Reversibility of Loewner energy via local reversals

Jinwoo Sung

University of Washington

Random Explorations Workshop

Contents

- Introduction to Loewner energy
  - Loewner Chains, SLE
- Reversibility of Loewner energy
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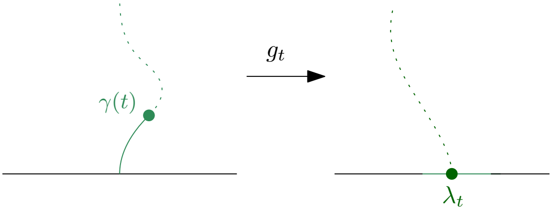
Chordal Loewner chains

- γ simple chord (crosscut) in $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ from 0 to ∞



$$g_t : \mathbb{H} \setminus \gamma((0,t]) \to \mathbb{H} \text{ with } g_t(z) = z + o(1) \text{ as } z \to \infty$$

- Capacity parameterization: γ can be continuously parameterized by $t \in (0, \infty)$ such that
$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \text{ as } z \to \infty$$
- Carathéodory's theorem $\Rightarrow$ g_t extends continuously to $\gamma(t)$.

$$\textbf{Driving function} \text{ of } \gamma \text{ is } t \mapsto \lambda_t := g_t(\gamma(t))$$

γ can be recovered from $(\lambda_t)_{t\in[0,\infty))}$ through the **Loewner equation**

$$z \notin \gamma((0,t]) \Leftrightarrow \text{ODE (in } t\text{) } \partial_t g_t(z) = \frac{2}{g_t(z)-\lambda_t},\ \ g_0(z) = z \text{ can be solved on } [0,t]$$

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When does a continuous function $(\lambda_t)_{t\in[0,\infty)}$ encode a simple chord in $\mathbb{H}$ from 0 to ∞?

- $\lambda \in C^{1/2}$ with $\|\lambda\|_{C^{1/2}} < 4$
  by Marshall–Rohde and Lind
  - K-quasichord with $K = K(\|\lambda\|_{C^{1/2}})$
  - Quantitative bound for K by Schoug–Shekhar–Viklund

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- $\lambda_t = \sqrt{\kappa} B_t$ with $\kappa \in (0,4]$ and B_t standard Brownian motion (a.s.) by Rohde–Schramm
  - Gives *Schramm–Loewner evolution* (SLE_κ)

Loewner energy

First studied by Yilin Wang (also Friz–Shekhar)

$$I(\gamma) := \frac{1}{2}\int_0^{\infty} \left|\frac{d\lambda_t}{dt}\right|^2 dt$$

if λ is absolutely continuous in t, and $I(\gamma) = \infty$ otherwise.

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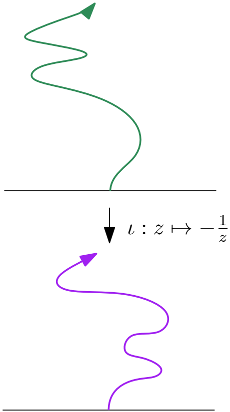


Theorem (Loewner energy reversibility; Wang '16; S. '24)

Let $\iota(z) = -\frac{1}{z}$. For any crosscut γ in $\mathbb{H}$ from 0 to ∞,

$$I(\gamma) = I(\iota(\gamma)).$$

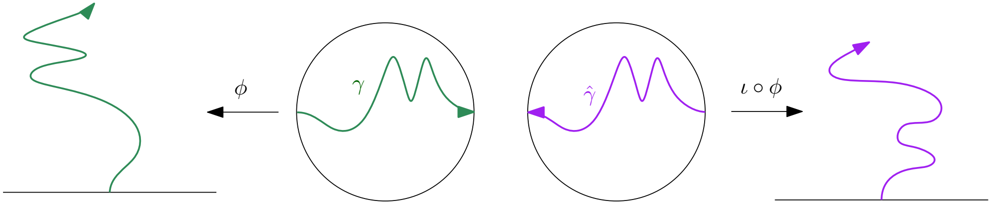
Remark: $I(\gamma) = 0$ if and only if $\gamma = i\mathbb{R}_+$. Only case that is easy to see.

