Trivariate Hypergeometric Formulas for Pure Partition Functions of Multiple 3-SLE$_\kappa$

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Main Results

For three curves, Möbius covariance leaves three independent variables:

$$(r_1, r_2, r_3) \in (0,1)^3.$$

We construct explicit trivariate hypergeometric formulas for the pure partition functions of multiple 3-SLE_κ. We set $\beta = 4/\kappa$.

Rainbow patterns

We identify these formulas with the corresponding pure partition functions for all $\beta > 1/2$, i.e., $\kappa \in (0,8)$.

Neighbor patterns

We identify these formulas with the corresponding pure partition functions for all $\beta \geq 2/3$, i.e., $\kappa \in (0,6]$.

Multiple SLE Partition Functions

Let

$$\mathfrak{X}_{2N} = \{(x_1, \dots, x_{2N}) \in \mathbb{R}^{2N} : x_1 < \cdots < x_{2N}\}.$$

An N-SLE$_\kappa$ partition function is $\mathcal{Z} : \mathfrak{X}_{2N} \to (0, \infty)$ satisfying the BPZ equations:

$$\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j \neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{2h}{(x_j - x_i)^2}\right)\right]\mathcal{Z} = 0, \quad h := \frac{6-\kappa}{2\kappa}$$

together with Möbius covariance with exponent h:

$$\mathcal{Z}(\mathbf{x}) = \prod_{i=1}^{2N} \phi'(x_i)^h\, \mathcal{Z}(\phi(x_1), \dots, \phi(x_{2N})).$$

Pure Partition Functions

- Pure partition functions are partition functions indexed by link patterns

$$\alpha \in \mathrm{LP}_N := \{\text{non-crossing pairings of } \{1,2,\ldots,2N\}\}.$$

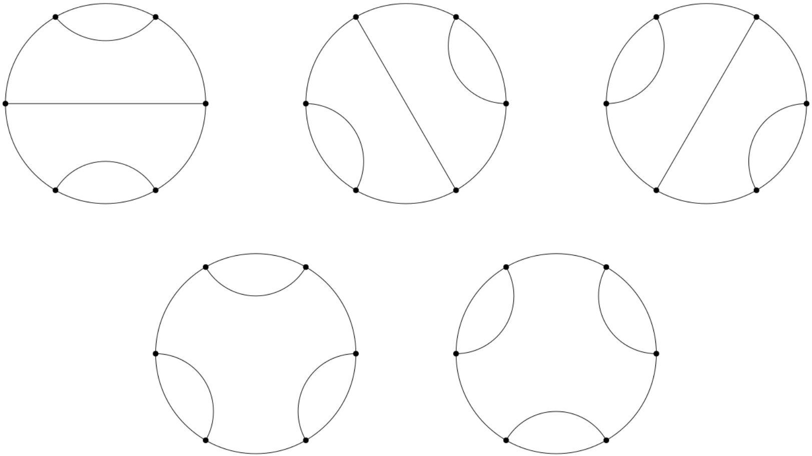
- They are singled out by fusion asymptotics:

$$\lim_{x_j,x_{j+1}\to\xi} \frac{\mathcal{Z}_\alpha}{|x_j - x_{j+1}|^{-2h}} = \begin{cases} \mathcal{Z}_{\alpha/\{j,j+1\}}, & \text{if } \{j,j+1\} \in \alpha, \\ 0, & \text{if } \{j,j+1\} \notin \alpha, \end{cases}$$

together with the base case $\mathcal{Z}_{\{\{1,2\}\}} = |x_1 - x_2|^{-2h}$ when $N = 1$.

This is the standard BPZ–Möbius–fusion characterization; see, e.g., [PW19]. Probabilistically, $\mathcal{Z}_\alpha$ determines the drift of the Loewner driving function for each one-curve marginal.

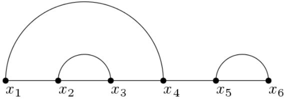
The Five Link Patterns in LP_3



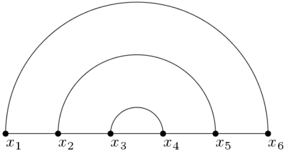
Top row: rainbow patterns. Bottom row: neighbor patterns.

