Liouville Quantum Duality and Random Planar Maps

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arXiv:2507.12203   *arXiv:2604.24180*

Random Explorations-IMSI

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Block-weighted planar maps

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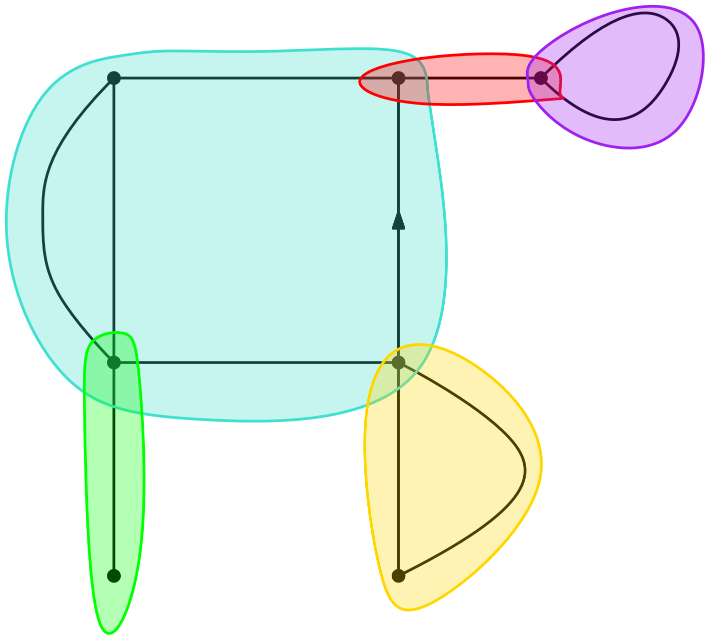
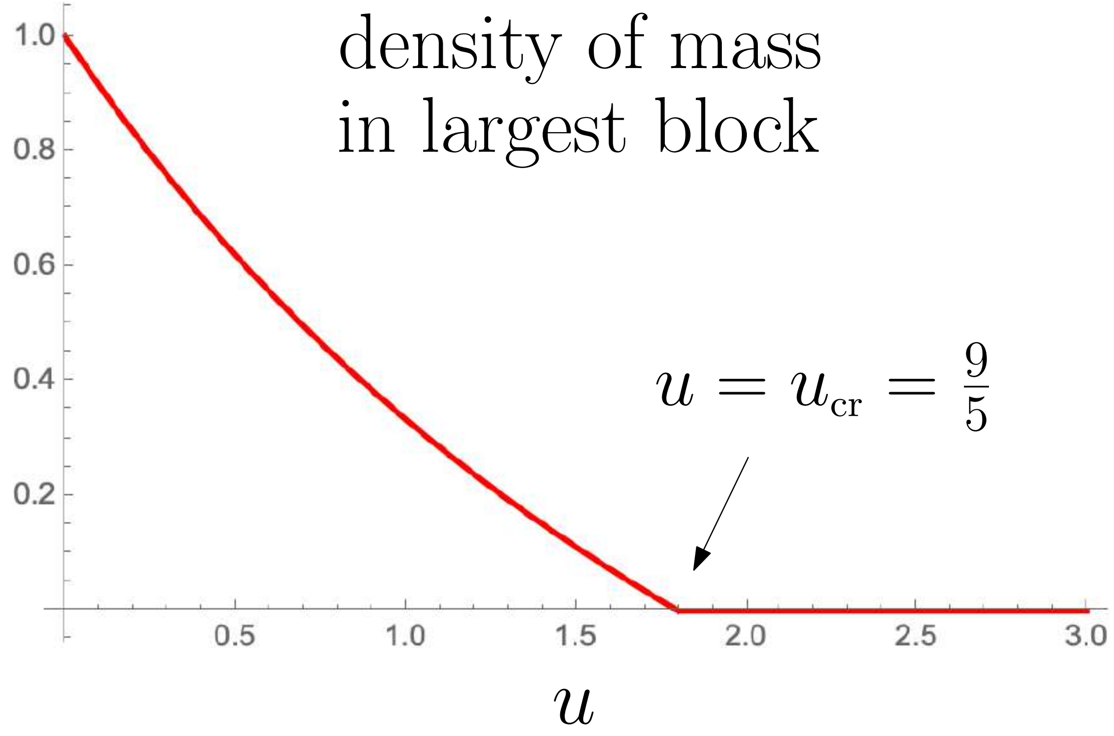


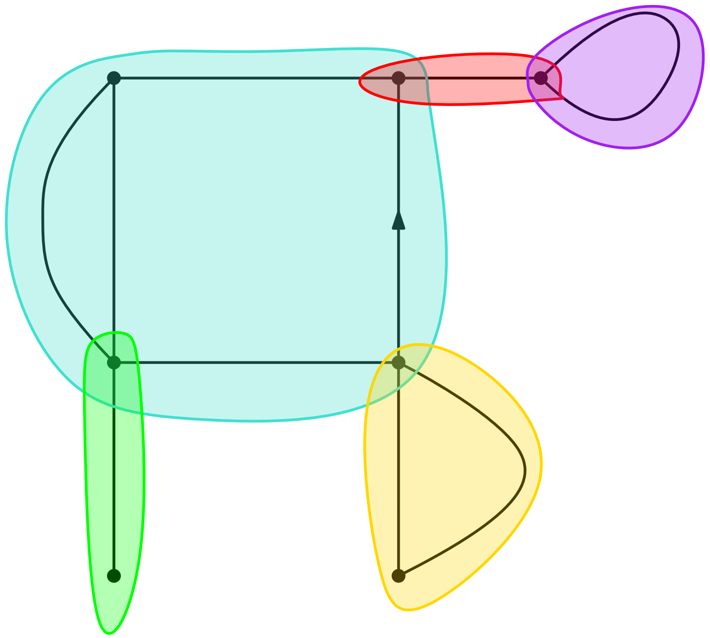
A phase transition in block-weighted random maps