Given an arbitrary **simply connected** domain D and an **oriented simple** chord (crosscut) γ crossing from prime end a to prime end b, we define

$$I(\gamma) := I(\phi(\gamma))$$

where $\phi : D \to \mathbb{H}$ is a conformal map such that $\phi(a) = 0$ and $\phi(b) = \infty$.

- Defined unambiguously b/c Loewner energy invariant under dilation ($I(\gamma) = I(c\gamma)$ for $c > 0$)



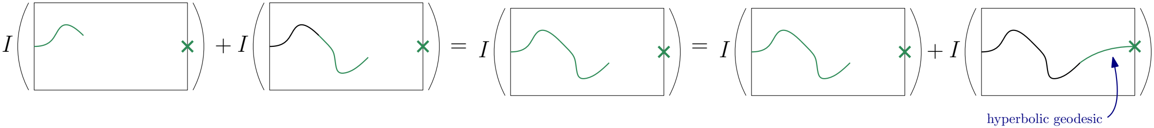
Going forward, we denote the orientation reversal of γ as $\hat{\gamma}$.

Basic properties of Loewner energy:

- $I(\gamma) = 0$ if and only if γ is a **hyperbolic geodesic** ($i\mathbb{R}_+$ on $\mathbb{H}$)
- Define the energy of the initial part of a simple chord γ going $0 \to \mathbb{H} \to \infty$ as

$$I(\gamma([0,T])) = \int_0^T \left|\dot{\lambda}_t\right|^2 dt$$

and to be conformally invariant on other domains. Satisfies **additivity**.



Loewner energy enjoys a wide variety of equivalent descriptions.

- Large deviation rate function of SLE_κ as $\kappa \to 0$

$$\text{“}\mathbb{P}^{\mathrm{SLE}_\kappa}(\gamma \in \mathcal{A}) \overset{\kappa \to 0}{\approx} \exp\left(-\frac{1}{\kappa} \inf_{\gamma \in \mathcal{A}} I(\gamma)\right)$$

  where $\mathcal{A}$ is some collection of simple chords in $\mathbb{H}$ from 0 to ∞.

  - First proved by Yilin Wang ('16).
  - Allowed collections $\mathcal{A}$ strengthened in works of Peltola–Wang, Guskov, Abuzaid–Peltola
  - Other SLE variants: Krusell, Abuzaid–Healey–Peltola, Chen–Huang–Wu, Ang–Park–Wang

Wang ('16) proved Loewner energy reversibility using:

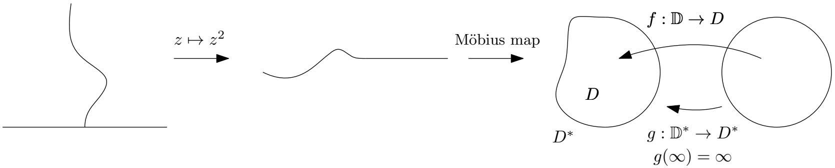
- Loewner energy = LDP rate function of chordal SLE_κ as $\kappa \to 0$
- Chordal SLE_κ reversibility: If $\gamma \sim \mathrm{SLE}_\kappa$ in $\mathbb{H}$ from 0 to ∞, then $\gamma \overset{d}{=} \iota(\gamma)$.
  - Proof by Dapeng Zhan ('08) using SLE commutation relations
  - Proof by Miller–Sheffield ('12) using imaginary geometry
  - Proof by Lawler–Yearwood ('21), "in the same spirit as Zhan's proof"

Intepretations of the loop version of Loewner energy (introduced by Rohde and Wang):

- Wang('18): Equivalent to the unique Kähler potential of the Weil–Petersson metric on universal Teichmüller space introduced by Takhtajan–Teo.

$$I(\gamma) = \frac{1}{\pi}\int_{\mathbb{D}} |\nabla \log|f'||^2\, dA + \frac{1}{\pi}\int_{\mathbb{D}^*} |\nabla \log|g'||^2\, dA + 4\log\left|\frac{f'(0)}{g'(\infty)}\right|$$

where



  - Related to the imaginary geometry proof of SLE reversibility.
    Cf. SLE/GFF vs. finite energy dictionary of Viklund and Wang

Proof overview

- We identify a crosscut by a **dense set of points** it passes through.
  - Lemma: If the closure of $\{z_1, z_2, \dots\} \in D$ is a crosscut γ, then it is the unique crosscut containing all of these points.
  - Follows directly from Jordan curve theorem.