The Same Link Patterns on the Real Line

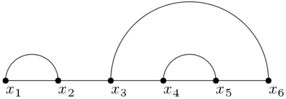
In the computations we work on 6 ordered real points.



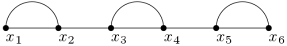
$\{\{1,4\},\{2,3\},\{5,6\}\}$



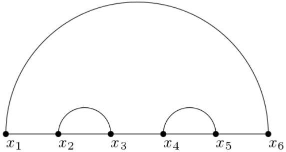
$\{\{1,6\},\{2,5\},\{3,4\}\}$



$\{\{1,2\},\{3,6\},\{4,5\}\}$



$\{\{1,2\},\{3,4\},\{5,6\}\}$



$\{\{1,6\},\{2,3\},\{4,5\}\}$

Top row: rainbow patterns. Bottom row: neighbor patterns.

$N = 2$ and $N = 3$ Are Special

For $2N$ ordered boundary points, Möbius covariance removes three degrees of freedom since the group of Möbius transformations has dimension 3. Thus the moduli space has dimension

$$2N - 3.$$

On the other hand, for a link pattern with N links, each pair of links gives an unoriented cross-ratio. Hence the number of such pairwise cross-ratios is

$$\binom{N}{2}.$$

The two numbers agree exactly when $N = 2$ or $N = 3$. Therefore, in the case $N = 3$, the three pairwise cross-ratios can be used as independent coordinates.

2-SLE Partition Function

For two non-crossing boundary pairs, define the unoriented cross-ratio

$$C_{\{p_1,q_1\},\{p_2,q_2\}} = \min\left\{\frac{|p_1-p_2||q_1-q_2|}{|p_1-q_2||q_1-p_2|}, \frac{|p_1-q_2||q_1-p_2|}{|p_1-p_2||q_1-q_2|}\right\} \in (0,1).$$

For

$$\alpha = \{\{a_1,b_1\},\{a_2,b_2\}\} \in \mathrm{LP}_2, \qquad r = r(x) = C_{\{x_{a_1},x_{b_1}\},\{x_{a_2},x_{b_2}\}},$$

the pure partition function is ([BBK05], see also [Wu20]):

$$\mathcal{Z}_\alpha(x) = \prod_{j=1}^{2} |x_{a_j} - x_{b_j}|^{-2h} \cdot r^{2/\kappa} \frac{f_\beta(r)}{f_\beta(1)},$$

where

$$f_\beta(r) = {}_2F_1(1-\beta,\beta;2\beta;r), \qquad \beta = \frac{4}{\kappa}.$$

The Hypergeometric Prototype

The Gaussian hypergeometric function ${}_2F_1(a,b;c;\cdot)$ is the analytic solution near 0, normalized by $f(0)=1$, of $\mathcal{L}_x^{(a,b;c)}f=0$, where

$$\mathcal{L}_x^{(a,b;c)} := x(1-x)\partial_x^2 + \big(c-(a+b+1)x\big)\partial_x - ab.$$

In the 2-SLE partition function, $(a,b,c)=(1-\beta,\beta,2\beta)$, $\beta=\frac{4}{\kappa}$. When $\beta>\frac{1}{2}$, Gauss's identity gives ${}_2F_1(1-\beta,\beta;2\beta;1)\in(0,\infty)$.

Keep both f_β and this differential operator in mind; they will appear again when we write the reduced equations for the three-variable problem.

Reduction to Three Variables

For

$$\alpha = \{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}\} \in \mathrm{LP}_3,$$

set

$$r_\ell = C_{\{x_{a_j}, x_{b_j}\}, \{x_{a_k}, x_{b_k}\}}, \qquad \{j, k, \ell\} = \{1, 2, 3\},$$

and

$$\mathbf{r} = (r_1, r_2, r_3) \in (0, 1)^3.$$

We write

$$\mathcal{Z}_\alpha(\mathbf{x}) = C_\alpha \prod_{j=1}^{3} |x_{a_j} - x_{b_j}|^{-2h} \cdot \prod_{j=1}^{3} r_j^{2/\kappa} \cdot F(\mathbf{r}).$$

A Sufficient Reduced System

We impose the following system on F, with an auxiliary function G, for every $\{j,k,\ell\}=\{1,2,3\}$.

