box with a black curve segment} \\ \text{green 'x' on the right boundary} \end{array}\right]\right) = I\left(\left[\begin{array}{c} \text{box with a combined green and black curve segment} \\ \text{green 'x' on the right boundary} \end{array}\right]\right) = I\left(\left[\begin{array}{c} \text{box with a combined green and black curve segment} \\ \text{green 'x' on the right boundary} \end{array}\right]\right) + I\left(\left[\begin{array}{c} \text{box with a green curve segment} \\ \text{green 'x' on the right boundary} \end{array}\right]\right)$$

hyperbolic geodesic 

Loewner energy enjoys a wide variety of equivalent descriptions.

- Large deviation rate function of SLE_κ as $\kappa \rightarrow 0$

$$\mathbb{P}^{\text{SLE}_\kappa}(\gamma \in \mathcal{A}) \stackrel{\kappa \rightarrow 0}{\approx} \exp\left(-\frac{1}{\kappa} \inf_{\gamma \in \mathcal{A}} I(\gamma)\right)$$

where $\mathcal{A}$ is some collection of simple chords in $\mathbb{H}$ from 0 to ∞ .

- ▶ First proved by Yilin Wang ('16).
- ▶ Allowed collections $\mathcal{A}$ strengthened in works of Peltola–Wang, Guskov, Abuzaid–Peltola
- ▶ Other SLE variants: Krusell, Abuzaid–Healey–Peltola, Chen–Huang–Wu, Ang–Park–Wang

Wang ('16) proved Loewner energy reversibility using:

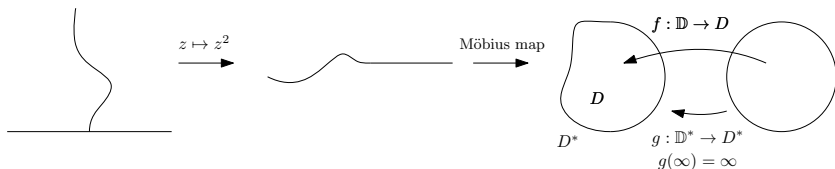
- Loewner energy = LDP rate function of chordal SLE_κ as $\kappa \rightarrow 0$
- Chordal SLE_κ reversibility: If $\gamma \sim \text{SLE}_\kappa$ in $\mathbb{H}$ from 0 to ∞ , then $\gamma \stackrel{d}{=} \iota(\gamma)$.
 - ▶ Proof by Dapeng Zhan ('08) using SLE commutation relations
 - ▶ Proof by Miller–Sheffield ('12) using imaginary geometry
 - ▶ Proof by Lawler–Yearwood ('21), “in the same spirit as Zhan’s proof”

Interpretations of the loop version of Loewner energy (introduced by Rohde and Wang):

- Wang('18): Equivalent to the unique Kähler potential of the Weil–Petersson metric on universal Teichmüller space introduced by Takhtajan–Teo.

$$I(\gamma) = \frac{1}{\pi} \int_{\mathbb{D}} |\nabla \log |f'||^2 dA + \frac{1}{\pi} \int_{\mathbb{D}^*} |\nabla \log |g'||^2 dA + 4 \log \left| \frac{f'(0)}{g'(\infty)} \right|$$

where



- Related to the imaginary geometry proof of SLE reversibility.
Cf. SLE/GFF vs. finite energy dictionary of Viklund and Wang

Proof overview

- We identify a crosscut by a **dense set of points** it passes through.
 - ▶ Lemma: If the closure of $\{z_1, z_2, \dots\} \in D$ is a crosscut γ , then it is the unique crosscut containing all of these points.
 - ▶ Follows directly from Jordan curve theorem.

Key idea

Given finite $\mathbf{z} = \{z_1, \dots, z_n\} \in D$, $\exists$ crosscut $\gamma^{\mathbf{z}}$ going $a \rightarrow D \rightarrow b$ such that

$$I(\gamma^{\mathbf{z}}) = \inf\{I(\gamma) : \gamma \text{ crosscut going } a \rightarrow D \rightarrow b, \mathbf{z} \subset \gamma\}.$$

Moreover, $I(\gamma^{\mathbf{z}}) = I(\hat{\gamma}^{\mathbf{z}})$.