Key idea

Given finite $\mathbf{z} = \{z_1, \dots, z_n\} \in D$, $\exists$ crosscut $\gamma^{\mathbf{z}}$ going $a \to D \to b$ such that

$$I(\gamma^{\mathbf{z}}) = \inf\{I(\gamma) : \gamma \text{ crosscut going } a \to D \to b, \mathbf{z} \subset \gamma\}.$$

Moreover, $I(\gamma^{\mathbf{z}}) = I(\hat{\gamma}^{\mathbf{z}})$.

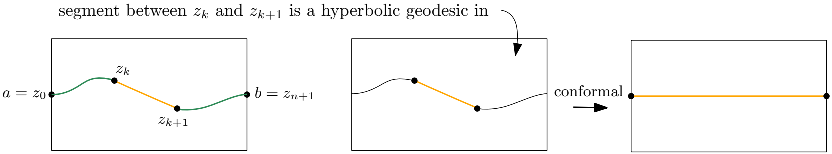
Input 1 (consequence of the driving function definition of I)

$\forall C < \infty$, $\{\gamma : \text{crosscut going } -1 \to \mathbb{D} \to 1 \text{ with } I(\gamma) \le C\}$ is **compact** in the Hausdorff metric on compact subsets of $\overline{\mathbb{D}}$ for any $C < \infty$.

- If $\gamma^n \to \gamma$ in Hausdorff distance, then $I(\gamma) \le \liminf_{n\to\infty} I(\gamma^n)$.
- Gives existence of $\gamma^{\mathbf{z}}$.
- First appears in Peltola–Wang. Follows essentially from the result of Wang('16): If $I(\gamma) \le C < \infty$, then $\exists$ K-quasiconformal $\phi : \mathbb{D} \to \mathbb{D}$ with $K = K(C)$ such that $\gamma = \phi((-1, 1))$.

Definition: A crosscut γ going $a \to D \to b$ is a **piecewise geodesic w.r.t. $\mathbf{z} \subset D$** if

- $\mathbf{z} = \{z_1, \ldots, z_n\} \subset \gamma$
- $\forall 0 \leq k \leq n$,



Fact: γ is a piecewise geodesic w.r.t. $\mathbf{z} \Leftrightarrow \hat{\gamma}$ is a piecewise geodesic w.r.t. $\mathbf{z}$.

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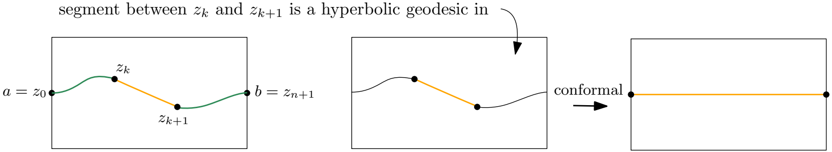

Fact: γ is a piecewise geodesic w.r.t. $\mathbf{z} \Leftrightarrow \hat{\gamma}$ is a piecewise geodesic w.r.t. $\mathbf{z}$.

Key proposition

Let γ be a crosscut passing through $\mathbf{z}$.

1. $I(\gamma) = I(\gamma^{\mathbf{z}}) \Rightarrow \gamma$ is piecewise geodesic w.r.t. $\mathbf{z}$.
2. If γ is a piecewise geodesic, then $I(\gamma) = I(\hat{\gamma})$.

Rohde and Wang: Introduced piecewise geodesic Jordan curves and proved (1) for loops

Proof of reversibility given the key proposition

Let crosscut γ going $-1 \to \mathbb{D} \to 1$ be given. Assume $I(\hat{\gamma}) < \infty$.

- Pick a sequence of finite sets $\mathbf{z}_1 \subset \mathbf{z}_2 \subset \cdots \subset \gamma$ such that $\cup_n \mathbf{z}_n$ is dense in γ.
- Pick $\gamma^n := \gamma^{\mathbf{z}}_n$.

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- By the key proposition, $I(\gamma^n) = I(\hat{\gamma}^n)$ and $\hat{\gamma}^n$ minimizes Loewner energy among crosscuts going $1 \to \mathbb{D} \to -1$ passing through $\mathbf{z}_n$.