For both rainbow and neighbor patterns:

$$\mathcal{L}_{r_\ell}^{(1-\beta,\beta;2\beta)} F = (1-r_j)(1-r_k)G. \tag{C}$$

For rainbow patterns:

$$(2r_j r_k \partial_j \partial_k + \beta r_j \partial_j + \beta r_k \partial_k - \beta r_\ell \partial_\ell) F = \big(r_\ell(1+r_j r_k) - (r_j + r_k)\big) G. \tag{R}$$

For neighbor patterns:

$$\big(2r_j r_k \partial_j \partial_k + \beta r_j \partial_j + \beta r_k \partial_k + \beta r_\ell \partial_\ell + \beta^2\big) F = \big(r_\ell(r_j + r_k) - (1 + r_j r_k)\big) G. \tag{N}$$

(C)+(R), or (C)+(N), is a **sufficient** condition for $\mathcal{Z}_\alpha$ to satisfy the BPZ equations.

The Series Ansatz

Assume that F and G are analytic at the origin ($\mathbf{n} = (n_1, n_2, n_3)$, $\mathbf{r}^{\mathbf{n}} = r_1^{n_1} r_2^{n_2} r_3^{n_3}$):

$$F(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{N}_0^3} A_{\mathbf{n}}\mathbf{r}^{\mathbf{n}}, \qquad G(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{N}_0^3} B_{\mathbf{n}}\mathbf{r}^{\mathbf{n}}.$$

The system for F and G gives nearest-neighbor recurrence relations for the coefficient arrays A and B.

With the normalization $A_{\mathbf{0}}^{(\beta)} = 1$, the recurrence system has a unique formal solution, for both rainbow and neighbor patterns and for all $\beta > 0$.

The resulting arrays A and B are fully symmetric in the three indices. For rainbow patterns, this symmetry is not geometrically obvious.

$\boxed{\text{This is formal algebra; convergence and boundary behavior come later.}}$

Nearest-Neighbor Recurrences

Comparing coefficients in (C), (R), and (N) gives, for $\{j,k,\ell\}=\{1,2,3\}$ and $\mathbf{n}\in\mathbb{Z}^3$,

$$(n_\ell+1)(n_\ell+2\beta)A_{\mathbf{n}+e_\ell}-(n_\ell+\beta)(n_\ell+1-\beta)A_{\mathbf{n}} \\ = B_{\mathbf{n}}-B_{\mathbf{n}-e_j}-B_{\mathbf{n}-e_k}+B_{\mathbf{n}-e_j-e_k}. \tag{C'}$$

$$(2n_jn_k+\beta n_j+\beta n_k-\beta n_\ell)A_{\mathbf{n}} \\ = B_{\mathbf{n}-e_\ell}+B_{\mathbf{n}-e_\ell-e_j-e_k}-B_{\mathbf{n}-e_j}-B_{\mathbf{n}-e_k}. \tag{R'}$$

$$(2n_jn_k+\beta n_j+\beta n_k+\beta n_\ell+\beta^2)A_{\mathbf{n}} \\ = B_{\mathbf{n}-e_\ell-e_j}+B_{\mathbf{n}-e_\ell-e_k}-B_{\mathbf{n}}-B_{\mathbf{n}-e_j-e_k}. \tag{N'}$$

Here we use the convention that, when $\mathbf{n}\notin\mathbb{N}_0^3$, $A_{\mathbf{n}}=B_{\mathbf{n}}=0$.

$\boxed{(C')+(R') \text{ or } (C')+(N') \text{ recursively determines the formal series.}}$

Two-Variable Degenerations

The trivariate series is not one of the standard Appell–Lauricella functions. However, when one variable, say r_3, is set to zero, the trivariate series degenerates to classical two-variable hypergeometric functions.