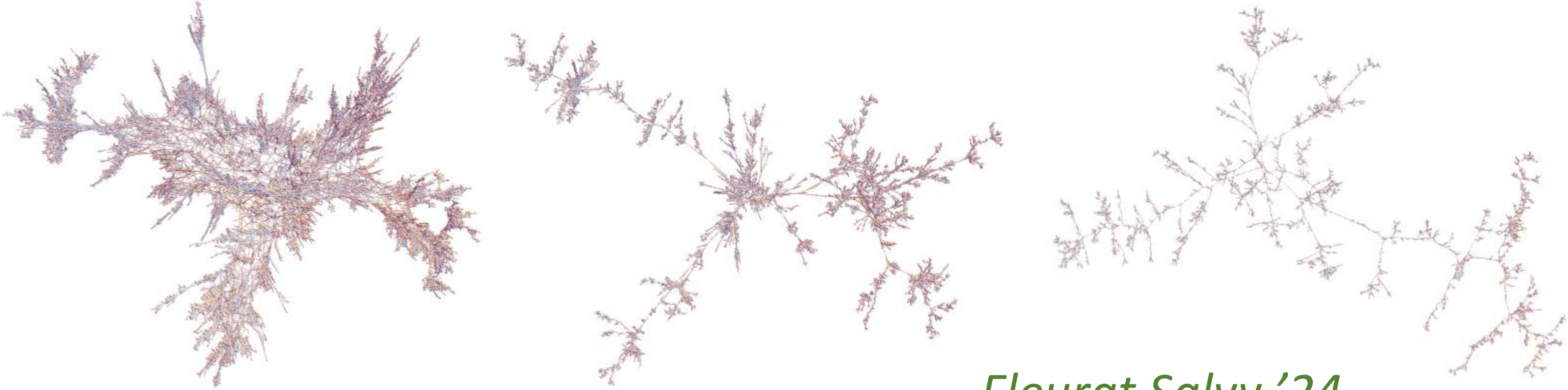
William Fleurat* Zéphyr Salvy†

(Rooted) planar maps with a weight u per 2-connected block

-> **transition** at $u = 9/5$





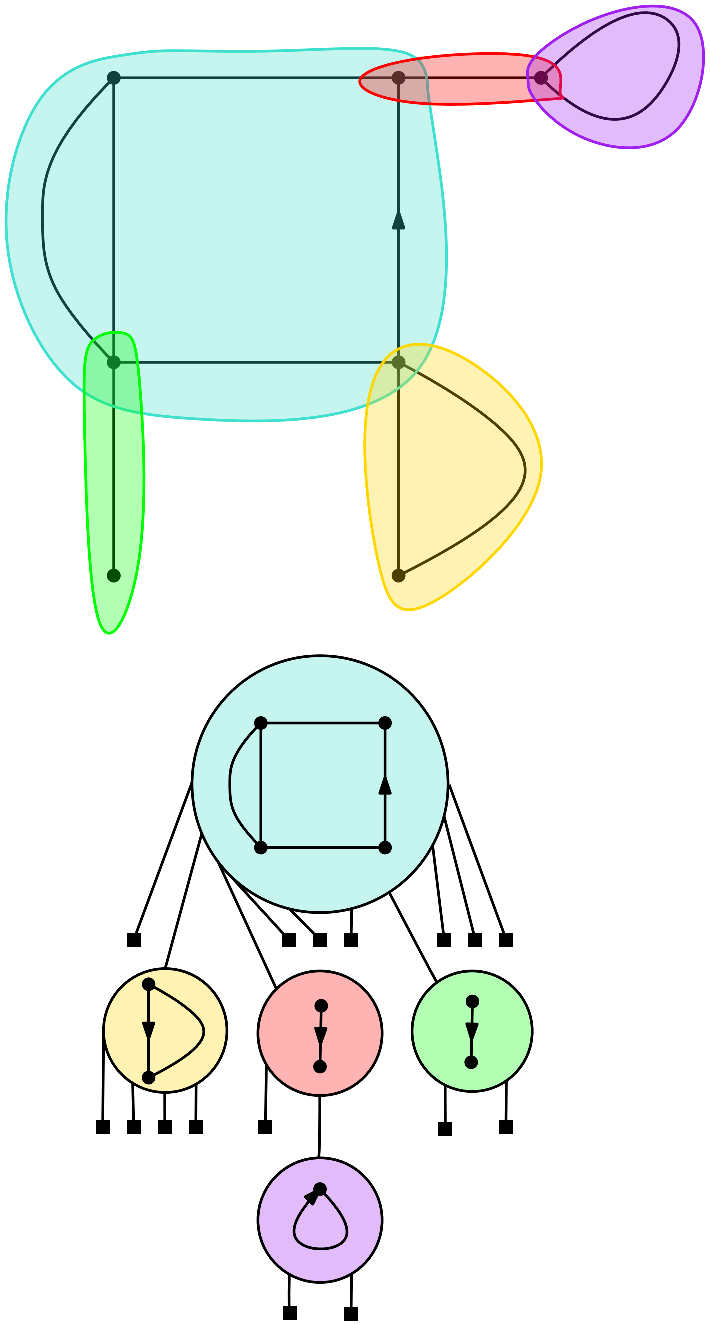


Fleurat Salvy '24

Maps with n edges

$$M_n \propto \begin{cases} \rho(u)^{-n}\ n^{-5/2} & u < 9/5 \\ \rho(u)^{-n}\ n^{-5/3} & u = 9/5 \\ \rho(u)^{-n}\ n^{-3/2} & u > 9/5 \end{cases}$$

Analysis of the **block-tree** = tree whose inner vertices are the blocks and whose edges code for the connexions of these blocks