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$$\begin{array}{ccccccccc} I(\gamma^1) & \leq & I(\gamma^2) & \leq & \cdots & & \\ \shortparallel & & \shortparallel & & & & \\ I(\hat{\gamma}^1) & \leq & I(\hat{\gamma}^2) & \leq & \cdots & \leq & I(\hat{\gamma}) \end{array}$$

- Passing to a subsequence, $\gamma^n \to \eta$ in Hausdorff metric, so $I(\eta) \leq \liminf_{n\to\infty} I(\gamma^n) \leq I(\hat{\gamma})$. Here, η is a crosscut going $-1 \to \mathbb{D} \to 1$ with $\cup_n \mathbf{z}_n \subset \eta \Rightarrow \gamma \subseteq \eta$.

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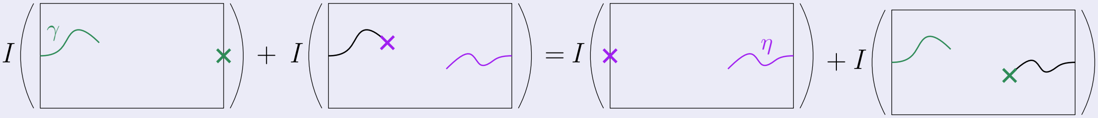
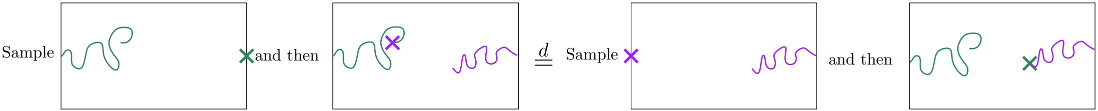
- By Jordan curve theorem, $\eta = \gamma$. Hence, $I(\gamma) \leq I(\hat{\gamma})$ and also $I(\hat{\gamma}) \leq I(\gamma)$. $\square$

Input 2 (consequence of the driving function definition of I)

Loewner energy commutation relation (Wang, '16)



Cf. SLE commutation relation (Dubédat)

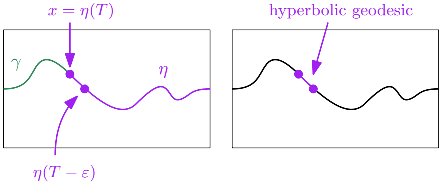


Hurdle for reversibility: For both cases, curves may not intersect, **including at endpoints**.

“Commutation inequality”

γ, η simple curves from distinct prime ends, **same endpoint** $x \in D$. $\eta : (0, T] \to D$.

Definition: η has a **geodesic tip** w.r.t. γ if $\exists\, \varepsilon > 0$ s.t.



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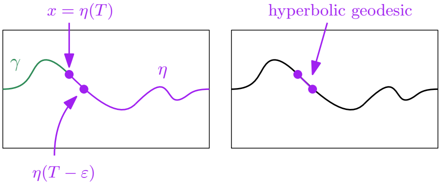
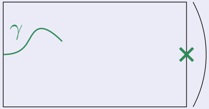
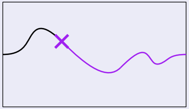
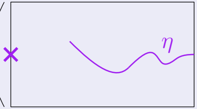
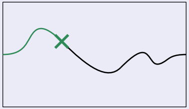
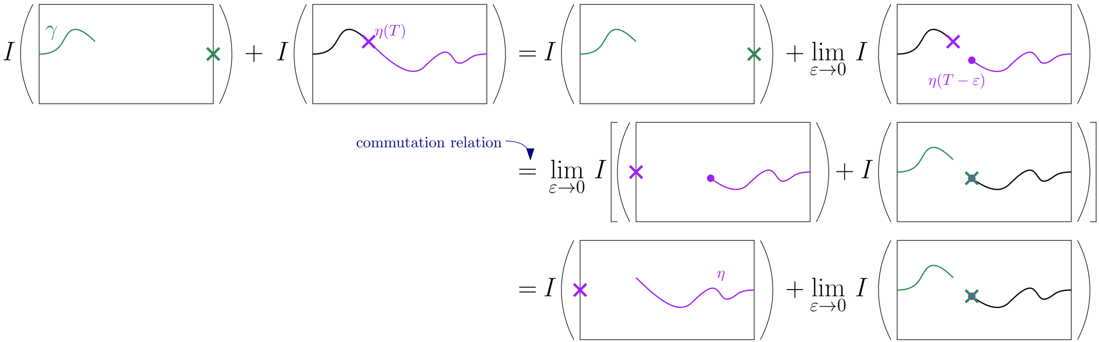