Rainbow case

The degeneration is described by Appell F_1:

$$F(r_1, r_2, 0) = F_1(1-\beta; \beta, \beta; 3\beta; r_1, r_2).$$

Neighbor case

The degeneration is described by Horn G_2:

$$F(r_1, r_2, 0) = G_2(\beta, \beta; 1-2\beta, 1-2\beta; -r_1, -r_2).$$

Rainbow Case: a Key Estimate

Set

$$|\mathbf{n}| = n_1 + n_2 + n_3, \qquad a_{\mathbf{n}} = \max(n_1, n_2, n_3),$$
$$b_{\mathbf{n}} = \operatorname{med}(n_1, n_2, n_3), \qquad c_{\mathbf{n}} = \min(n_1, n_2, n_3).$$

The recurrence relations imply the following coefficient estimates.

Case 1: $\beta > 1$. For any $p \in (3, 2\beta + 1)$,

$$|A_{\mathbf{n}}| \lesssim_{\beta,p} (|\mathbf{n}| + 1)^{-p}.$$

Case 2: $\beta \in (1/2, 1)$. For any $s \in \left(1, (\beta + \frac{1}{2}) \wedge (2 - \beta)\right)$,

$$|A_{\mathbf{n}}| \lesssim_{\beta,s} (a_{\mathbf{n}} + 1)^{-s}(b_{\mathbf{n}} + 1)^{-1}(c_{\mathbf{n}} + 1)^{-s}.$$

Case 3: $\beta = 1$. This is simple: $A_{\mathbf{n}} = 0$ when $\mathbf{n} \neq \mathbf{0}$.

Rainbow Case: Absolute Summability

For $\beta > 1$, choose $p \in (3, 2\beta + 1)$. Since

$$\#\{\mathbf{n} \in \mathbb{N}_0^3 : |\mathbf{n}| = N\} = \frac{(N+1)(N+2)}{2} \lesssim (N+1)^2,$$

we get

$$\sum_{\mathbf{n} \in \mathbb{N}_0^3} |A_{\mathbf{n}}| \lesssim \sum_{N=0}^{\infty} (N+1)^2 (N+1)^{-p} = \sum_{N=0}^{\infty} (N+1)^{2-p} < \infty.$$

For $\beta = 1$,

$$\sum_{\mathbf{n}} |A_{\mathbf{n}}| = |A_{\mathbf{0}}| = 1 < \infty.$$

Rainbow Case: Absolute Summability

If $\beta \in (1/2, 1)$, choose $s \in (1, (\beta + \frac{1}{2}) \wedge (2 - \beta))$. Then

$$\sum_{\mathbf{n}} |A_{\mathbf{n}}| \lesssim \sum_{a=0}^{\infty} \sum_{b=0}^{a} \sum_{c=0}^{b} (a+1)^{-s} (b+1)^{-1} (c+1)^{-s}$$

$$\lesssim \sum_{a=0}^{\infty} \sum_{b=0}^{a} (a+1)^{-s} (b+1)^{-1} \lesssim \sum_{a=0}^{\infty} (a+1)^{-s} \log(a+1) < \infty.$$

Thus, for all $\beta > 1/2$, i.e., $\kappa \in (0, 8)$, $\sum_{\mathbf{n}} |A_{\mathbf{n}}| < \infty$. So $\sum_{\mathbf{n}} A_{\mathbf{n}} \mathbf{r}^{\mathbf{n}}$ converges uniformly and defines a function F, which is analytic in $(-1, 1)^3$ and continuous on $[-1, 1]^3$.

Using (C') and (R'), we conclude that $\sum_{\mathbf{n}} B_{\mathbf{n}} \mathbf{r}^{\mathbf{n}}$ converges locally uniformly in $(-1, 1)^3$ and defines an analytic function G in $(-1, 1)^3$. Thus, F and G are solutions of the system (C)+(R).

Rainbow Case: the Side Face $r_3 = 1$

Absolute summability for $\beta > 1/2$ allows summation in the third index:

$$A^{\Sigma}_{m,n} := \sum_{q=0}^{\infty} A_{m,n,q}, \qquad F^{\Sigma}(r_1, r_2) := F(r_1, r_2, 1) = \sum_{m,n\geq 0} A^{\Sigma}_{m,n} r_1^m r_2^n.$$

Summing the coefficient recurrences in the n_3-direction gives the side-face equation

$$(1 - r_1)\mathcal{L}_{r_1}^{(1-\beta,\beta;2\beta)} F^{\Sigma} = (1 - r_2)\mathcal{L}_{r_2}^{(1-\beta,\beta;2\beta)} F^{\Sigma}.$$

At the coefficient level, this is a two-dimensional recurrence for A^{Σ}, and it determines the whole array from its values on one axis.