Back to physics

NEW CRITICAL BEHAVIOR IN $d=0$ LARGE-N MATRIX MODELS

SUMIT R. DAS, AVINASH DHAR, ANIRVAN M. SENGUPTA and SPENTA R. WADIA

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

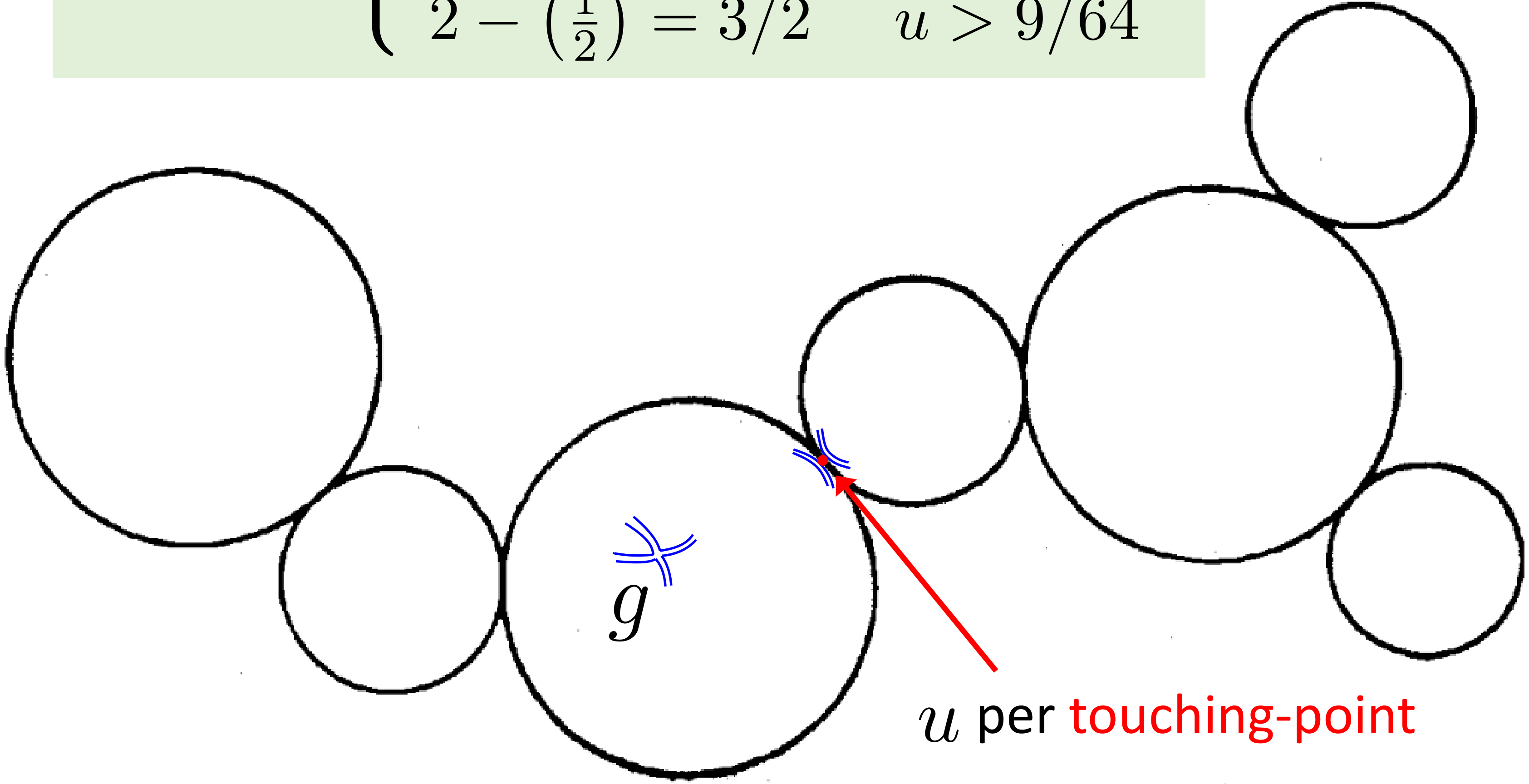
Received 9 January 1990

The non-perturbative formulation of 2-dimensional quantum gravity in terms of the large-N limit of matrix models is studied to include the effects of higher order curvature terms. This leads to matrix models whose potential contains a symmetry breaking term of the form $\mathrm{Tr}\,\phi A \phi A$, where A is a given fixed matrix. This is studied in $d=0$ dimensions and effectively induces additional terms of the form $(\mathrm{Tr}\,\phi^k)^2$ in the one matrix potential. An exact solution to leading order of the potential $V(\phi) = 1/2\,\mathrm{Tr}\,\phi^2 + g/N\,\mathrm{Tr}\,\phi^4 + g'/N^2\,(\mathrm{Tr}\,\phi^2)^2$ is presented leading to 3 phases: $\gamma = -1/2$ (smooth surfaces), $\gamma = 1/2$ (branched polymer) and $\gamma = 1/3$ (intermediate phase). Including a $\mathrm{Tr}\,\phi^6$ term in the potential gives rise to an additional phase with $\gamma = 1/4$. It is conjectured that for the general polynomial potential there are phases with $\gamma = 1/n$, $n = 2, 3, \ldots$. The $\gamma > 0$ phases may correspond to $c > 1$ matter coupled to 2-dimensional gravity.

$$Z_n(u) \propto \frac{g_c^{-n}}{n^{2-\gamma_S}}$$

$$2-\gamma_S = \begin{cases} 2-(-\frac{1}{2}) = 5/2 & u < 9/64 \\ 2-(\frac{1}{3}) = 5/3 & u = 9/64 \\ 2-(\frac{1}{2}) = 3/2 & u > 9/64 \end{cases}$$

Baby-universes



Touching random surfaces and Liouville gravity

Igor R. Klebanov
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 25 July 1994)

Large N matrix models modified by terms of the form $g(\mathrm{Tr}\Phi^n)^2$ generate random surfaces which touch at isolated points. Matrix model results indicate that, as g is increased to a special value g_t, the string susceptibility exponent suddenly jumps from its conventional value γ to $\gamma/(\gamma-1)$. We study this effect in Liouville gravity and attribute it to a change of the interaction term from $Oe^{\alpha_+\phi}$ for $g < g_t$ to $Oe^{\alpha_-\phi}$ for $g = g_t$ (α_+ and α_- are the two roots of the conformal invariance condition for the Liouville dressing of a matter operator O). Thus, the new critical behavior is explained by the unconventional branch of Liouville dressing in the action.

PACS number(s): 11.25.Pm, 11.25.Sq

$$\gamma_S \to \gamma'_S = \frac{\gamma_S}{\gamma_S - 1}$$

i.e., $\boxed{(1-\gamma_S)(1-\gamma'_S) = 1}$

Liouville quantum gravity

Kahane '85, D - Sheffield '09 '11
Rhodes Vargas '11, Miller Sheffield, Gwynne Holden Sun,
David Kupiainen Rhodes Vargas, ...

GFF h averaged over a circle of radius ε around z : $h_\varepsilon(z)$

$\gamma-$LQG: random measure for $0 \leq \gamma < 2$

$$\mu_\gamma(dz) := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz$$

describes the continuum limit of maps in the universality class of central charge $c = 1 - 6\left(\frac{\gamma}{2} - \frac{2}{\gamma}\right)^2$

$$\gamma_{\mathrm{S}} = 1 - \frac{4}{\gamma^2} < 0$$

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$$\gamma_{\mathrm{S}} = 1 - \frac{4}{\gamma^2} < 0$$

2nd solution $\gamma' = 4/\gamma > 2$

$$\gamma'_{\mathrm{S}} = 1 - \frac{4}{\gamma'^2} > 0$$

Liouville quantum duality

Kahane '85, D - Sheffield '09 '11

$\gamma' -$ LQG for $\gamma' \geq 2$

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma'^2/2} e^{\gamma' h_\varepsilon(z)} dz = 0$$

Liouville quantum duality

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D - Rhodes Sheffield Vargas '13