Key lemma

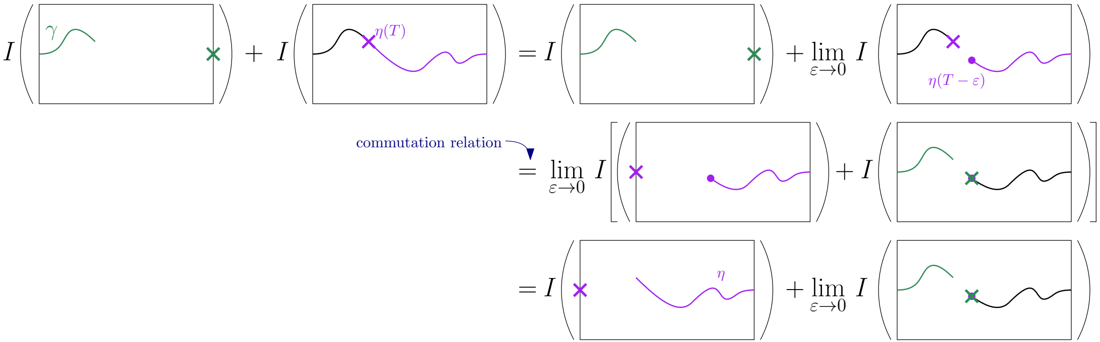
If η has a geodesic tip w.r.t. γ, then

$I\Big($  $\Big) + I\Big($  $\Big) \geq I\Big($  $\Big) + I\Big($  $\Big)$

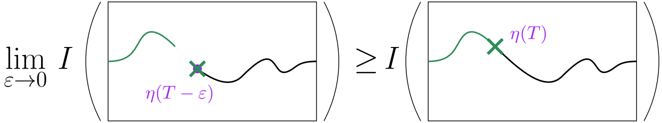
Proof of key lemma:

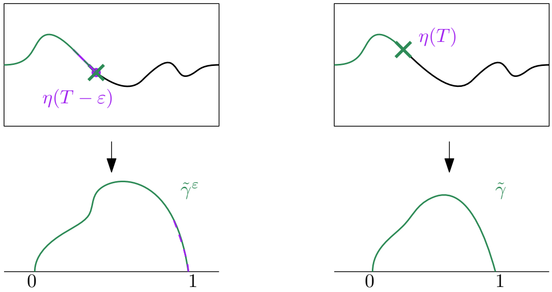


Proof of key lemma:

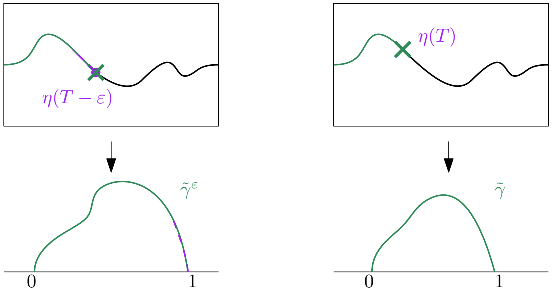


Lemma follows if

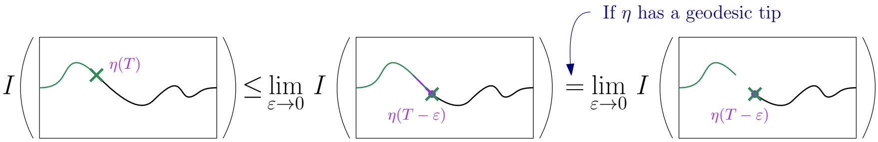




Claim: Given any $\varepsilon_n \to 0$, there exists a further subsequence ε_{n_k} such that $\tilde{\gamma}^{\varepsilon_{n_k}} \to \tilde{\gamma}$ in Hausdorff distance.