The axis values are identified by the Appell boundary degenerations and Gauss's identity. Hence

$$F(r_1, r_2, 1) = \frac{\Gamma(3\beta)\Gamma(3\beta - 1)}{\Gamma(2\beta)\Gamma(4\beta - 1)} f_\beta(r_1) f_\beta(r_2), \quad f_\beta(r) := {}_2F_1(1 - \beta, \beta; 2\beta; r).$$

Neighbor Case: Local Convergence

For the neighbor patterns, we are only able to prove that the power series

$$\sum_{\mathbf{n}} A_{\mathbf{n}} \mathbf{r}^{\mathbf{n}}, \qquad \sum_{\mathbf{n}} B_{\mathbf{n}} \mathbf{r}^{\mathbf{n}}.$$

converge in a small polydisc, and define analytic functions F and G solving (C)+(N) there.

We do not expect convergence up to the unit cube. In fact, when $\beta = 1$, the polyradius cannot exceed $1/3$.

rainbow	:	coefficient estimates give convergence up to the unit cube,
neighbor	:	local convergence must be analytically continued.

Pfaff Systems: the Basic Mechanism

For the neighbor case, set

$$\mathbf{Y} = (F_1, F_2, F_3, F, G)^{\mathrm{T}}, \qquad F_i = \partial_i F.$$

For each direction r_ℓ, the goal is to write

$$\partial_\ell \mathbf{Y} = M^{(\ell)}(\mathbf{r})\mathbf{Y}. \tag{1}$$

The components are handled as follows:

- $\partial_\ell F = F_\ell$ is immediate.
- The derivatives $\partial_\ell F_j = F_{j\ell}$ are second derivatives of F. The diagonal ones come from (C), and the mixed ones from (N).
- The difficult part is $\partial_\ell G$. We differentiate the equations (C) and (N), eliminate the higher derivatives of F, and solve for $\partial_\ell G$.

In this way, the second-order system is converted into a first-order system.

Why Different Pfaff Systems Are Useful

The same idea can be applied after multiplying the components by weights:

$$\mathbf{Y}_R = \left(R_1 F_1, R_2 F_2, R_3 F_3, R_4 F, R_5 G\right)^{\mathrm{T}}.$$

Different choices of the weights R_i produce different first-order systems.

In the proof we use:

- one Pfaff system for analytic continuation to a neighborhood of $[0,1)^3$;
- four Pfaff systems for continuous extension to $[0,1]^3$.

The point is that, after the second-order system is converted into first-order systems, analytic coefficients give analytic continuation, and size bounds for M give Gronwall estimates. This is the mechanism behind analytic continuation and continuous extension.

The Bulk System: Analytic Continuation

For analytic continuation, we use the unweighted Pfaff vector

$$\mathbf{Y} = (F_1, F_2, F_3, F, G)^{\mathrm{T}}, \qquad \partial_\ell \mathbf{Y} = M^{(\ell)}(\mathbf{r})\mathbf{Y}.$$

The local power series gives an analytic germ of $\mathbf{Y}$ near the origin. From any point where $\mathbf{Y}$ is already analytic, we continue it by solving this Pfaff system along coordinate directions.

Since the matrices $M^{(\ell)}$ are analytic in the bulk, standard analytic dependence for ODEs extends $\mathbf{Y}$ analytically along these coordinate flows. Iterating the flows and removing the possible singularities gives analytic continuation of $\mathbf{Y}$ to a complex neighborhood of $[0,1)^3$.

Why Boundary Regularity Is Needed

The analytic extension of F to a neighborhood of $[0,1)^3$ is still not enough to identify $\mathcal{Z}_\alpha$ as a *pure* partition function.

To verify fusion asymptotics, we let two adjacent marked points merge. When the two points are linked, the corresponding limit sends two cross-ratios to 1.