Input 1 (consequence of the driving function definition of I)

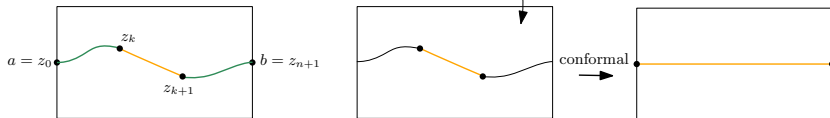
$\forall C < \infty$, $\{\gamma : \text{crosscut going } -1 \rightarrow \mathbb{D} \rightarrow 1 \text{ with } I(\gamma) \leq C\}$ is **compact** in the Hausdorff metric on compact subsets of $\overline{\mathbb{D}}$ for any $C < \infty$.

- If $\gamma^n \rightarrow \gamma$ in Hausdorff distance, then $I(\gamma) \leq \liminf_{n \rightarrow \infty} I(\gamma^n)$.
- Gives existence of γ^Z .
- First appears in Peltola–Wang. Follows essentially from the result of Wang('16):
If $I(\gamma) \leq C < \infty$, then $\exists K$ -quasiconformal $\phi : \mathbb{D} \rightarrow \mathbb{D}$ with $K = K(C)$ such that $\gamma = \phi((-1, 1))$.

Definition: A crosscut γ going $a \rightarrow D \rightarrow b$ is a **piecewise geodesic w.r.t. $\mathbf{z} \subset D$** if

- $\mathbf{z} = \{z_1, \dots, z_n\} \subset \gamma$
- $\forall 0 \leq k \leq n,$

segment between z_k and z_{k+1} is a hyperbolic geodesic in

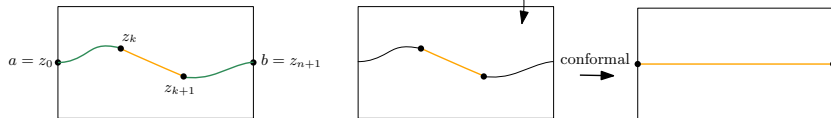


Fact: γ is a piecewise geodesic w.r.t. $\mathbf{z} \Leftrightarrow \hat{\gamma}$ is a piecewise geodesic w.r.t. $\mathbf{z}$.

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Fact: γ is a piecewise geodesic w.r.t. $\mathbf{z} \Leftrightarrow \hat{\gamma}$ is a piecewise geodesic w.r.t. $\mathbf{z}$.

Key proposition

Let γ be a crosscut passing through $\mathbf{z}$.

- 1 $I(\gamma) = I(\gamma^{\mathbf{z}}) \Rightarrow \gamma$ is piecewise geodesic w.r.t. $\mathbf{z}$.
- 2 If γ is a piecewise geodesic, then $I(\gamma) = I(\hat{\gamma})$.

Rohde and Wang: Introduced piecewise geodesic Jordan curves and proved (1) for loops

Proof of reversibility given the key proposition

Let crosscut γ going $-1 \rightarrow \mathbb{D} \rightarrow 1$ be given. Assume $I(\hat{\gamma}) < \infty$.

- Pick a sequence of finite sets $\mathbf{z}_1 \subset \mathbf{z}_2 \subset \cdots \subset \gamma$ such that $\cup_n \mathbf{z}_n$ is dense in γ .
- Pick $\gamma^n := \gamma_n^{\mathbf{z}}$.

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- By the key proposition, $I(\gamma^n) = I(\hat{\gamma}^n)$ and $\hat{\gamma}^n$ minimizes Loewner energy among crosscuts going $1 \rightarrow \mathbb{D} \rightarrow -1$ passing through $\mathbf{z}_n$.