Claim: Given any $\varepsilon_n \to 0$, there exists a further subsequence ε_{n_k} such that $\tilde{\gamma}^{\varepsilon_{n_k}} \to \tilde{\gamma}$ in Hausdorff distance. Then,



$\square$

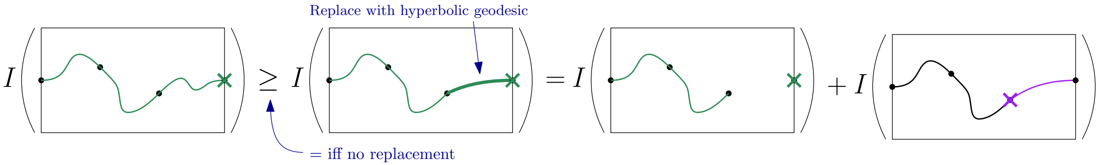
Proof of the key proposition

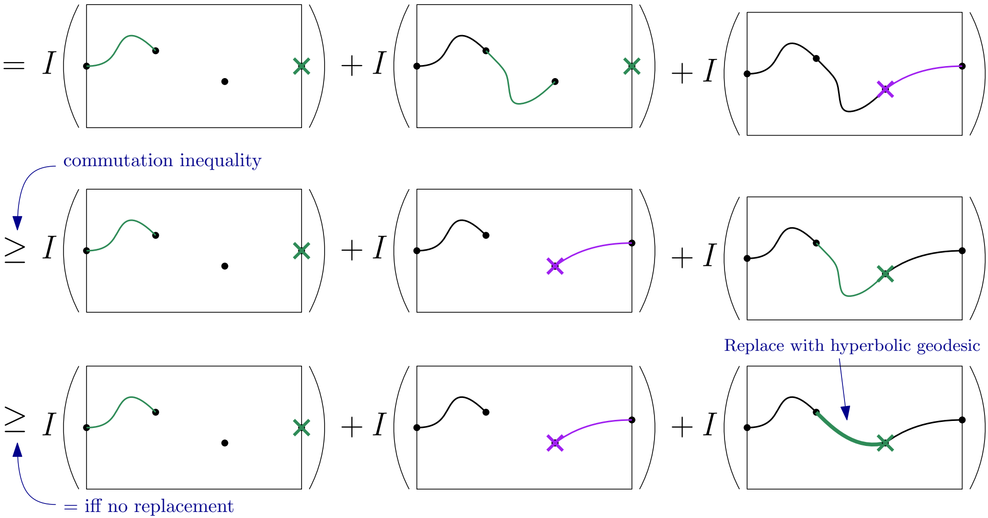
Key proposition

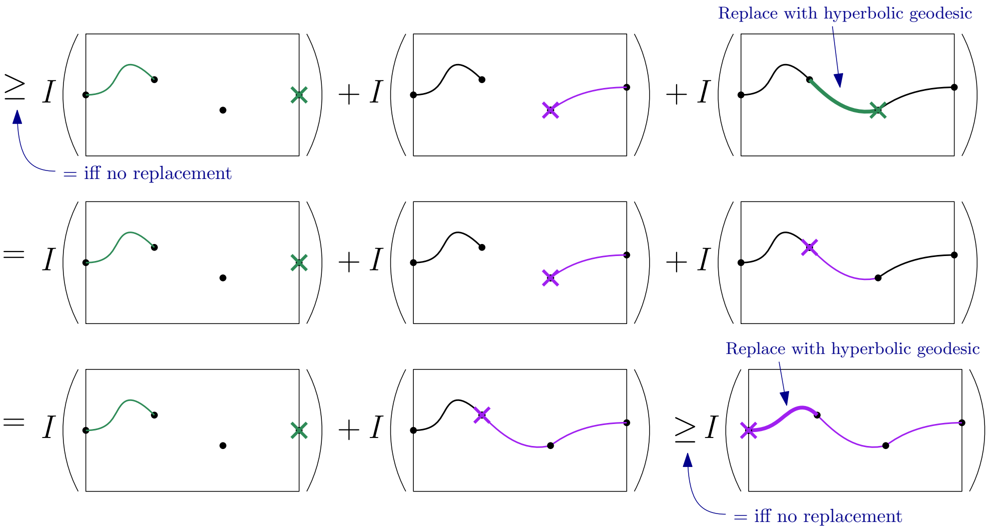
Let γ be a crosscut passing through $\mathbf{z}$.

1. $I(\gamma) = I(\gamma^{\mathbf{z}}) \Rightarrow \gamma$ is piecewise geodesic w.r.t. $\mathbf{z}$.
2. If γ is a piecewise geodesic, then $I(\gamma) = I(\hat{\gamma})$.