$\gamma = 2$

$$\mu_\gamma(dz) := \lim_{\varepsilon \to 0} \sqrt{\log(1/\varepsilon)}\ \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz$$

Liouville quantum duality $\gamma\gamma' = 4$

D - Sheffield '09 '11
Barral Jin Rhodes Vargas '13

$\gamma' -$ LQG for $\gamma' > 2$

$$\cancel{\lim_{\varepsilon \to 0} \varepsilon^{\gamma'^2/2} e^{\gamma' h_\varepsilon(z)} dz = 0}$$

Additional random set of *atoms* with localized quantum area η

$$\mu_{\gamma'}(dz) := \int_0^\infty \eta \mathcal{N}_{\gamma'}(dz, d\eta)$$

$\mathcal{N}_{\gamma'}$ Poisson random measure of intensity $\mu_\gamma(dz) d\eta / \eta^{1+1/\alpha}$

$$\alpha = 4/\gamma^2 > 1$$

Liouville quantum duality $\gamma\gamma' = 4$

D - Sheffield '09 '11
Barral Jin Rhodes Vargas '13

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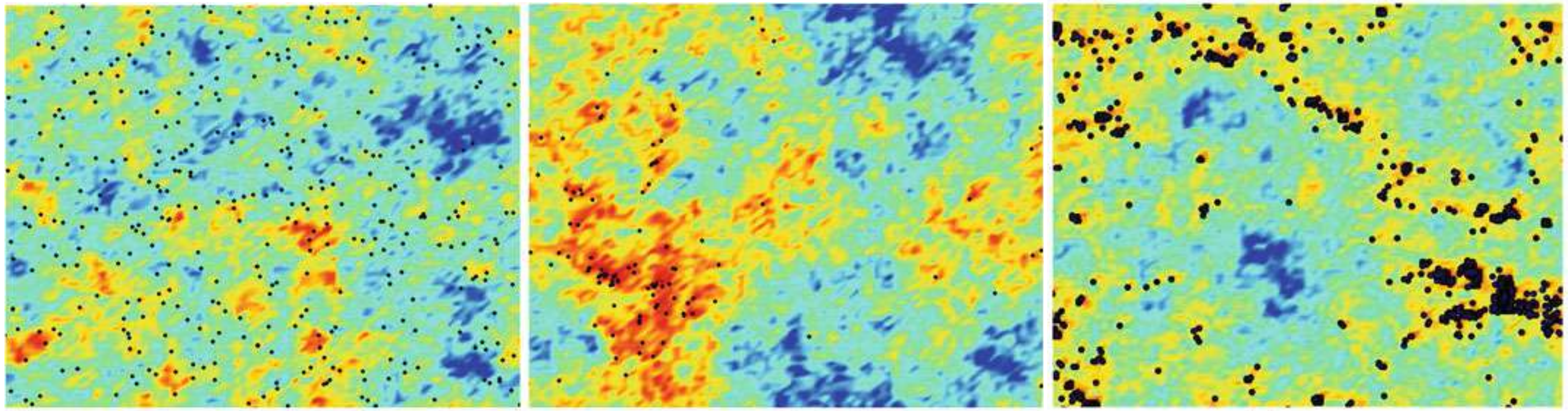

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D - Sheffield '09 '11
Barral Jin Rhodes Vargas '13

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$$\alpha = 4/\gamma^2 > 1$$

$$\boxed{\mathbb{E} \exp[-\lambda \, \mu_{\gamma'}(A)] = \mathbb{E} \exp \left[\Gamma(-\alpha') \lambda^{\alpha'} \mu_\gamma(A) \right]} \qquad \alpha\alpha' = 1$$

KPZ relation: Fractal set of Hausdorff dimension $D_{\mathrm{H}} = 2 - 2x$

Euclidean vs quantum scaling dimensions (x, Δ_γ)

D - Sheffield '09 '11
Rhodes Vargas '11
Barral Jin Rhodes Vargas '13

$$x = \frac{\gamma^2}{4}\Delta_\gamma^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta_\gamma \qquad \Delta_\gamma \geq 0$$