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$$\begin{array}{ccccccc} I(\gamma^1) & \leq & I(\gamma^2) & \leq & \cdots \\ \parallel & & \parallel & & \\ I(\hat{\gamma}^1) & \leq & I(\hat{\gamma}^2) & \leq & \cdots & \leq & I(\hat{\gamma}) \end{array}$$

- Passing to a subsequence, $\gamma^n \rightarrow \eta$ in Hausdorff metric, so $I(\eta) \leq \liminf_{n \rightarrow \infty} I(\gamma^n) \leq I(\hat{\gamma})$.
Here, η is a crosscut going $-1 \rightarrow \mathbb{D} \rightarrow 1$ with $\cup_n \mathbf{z}_n \subset \eta \Rightarrow \gamma \subseteq \eta$.

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Here, η is a crosscut going $-1 \rightarrow \mathbb{D} \rightarrow 1$ with $\cup_n \mathbf{z}_n \subset \eta \Rightarrow \gamma \subseteq \eta$.
- By Jordan curve theorem, $\eta = \gamma$. Hence, $I(\gamma) \leq I(\hat{\gamma})$ and also $I(\hat{\gamma}) \leq I(\gamma)$. $\square$

Input 2 (consequence of the driving function definition of I)

Loewner energy commutation relation (Wang, '16)

$$I\left(\boxed{\begin{array}{c} \gamma \\ \times \end{array}}\right) + I\left(\boxed{\begin{array}{c} \text{black curve} \times \text{purple curve} \end{array}}\right) = I\left(\boxed{\begin{array}{c} \times \text{purple curve} \eta \end{array}}\right) + I\left(\boxed{\begin{array}{c} \gamma \times \text{black curve} \end{array}}\right)$$

Cf. SLE commutation relation (Dubédat)

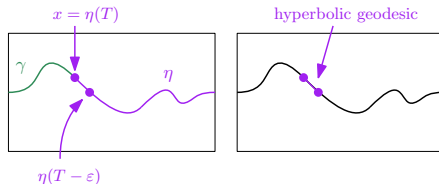
$$\text{Sample } \boxed{\begin{array}{c} \gamma \\ \times \end{array}} \text{ and then } \boxed{\begin{array}{c} \gamma \times \text{purple curve} \end{array}} \stackrel{d}{=} \text{Sample } \boxed{\begin{array}{c} \times \text{purple curve} \end{array}} \text{ and then } \boxed{\begin{array}{c} \gamma \times \text{purple curve} \end{array}}$$

Hurdle for reversibility: For both cases, curves may not intersect, **including at endpoints**.

“Commutation inequality”

γ, η simple curves from distinct prime ends, **same endpoint** $x \in D$. $\eta : (0, T] \rightarrow D$.

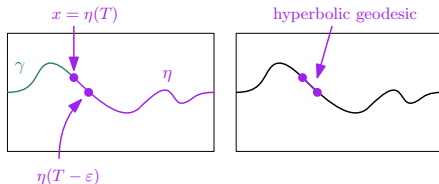
Definition: η has a **geodesic tip** w.r.t. γ if $\exists \varepsilon > 0$ s.t.



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Definition: η has a **geodesic tip** w.r.t. γ if $\exists \varepsilon > 0$ s.t.



Key lemma

If η has a geodesic tip w.r.t. γ , then

$$I\left(\begin{array}{|c|} \hline \gamma \\ \hline \end{array}\right) \times + I\left(\begin{array}{|c|} \hline \text{hyperbolic geodesic} \\ \hline \end{array}\right) \geq I\left(\begin{array}{|c|} \hline \eta \\ \hline \end{array}\right) \times + I\left(\begin{array}{|c|} \hline \gamma \\ \hline \end{array}\right) \times$$

Proof of key lemma:

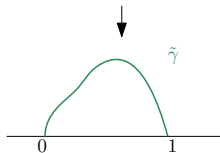
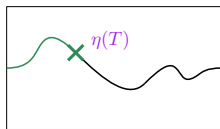
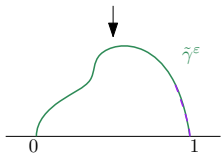
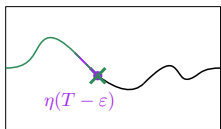
$$\begin{aligned}
 I\left(\left[\begin{array}{c} \gamma \\ \hline \end{array}\right] \times\right) + I\left(\left[\begin{array}{c} \text{black curve} \quad \eta(T) \\ \hline \end{array}\right]\right) &= I\left(\left[\begin{array}{c} \text{green curve} \\ \hline \end{array}\right] \times\right) + \lim_{\varepsilon \rightarrow 0} I\left(\left[\begin{array}{c} \text{black curve} \quad \times \quad \text{purple curve} \\ \hline \end{array}\right] \right. \\
 &\quad \left. \begin{array}{c} \eta(T - \varepsilon) \end{array}\right] \right) \\
 &\stackrel{\text{commutation relation}}{=} \lim_{\varepsilon \rightarrow 0} I\left[\left(\left[\begin{array}{c} \times \\ \hline \end{array}\right] \left[\begin{array}{c} \text{purple curve} \\ \hline \end{array}\right]\right) + I\left(\left[\begin{array}{c} \text{green curve} \\ \hline \end{array}\right] \left[\begin{array}{c} \times \\ \hline \end{array}\right]\right)\right] \\
 &= I\left(\left[\begin{array}{c} \times \\ \hline \end{array}\right] \left[\begin{array}{c} \text{purple curve} \quad \eta \\ \hline \end{array}\right]\right) + \lim_{\varepsilon \rightarrow 0} I\left(\left[\begin{array}{c} \text{green curve} \\ \hline \end{array}\right] \left[\begin{array}{c} \times \\ \hline \end{array}\right]\right)
 \end{aligned}$$

Proof of key lemma:

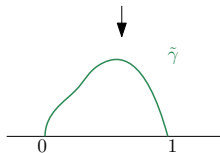
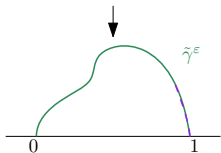
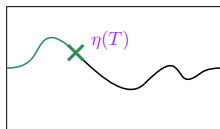
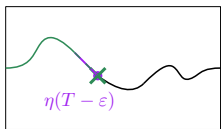
$$\begin{aligned}
 I\left(\left[\begin{array}{c} \gamma \\ \hline \end{array}\right] \times\right) + I\left(\left[\begin{array}{c} \eta(T) \\ \hline \end{array}\right]\right) &= I\left(\left[\begin{array}{c} \gamma \\ \hline \end{array}\right] \times\right) + \lim_{\varepsilon \rightarrow 0} I\left(\left[\begin{array}{c} \times \\ \eta(T-\varepsilon) \end{array}\right]\right) \\
 &\stackrel{\text{commutation relation}}{=} \lim_{\varepsilon \rightarrow 0} I\left[\left(\left[\begin{array}{c} \times \\ \hline \end{array}\right] \bullet\right) + I\left(\left[\begin{array}{c} \gamma \\ \hline \end{array}\right] \times\right)\right] \\
 &= I\left(\left[\begin{array}{c} \times \\ \hline \end{array}\right] \eta\right) + \lim_{\varepsilon \rightarrow 0} I\left(\left[\begin{array}{c} \gamma \\ \hline \end{array}\right] \times\right)
 \end{aligned}$$

Lemma follows if

$$\lim_{\varepsilon \rightarrow 0} I\left(\left[\begin{array}{c} \gamma \\ \hline \end{array}\right] \times \eta(T-\varepsilon)\right) \geq I\left(\left[\begin{array}{c} \gamma \\ \hline \end{array}\right] \times \eta(T)\right)$$



Claim: Given any $\varepsilon_n \rightarrow 0$, there exists a further subsequence ε_{n_k} such that $\tilde{\gamma}^{\varepsilon_{n_k}} \rightarrow \tilde{\gamma}$ in Hausdorff distance.



Claim: Given any $\varepsilon_n \rightarrow 0$, there exists a further subsequence ε_{n_k} such that $\tilde{\gamma}^{\varepsilon_{n_k}} \rightarrow \tilde{\gamma}$ in Hausdorff distance. Then,

$$I \left(\left[\begin{array}{c} \text{Diagram of } \eta(T) \end{array} \right] \right) \leq \lim_{\varepsilon \rightarrow 0} I \left(\left[\begin{array}{c} \text{Diagram of } \eta(T - \varepsilon) \end{array} \right] \right) = \lim_{\varepsilon \rightarrow 0} I \left(\left[\begin{array}{c} \text{Diagram of } \eta(T - \varepsilon) \end{array} \right] \right)$$

If η has a geodesic tip



Proof of the key proposition

Key proposition

Let γ be a crosscut passing through z .