Base step of the induction:




= I( ) + I( ) + I( )
commutation inequality
≥ I( ) + I( ) + I( )
Replace with hyperbolic geodesic
≥ I( ) + I( ) + I( )
= iff no replacement


Replace with hyperbolic geodesic
$\geq I \; + I \; + I$
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$= I \; + I \; + I$
Replace with hyperbolic geodesic
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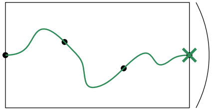
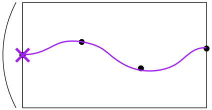
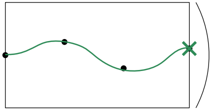
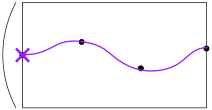
In conclusion,

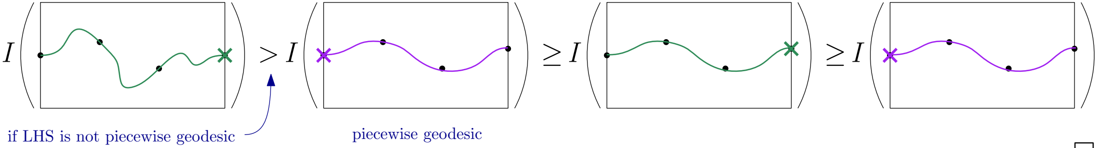
$I\Bigg($  $\Bigg) > I\Bigg($  $\Bigg) \geq I\Bigg($  $\Bigg) \geq I\Bigg($  $\Bigg)$

if LHS is not piecewise geodesic

piecewise geodesic

$\square$

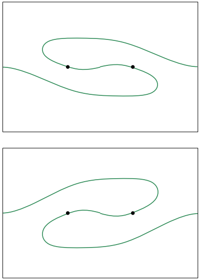
In conclusion,



$\square$

Isotopy class of the chord w.r.t. **z** (homotopy through chords fixing **z**) does not change during the reversal process of the proof.

Corollary: Existence of Loewner energy minimizer for each isotopy class w.r.t. **z**.



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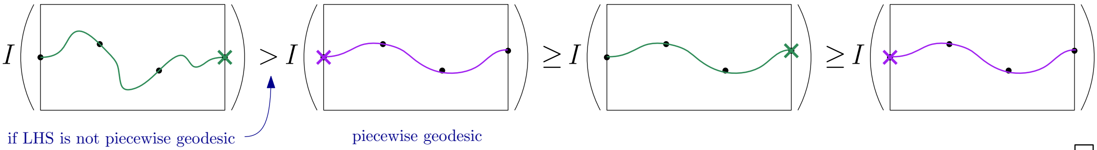

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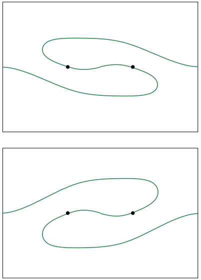
Isotopy class of the chord w.r.t. $\mathbf{z}$ (homotopy through chords fixing $\mathbf{z}$) does not change during the reversal process of the proof.

Corollary: Existence of Loewner energy minimizer for each isotopy class w.r.t. $\mathbf{z}$.

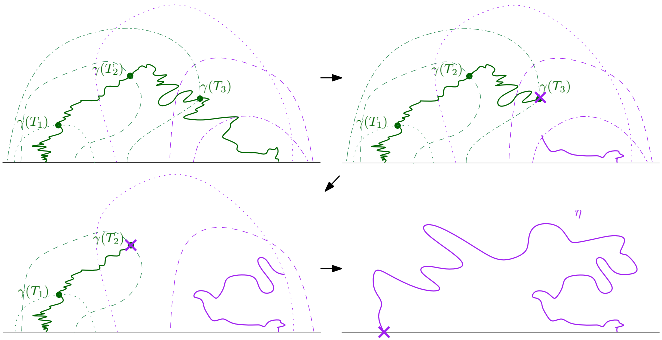


- Uniqueness for each isotopy class follows by straightforward adaptation of the proof of the loop case by Bonk–Junnila–Rohde–Wang
- Isotopy class of crosscuts w.r.t. $\mathbf{z} = \{z_1, \ldots, z_n\}$
  $\simeq$ Mapping class group of n-punctured disk
  $\simeq$ Braid group on n strands $\Leftarrow$ countably infinite for $n \geq 2$