Duality $\gamma\gamma' = 4$

Dual scaling dimension $\Delta_{\gamma'}$

$$\gamma'(\Delta_{\gamma'} - 1) = \gamma(\Delta_\gamma - 1)$$

KPZ relation: Fractal set of Hausdorff dimension $D_{\mathrm{H}} = 2 - 2x$

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Duality $\gamma\gamma' = 4$

Dual scaling dimension $\Delta_{\gamma'}$

$$\gamma'(\Delta_{\gamma'} - 1) = \gamma(\Delta_\gamma - 1)$$

In terms of $\gamma_{\mathrm{S}} = 1 - \frac{4}{\gamma^2}$

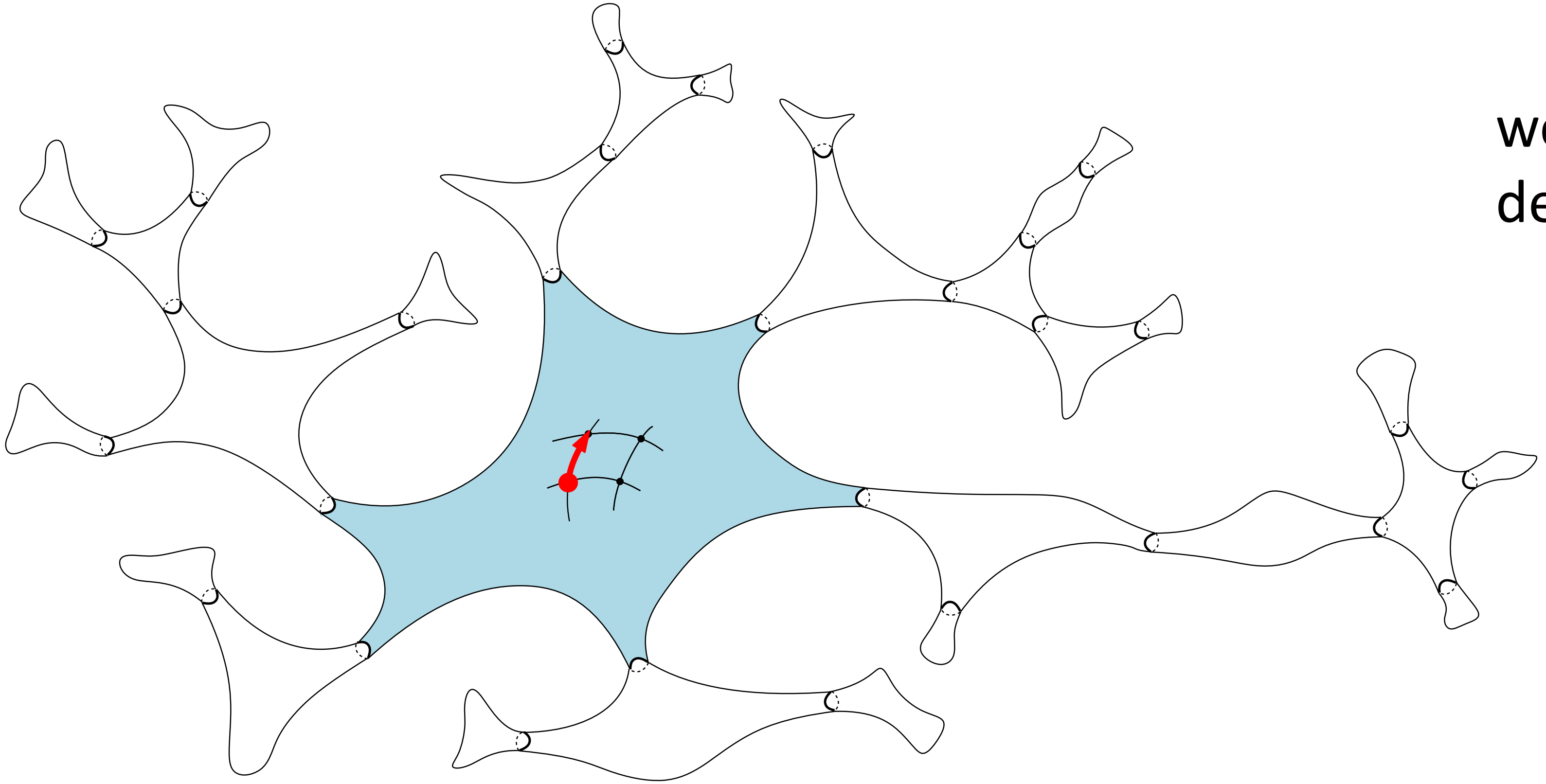
$$\Delta_{\gamma'} = \frac{\Delta_\gamma - \gamma_{\mathrm{S}}}{1 - \gamma_{\mathrm{S}}}$$

$$(1 - \gamma_{\mathrm{S}})(1 - \gamma'_{\mathrm{S}}) = 1$$

$$\Delta_\gamma = \frac{\Delta_{\gamma'} - \gamma'_{\mathrm{S}}}{1 - \gamma'_{\mathrm{S}}}$$

Analytic combinatorics

Flajolet Sedgewick '09



weight ϕ_i per block with i descending sub-blocks

$$\phi(\tau) = \sum_{i \geq 0} \phi_i \tau^i$$

y_n trees with n blocks

$$y(z) = \sum_{n \geq 1} y_n z^n$$

Substitution relation

$$y(z) = z\,\phi(y(z))$$

$$y(z) = z\,\phi(y(z))$$

Tree critical point where the mapping $z \mapsto y$ is no longer invertible

$$1 = z_c\,\phi'(y(z_c))$$

$\big(z_c, y_c = y(z_c)\big)$ determined by $\phi(y_c) = y_c\,\phi'(y_c) \quad , \quad z_c = \dfrac{y_c}{\phi(y_c)}$ (CRIT)

Singularity of $\phi(\tau)$

$$\phi_i \propto \tau_\phi^{-i}/i^{1+\alpha} \qquad 1 < \alpha < 2$$

$$\phi(\tau) = \phi(\tau_\phi) - (\tau_\phi - \tau)\phi'(\tau_\phi) + K(\tau_\phi - \tau)^\alpha + \cdots$$

Asymptotics for y_n

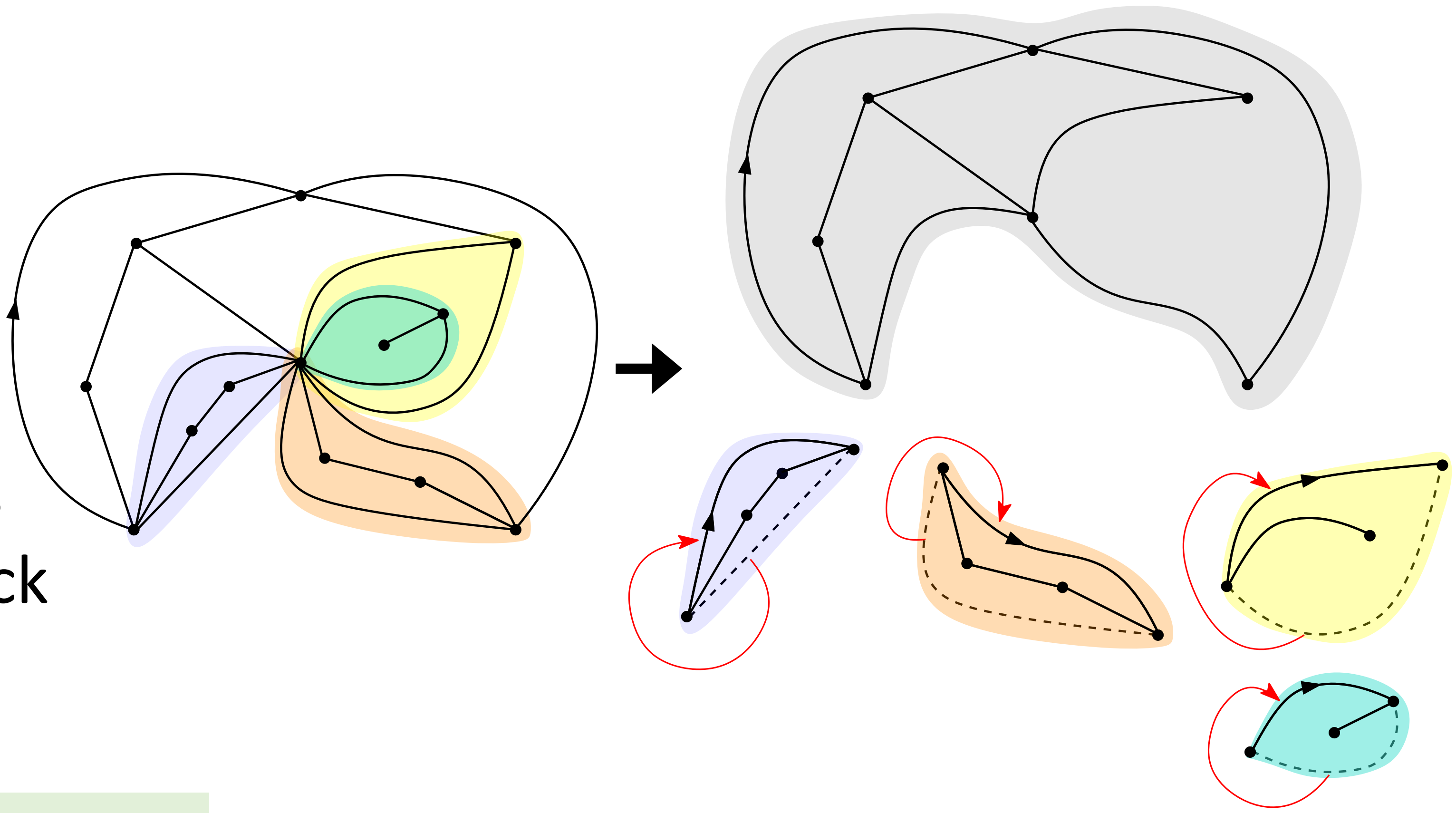
	subcritical	critical	supercritical
$1 < \alpha < 2$	$\dfrac{z_\phi^{-n}}{n^{1+\alpha}}$	$\dfrac{z_c^{-n}}{n^{1+1/\alpha}}$	$\dfrac{z_c^{-n}}{n^{3/2}}$
$\alpha = 2 \ (\times \log)$	$\dfrac{z_\phi^{-n}}{n^3}$	$\dfrac{z_c^{-n}}{n^{3/2}(\log n)^{1/2}}$	$\dfrac{z_c^{-n}}{n^{3/2}}$