Thus we need controlled boundary values of F on the fusion edges: $\{r_j = r_k = 1\}$. After normalization, these values should reduce to the normalized 2-SLE factor $f_\beta(r_\ell)/f_\beta(1)$.

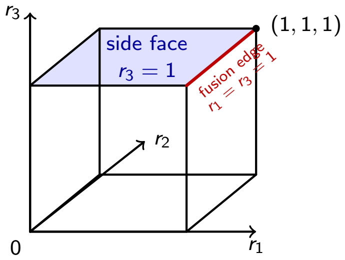
For positivity, we use an SLE martingale argument. This requires boundedness of F in $[0,1)^3$ and positive subsequential boundary limits.

A Convenient Sufficient Boundary Result

A convenient sufficient statement is

$$F \in C([0,1]^3), \qquad F\big|_{r_\ell=1} = C_\beta f_\beta(r_j) f_\beta(r_k), \quad \{j,k,\ell\} = \{1,2,3\}.$$

- This gives boundedness on $[0,1)^3$.
- Boundary positivity follows from the positivity of f_β (at $r_\ell = 1$) and Appell F_1 or Horn G_2 (at $r_\ell = 0$).
- The values on the fusion edges follows from the values on the side faces.
- Rainbow: directly from absolute summability and recurrences.
  Neighbor: separate boundary analysis.



Non-corner Estimates

The proof of the continuous extension of F to $[0,1]^3$ has three parts: non-corner estimates, side-face analysis, and corner propagation.

For the non-corner estimate, fix $L \in (0,1)$ and work in

$$D_L^{(\ell)} = \{\mathbf{r} \in [0,1)^3 : r_\ell \le L\}, \qquad \{j,k,\ell\} = \{1,2,3\}.$$

Using a weighted first-order system adapted to this region, one obtains

$$|F| + |F_\ell| \lesssim_L 1,$$

$$|F_j| + |F_k| + (1 - r_j r_k)|G| \lesssim_L (1 - r_j r_k)^{(2\beta-2)\wedge 0}, \qquad \beta > \frac{2}{3}.$$

At $\beta = 2/3$, logarithmic factors appear, for example $\log \frac{1-r_j r_k}{1-r_j}$.

As a consequence, F extends continuously to $\overline{D_L^{(\ell)}}$. By varying L and ℓ, we conclude that F extends continuously to $[0,1]^3 \setminus \{(1,1,1)\}$.

Side Face: a Boundary-adapted Pfaff System

To analyze a side face, for example $r_3 = 1$, we use a weighted Pfaff vector

$$\mathbf{Y}^{[3]} = \big(r_1 F_1,\ r_2 F_2,\ F_3,\ F,\ (1-r_1)(1-r_2)G\big)^T.$$

This vector satisfies a first-order normal system

$$\partial_3 \mathbf{Y}^{[3]} = M^{[3]}(\mathbf{r})\mathbf{Y}^{[3]},$$

whose coefficients have controlled regular-singular behavior as $r_3 \uparrow 1$. The other side faces are treated by symmetry.

The point of introducing $\mathbf{Y}^{[3]}$ is that it controls exactly the quantities needed to restrict equation (C) to the side face.

Side Face: Boundary Equations and Value

The control from the previous slide lets us pass to the limit in (C). Since the right-hand side of (C) contains the factor $1 - r_3$, on the face $r_3 = 1$, we get

$$\mathcal{L}_{r_1}^{(1-\beta,\beta;2\beta)} F|_{r_3=1} = 0, \qquad \mathcal{L}_{r_2}^{(1-\beta,\beta;2\beta)} F|_{r_3=1} = 0.$$

Thus the side trace separates. The axis values are fixed by the lower-dimensional degeneration and Gauss's identity, so

$$F|_{r_3=1} = C_\beta f_\beta(r_1) f_\beta(r_2), \qquad C_\beta = \frac{\Gamma(2\beta)\Gamma(3\beta-1)}{\Gamma(\beta)\Gamma(4\beta-1)}.$$

Here $f_\beta(r) = {}_2F_1(1-\beta,\beta;2\beta;r)$, as before.

Side Faces: Normal Estimates

The side-face value alone is not enough; one also needs quantitative control as $r_\ell \uparrow 1$.