SLE reversibility proof: Zhan

ν^n = Probability measure on coupling (γ, η) where

- Marginal law of the trace γ (resp. η) = chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to 1 (resp. 1 to 0)



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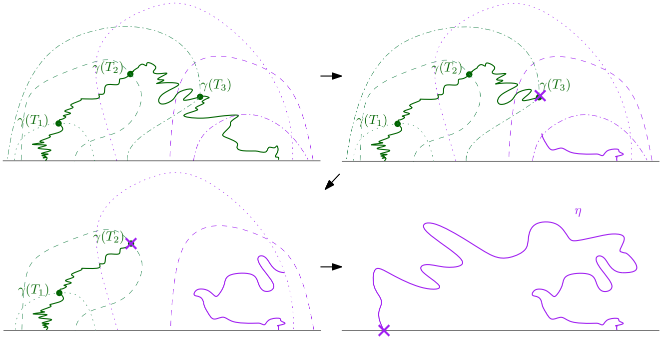
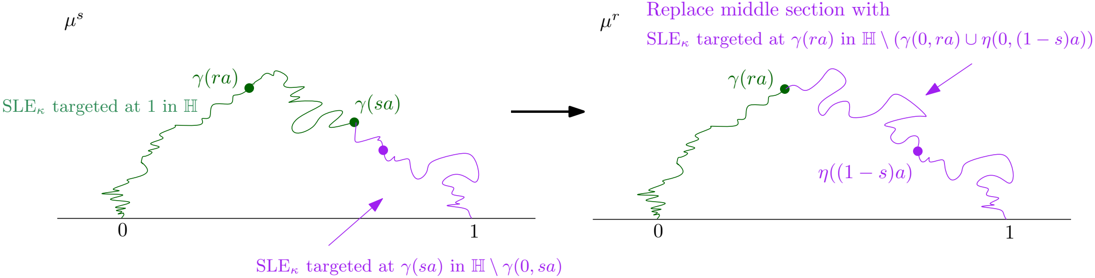


- n = number of pairs of disjoint hulls. As $n \to \infty$, next target point arbitrarily close.
- Prokhorov's theorem: ν^n converges weakly along a subsequence to ν^∞.
- For $(\gamma, \eta) \sim \nu^\infty$, $\hat{\gamma} = \eta$ a.s.

SLE reversibility proof: Lawler and Yearwood

Define μ^1 = Law of hull enclosed by γ, μ^0 = Law of hull enclosed by η.

Set a = hcap of entire hull. Preserved on each step.



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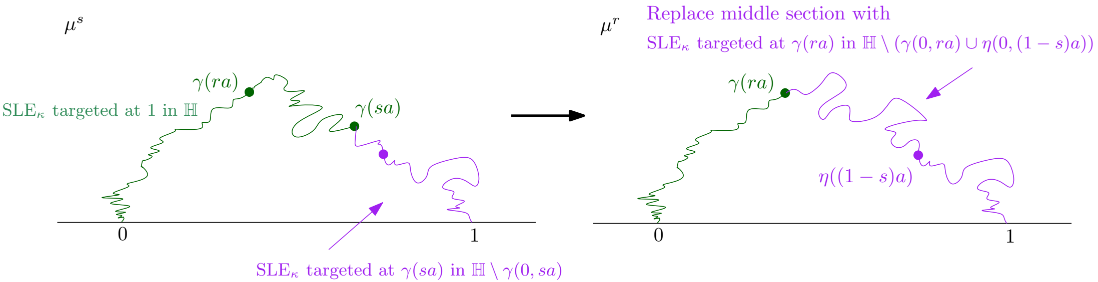

- Commutation relation $\Rightarrow$ RHS is μ^r. Gives coupling between μ^s and μ^r.
- ν^n = Coupling b/w μ^1 and μ^0 obtained by $\mu^1 \to \mu^{1-1/n} \to \cdots \mu^{1/n} \to \mu^0$
- $\exists\, c, \delta > 0$ such that $d_{\mathsf{Prokhorov}}(\mu^s, \mu^r) = c|s-r|^{1+\delta}$ under above coupling

  $\Rightarrow$ As $n \to \infty$, $d_{\mathsf{Prokhorov}}(\mu^1, \mu^0) \to 0$.

Happpy Birthday Greg!