$$\phi(\tau) = \phi(\tau_\phi) - (\tau_\phi - \tau)\phi'(\tau_\phi) - K(\tau_\phi - \tau)^2 \log(\tau_\phi - \tau) + \cdots$$

Block-weighted maps

$$M_u(g) = \sum_{n\geq 0} g^n\, m_n^{(u)}$$

Quadrangulations with n faces and a weight u per **simple** block



$$\boxed{M_u(g) = 1 + u\left[B\left(g\, M_u(g)^2\right) - 1\right]}$$

$$B(t) = \sum_{j\geq 0} b_j\, t^j$$

simple quadrangulations with j faces

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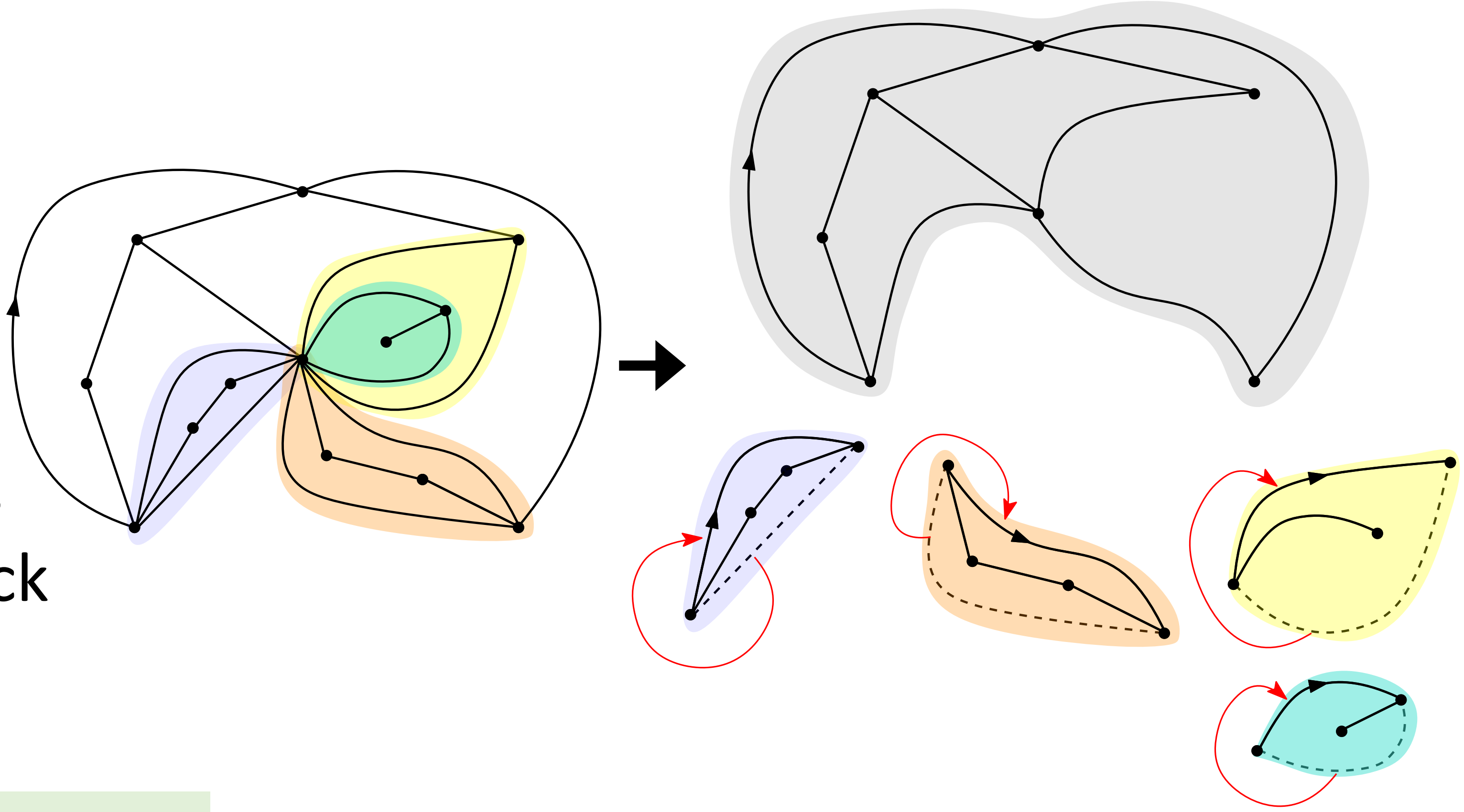