- 1 $I(\gamma) = I(\gamma^z) \Rightarrow \gamma$ is piecewise geodesic w.r.t. z .
- 2 If γ is a piecewise geodesic, then $I(\gamma) = I(\hat{\gamma})$.

Base step of the induction:

$$I\left(\begin{array}{|c|} \hline \text{Diagram 1: A green curve with a green 'x' at the right end.} \\ \hline \end{array}\right) \geq I\left(\begin{array}{|c|} \hline \text{Diagram 2: The same green curve, but the segment near the 'x' is replaced by a straight green line segment. An arrow points to this segment with the text 'Replace with hyperbolic geodesic' above it.} \\ \hline \end{array}\right) = I\left(\begin{array}{|c|} \hline \text{Diagram 3: The green curve ending at a green 'x' on the right boundary.} \\ \hline \end{array}\right) + I\left(\begin{array}{|c|} \hline \text{Diagram 4: A purple curve starting from a purple 'x' on the left boundary and ending at a black dot on the right boundary.} \\ \hline \end{array}\right)$$

= iff no replacement

$$= I \left(\begin{array}{|c|} \hline \text{Diagram 1: A rectangle containing a green curve from the left edge to a black dot, a black dot in the center, and a green 'X' on the right edge.} \\ \hline \end{array} \right) + I \left(\begin{array}{|c|} \hline \text{Diagram 2: A rectangle containing a black curve from the left edge to a black dot, a green curve from that dot to another black dot, and a green 'X' on the right edge.} \\ \hline \end{array} \right) + I \left(\begin{array}{|c|} \hline \text{Diagram 3: A rectangle containing a black curve from the left edge to a black dot, a black curve from that dot to a purple 'X', and a purple curve from the 'X' to the right edge.} \\ \hline \end{array} \right)$$

commutation inequality

$$\geq I \left(\begin{array}{|c|} \hline \text{Diagram 4: A rectangle containing a green curve from the left edge to a black dot, a black dot in the center, and a green 'X' on the right edge.} \\ \hline \end{array} \right) + I \left(\begin{array}{|c|} \hline \text{Diagram 5: A rectangle containing a black curve from the left edge to a black dot, a purple curve from that dot to the right edge, and a purple 'X' on the curve.} \\ \hline \end{array} \right) + I \left(\begin{array}{|c|} \hline \text{Diagram 6: A rectangle containing a black curve from the left edge to a black dot, a green curve from that dot to a green 'X', and a black curve from the 'X' to the right edge.} \\ \hline \end{array} \right)$$

Replace with hyperbolic geodesic

$$\geq I \left(\begin{array}{|c|} \hline \text{Diagram 4: A rectangle containing a green curve from the left edge to a black dot, a black dot in the center, and a green 'X' on the right edge.} \\ \hline \end{array} \right) + I \left(\begin{array}{|c|} \hline \text{Diagram 5: A rectangle containing a black curve from the left edge to a black dot, a purple curve from that dot to the right edge, and a purple 'X' on the curve.} \\ \hline \end{array} \right) + I \left(\begin{array}{|c|} \hline \text{Diagram 7: A rectangle containing a black curve from the left edge to a black dot, a green curve from that dot to a green 'X', and a black curve from the 'X' to the right edge. An arrow points to the green curve with the text 'Replace with hyperbolic geodesic'.} \\ \hline \end{array} \right)$$