Near the side face $r_\ell = 1$, the normal analysis gives

$$F(\mathbf{r}) = F(r_j, r_k)|_{r_\ell=1} + \text{normal correction} + \text{controlled error}.$$

The estimates also control the normal derivative of F and the function G in weighted forms:

$$(1 - r_\ell)|F_\ell|, \qquad (1 - r_1)(1 - r_2)(1 - r_3)|G|.$$

At the critical endpoint $\beta = 2/3$, logarithmic normal terms appear.

$\boxed{\text{These estimates provide the boundary input for the corner analysis.}}$

The Triple Corner: the First Regime

The last difficulty is the corner $(1,1,1)$. Set $t_j = 1 - r_j$. By symmetry, first work in the ordered region

$$t_3 \le t_2 \le t_1 \le \frac{1}{2}.$$

The analysis is separated by the ratio

$$\chi(\mathbf{r}) = \frac{t_1 t_2}{t_3}.$$

Let $\gamma = (2\beta - 1) \wedge 1$. When $\beta > 2/3$, a corner-adapted Pfaff system gives, for $y_0 := C_\beta f_\beta(1)^2$,

$$|F(\mathbf{r}) - y_0| + \sum_{j=1}^{3} t_j |F_j(\mathbf{r})| + t_1 t_2 t_3 |G(\mathbf{r})| \lesssim t_1^\gamma, \quad \text{in the region } \chi \ge 1/2.$$

At $\beta = 2/3$, we get similar estimates with logarithmic correction.

Corner Propagation: the Second Regime

It remains to treat the complementary corner region $\chi(\mathbf{r}) < \frac{1}{2}$

The main idea is that we connect the target point in $\chi < 1/2$ to the interface $\chi = 1/2$ by a Pfaff flow and then use the estimates in the first region and Grönwall estimate associated with the Pfaff system. For every $0 < \gamma' < \gamma$, this argument gives, for all $\beta \geq 2/3$,

$$|F(\mathbf{r}) - y_0| + \sum_{j=1}^{3} t_j |F_j(\mathbf{r})| + t_1 t_2 t_3 |G(\mathbf{r})| \lesssim t_1^{\gamma'} \quad \text{in the region } \chi < 1/2.$$

The estimate holds at $\beta = 2/3$ because the logarithmic factors are absorbed.

Final Stitching

As a result, for all $\beta \geq 2/3$, in the whole ordered sector $t_3 \leq t_2 \leq t_1 \leq 1/2$,

$$F(\mathbf{r}) \longrightarrow y_0 \qquad \text{as } \mathbf{r} \to (1,1,1).$$

By symmetry, we then conclude that $F(\mathbf{r}) \to y_0$ as $\mathbf{r} \to (1,1,1)$.

Since F has already been extended continuously to $[0,1]^3 \setminus \{(1,1,1)\}$, this gives a continuous extension of F to the whole cube $[0,1]^3$.

This is the boundary regularity needed for the neighbor patterns.

Positivity and Identification

It remains to prove $Z_\alpha > 0$. The argument below works for both types of patterns. By the BPZ equations, Itô's formula, and the Loewner equation, one obtains a local martingale

$$M_t = \text{observable built from } Z_\alpha.$$

The boundedness of F implies that the stopped local martingale is uniformly integrable and allows optional stopping: $M_0 = \mathbb{E}[M_\tau]$.

The boundary degenerations on $\{r_\ell = 0\}$ and the side-face values on $\{r_\ell = 1\}$ imply $F > 0$ on $\partial[0,1]^3$, which further implies that $M_\tau \geq 0$ and $\mathbb{P}[M_\tau > 0] > 0$.

Hence $M_0 > 0$. Since M_0 is a positive prefactor times $Z_\alpha(x)$, we get $Z_\alpha(x) > 0$.

Together with BPZ, Möbius covariance, and fusion asymptotics, this identifies the constructed Z_α with the pure partition function.

Integer Values of β

Integer values of β reveal additional algebraic structure.

Rainbow case. When $\beta \in \mathbb{N}$, there are only finitely many $\mathbf{n}$ such that $A_{\mathbf{n}} \neq 0$. Thus, F is a polynomial.