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simple quadrangulations with j faces

$$y(z) = z\,\phi(y(z))$$

$$z = \sqrt{g}\ , \quad y(z) = zM_u(z^2)$$

$$\phi(\tau) = 1 + u\left[B(\tau^2) - 1\right]$$

Block-weighted maps

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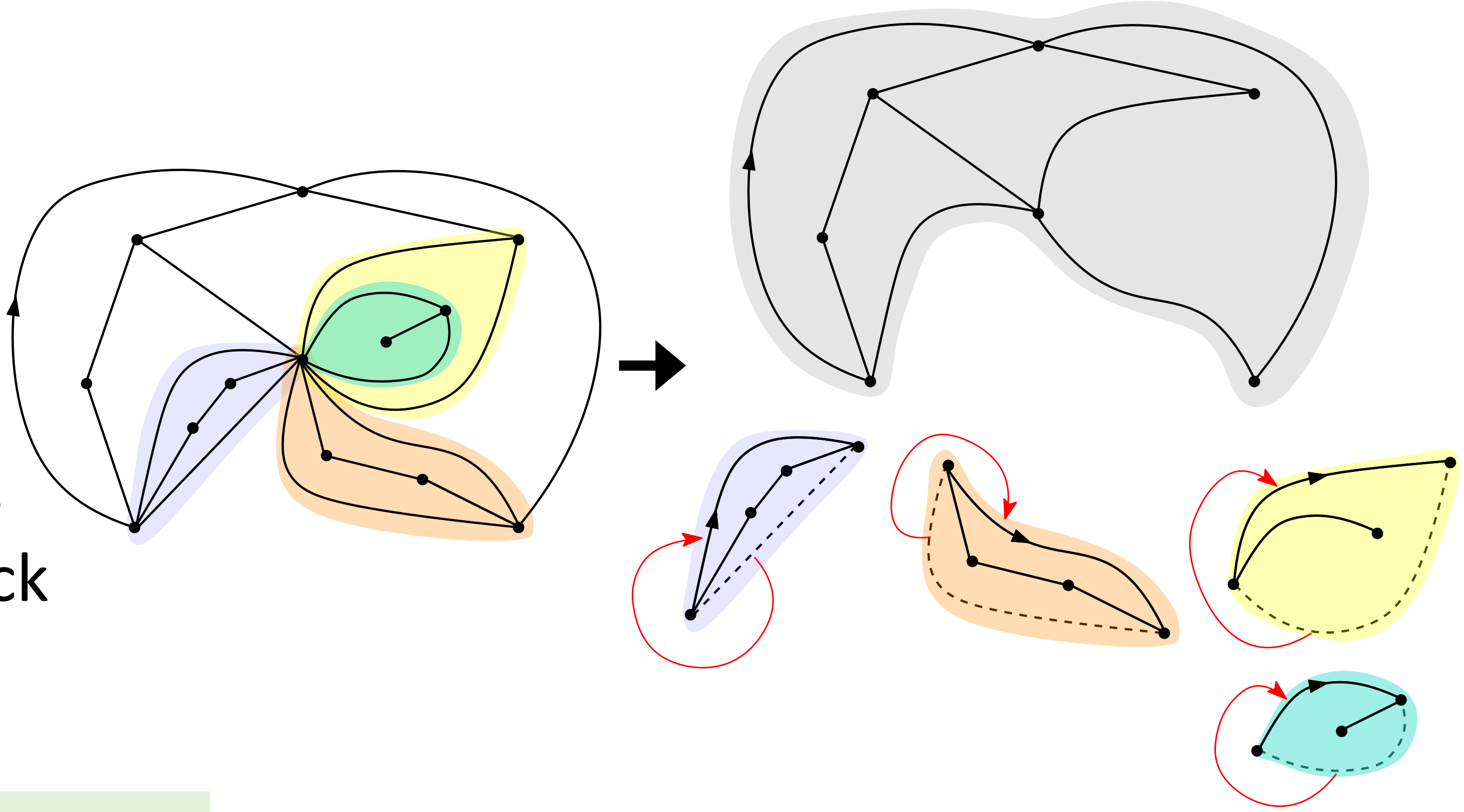
Quadrangulations with n faces
and a weight u per **simple** block



$$M_1(g) = B\left(g\, M_1(g)^2\right)$$

$$B(t) = \sum_{j\geq 0} b_j\, t^j$$

$$\boxed{M_u(g) = 1 + u\left[B\left(g\, M_u(g)^2\right) - 1\right]}$$

simple quadrangulations
with j faces

$$z = \sqrt{g}\ , \quad y(z) = z M_u(z^2)$$

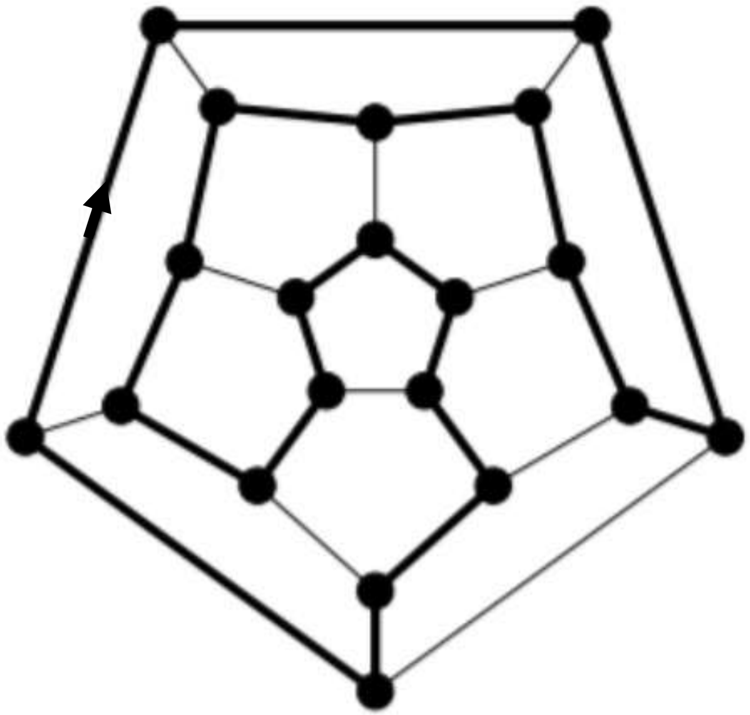
$$\boxed{y(z) = z\,\phi(y(z))}$$

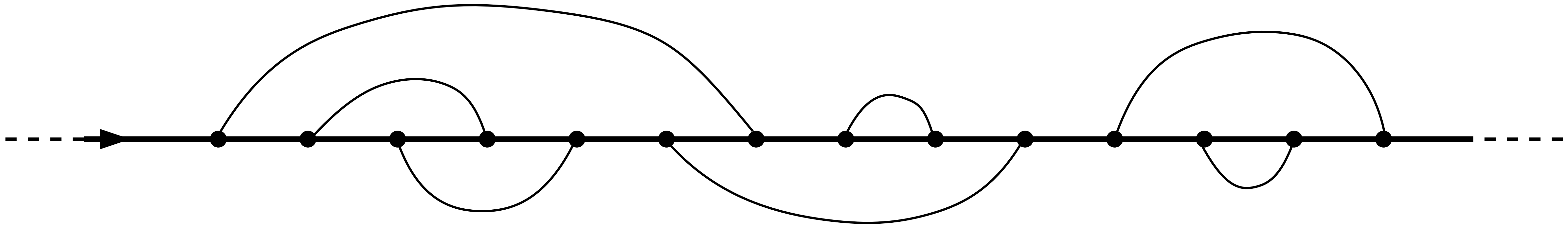
$$\phi(\tau) = 1 + u\left[B(\tau^2) - 1\right]$$