= iff no replacement

$$\begin{aligned}
&\geq I\left(\begin{array}{|c|} \hline \text{Diagram 1: Green curve, black dot, green X} \\ \hline \end{array}\right) + I\left(\begin{array}{|c|} \hline \text{Diagram 2: Black curve, purple X, purple curve} \\ \hline \end{array}\right) + I\left(\begin{array}{|c|} \hline \text{Diagram 3: Black curve, green segment, green X} \\ \hline \end{array}\right) \\
&\quad \uparrow \text{= iff no replacement} \qquad \qquad \qquad \text{Replace with hyperbolic geodesic} \\
&= I\left(\begin{array}{|c|} \hline \text{Diagram 1: Green curve, black dot, green X} \\ \hline \end{array}\right) + I\left(\begin{array}{|c|} \hline \text{Diagram 2: Black curve, purple X, purple curve} \\ \hline \end{array}\right) + I\left(\begin{array}{|c|} \hline \text{Diagram 3: Black curve, purple X, black segment, black dot} \\ \hline \end{array}\right) \\
&= I\left(\begin{array}{|c|} \hline \text{Diagram 1: Green curve, black dot, green X} \\ \hline \end{array}\right) + I\left(\begin{array}{|c|} \hline \text{Diagram 2: Black curve, purple X, purple curve} \\ \hline \end{array}\right) \geq I\left(\begin{array}{|c|} \hline \text{Diagram 4: Black curve, purple X, black segment, black dot, purple curve} \\ \hline \end{array}\right) \\
&\quad \qquad \qquad \text{Replace with hyperbolic geodesic} \qquad \qquad \qquad \uparrow \text{= iff no replacement}
\end{aligned}$$

In conclusion,

$$I\left(\begin{array}{|c|} \hline \text{if LHS is not piecewise geodesic} \\ \hline \end{array}\right) > I\left(\begin{array}{|c|} \hline \text{piecewise geodesic} \\ \hline \end{array}\right) \geq I\left(\begin{array}{|c|} \hline \text{if LHS is not piecewise geodesic} \\ \hline \end{array}\right) \geq I\left(\begin{array}{|c|} \hline \text{piecewise geodesic} \\ \hline \end{array}\right)$$

□

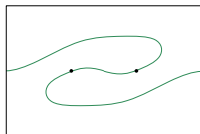
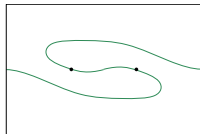
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Isotopy class of the chord w.r.t. $\mathbf{z}$ (homotopy through chords fixing $\mathbf{z}$) does not change during the reversal process of the proof.

Corollary: Existence of Loewner energy minimizer for each isotopy class w.r.t. $\mathbf{z}$.



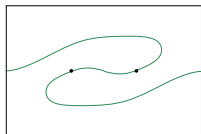
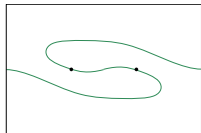
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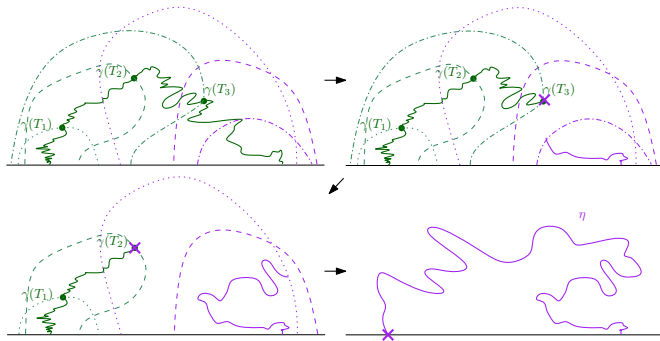


- Uniqueness for each isotopy class follows by straightforward adaptation of the proof of the loop case by Bonk–Junnla–Rohde–Wang
- Isotopy class of crosscuts w.r.t. $\mathbf{z} = \{z_1, \dots, z_n\}$
 $\simeq$ Mapping class group of n -punctured disk
 $\simeq$ Braid group on n strands $\Leftarrow$ countably infinite for $n \geq 2$