Neighbor case. We define the discriminant polynomial:

$$\Delta(\mathbf{r}) = (1 - r_1 r_2 - r_1 r_3 - r_2 r_3)^2 - 4 r_1 r_2 r_3 (r_1 + r_2 + r_3 - 2).$$

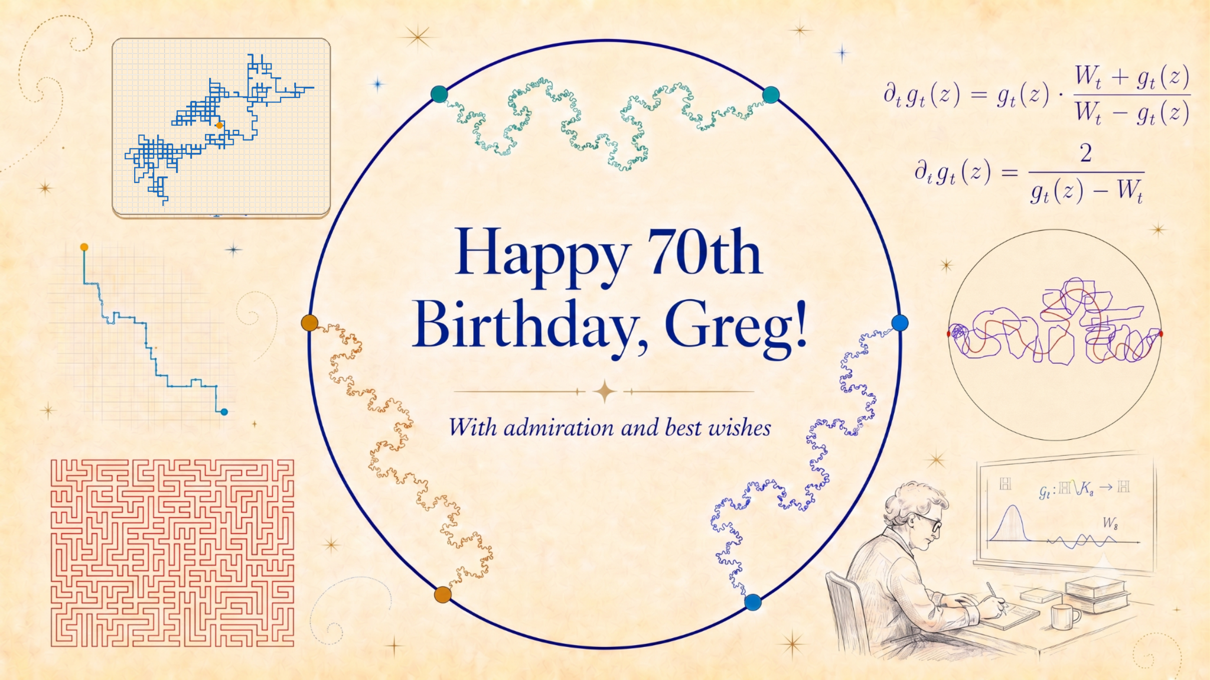
This is the nontrivial polynomial factor appearing in the denominators of the Pfaff-system coefficients.

We conjecture that, when $\beta \in \mathbb{N}$, F has the form:

$$F(\mathbf{r}) = \frac{P_\beta(\mathbf{r})\sqrt{\Delta(\mathbf{r})} - Q_\beta(\mathbf{r})}{D_\beta \cdot (r_1 r_2 r_3)^{2\beta-1}}, \qquad D_\beta := \frac{4(3\beta-2)!(4\beta-3)!((\beta-1)!)^3}{((2\beta-1)!)^2((2\beta-2)!)^3},$$

where P_β, Q_β are symmetric polynomials with $P_\beta(\mathbf{0}) = Q_\beta(\mathbf{0}) = 1$.

We have an algorithm to construct P_β, Q_β and have verified this conjecture for $1 \leq \beta \leq 16$ using computer algebra; the computational complexity is $O(\beta^6)$.



Happy 70th Birthday, Greg!
With admiration and best wishes
$\partial_t g_t(z) = g_t(z) \cdot \frac{W_t + g_t(z)}{W_t - g_t(z)}$
$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$

Thank you!

Questions and comments are welcome.

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