SLE reversibility proof: Zhan

ν^n = Probability measure on coupling (γ, η) where

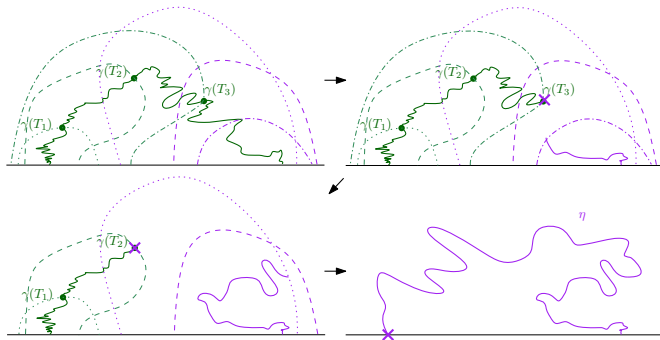
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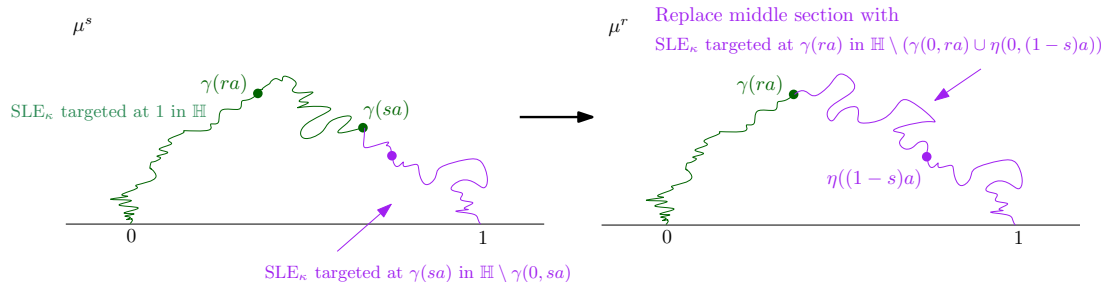


- n = number of pairs of disjoint hulls. As $n \rightarrow \infty$, next target point arbitrarily close.
- Prokhorov's theorem: ν^n converges weakly along a subsequence to ν^∞ .
- For $(\gamma, \eta) \sim \nu^\infty$, $\hat{\gamma} = \eta$ a.s.

SLE reversibility proof: Lawler and Yearwood

Define $\mu^1 = \text{Law of hull enclosed by } \gamma$, $\mu^0 = \text{Law of hull enclosed by } \eta$.

Set $a = \text{hcap of entire hull}$. Preserved on each step.

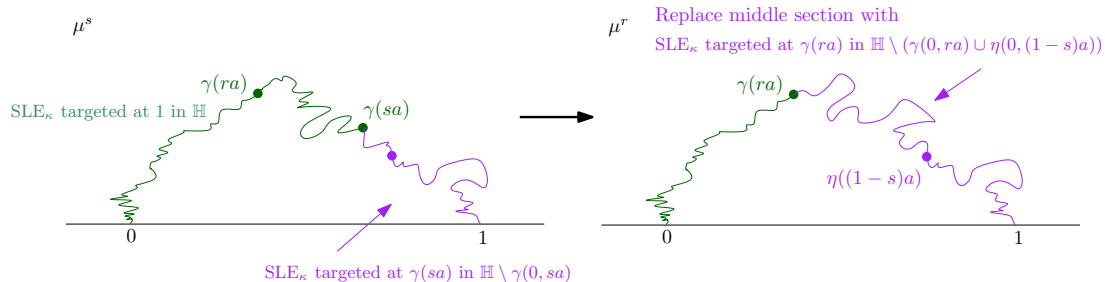


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- Commutation relation $\Rightarrow$ RHS is μ^r . Gives coupling between μ^s and μ^r .
- ν^n = Coupling b/w μ^1 and μ^0 obtained by $\mu^1 \rightarrow \mu^{1-1/n} \rightarrow \dots \mu^{1/n} \rightarrow \mu^0$
- $\exists c, \delta > 0$ such that $d_{\text{Prokhorov}}(\mu^s, \mu^r) = c|s - r|^{1+\delta}$ under above coupling
 $\Rightarrow$ As $n \rightarrow \infty$, $d_{\text{Prokhorov}}(\mu^1, \mu^0) \rightarrow 0$.



Happy Birthday Greg!