Other examples: maps equipped with a **Hamiltonian cycle**

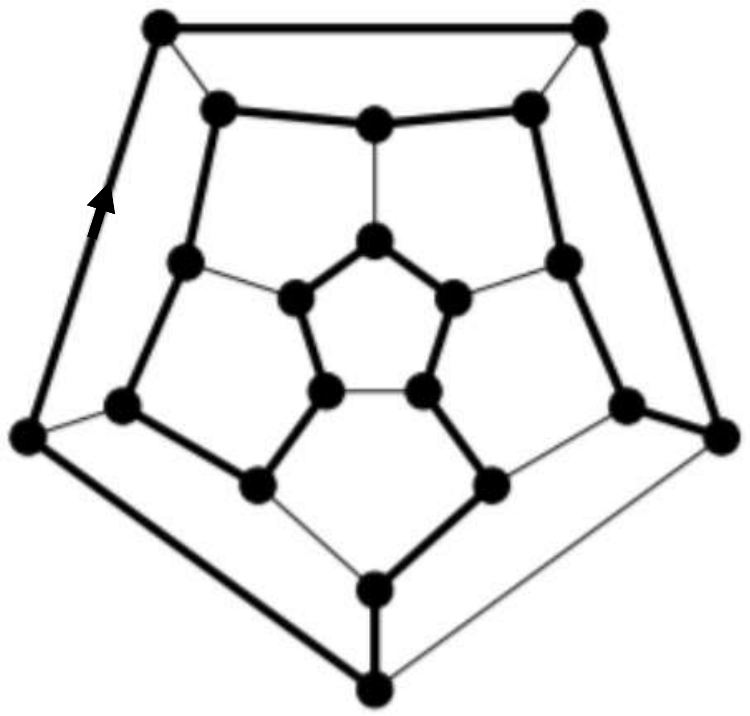
$$c = -2$$

Cubic maps: system of arches





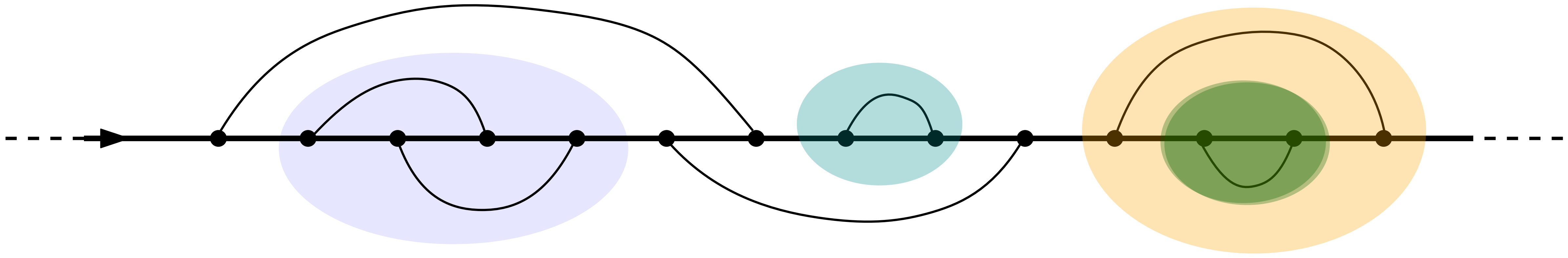
Other examples: maps equipped with a **Hamiltonian cycle**



$$M_u(g) = 1 + u\left[B\left(g\, M_u(g)^2\right) - 1\right]$$

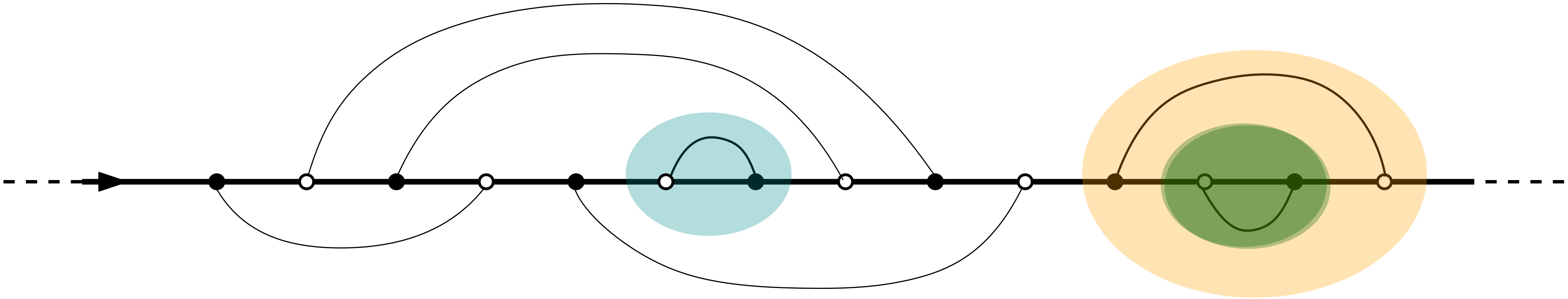
Cubic maps: system of arches

decomposed into **irreducible** blocks



Bicubic maps : system of arches decomposed into irreducible blocks

$$c = -1$$



different universality class!

Guitter Kristjansen Nielsen '99

Di Francesco - D - Golinelli Guitter '23

D - Golinelli Guitter '23

In general, we know the case $u = 1$

$$m_n^{(1)} \propto \frac{(g_1)^{-n}}{n^{2-\gamma_S}} \qquad 2 < 2 - \gamma_S < 3$$

$$m_n^{(u)} \propto \frac{g_{cr}(u)^{-n}}{n^{1+\alpha}} \quad \text{for} \quad u < u_{cr}$$

$$m_n^{(u)} \propto \frac{g_c(u)^{-n}}{n^{1+1/\alpha}} \quad \text{for} \quad u = u_{cr}$$

$$m_n^{(u)} \propto \frac{g_c(u)^{-n}}{n^{3/2}} \quad \text{for} \quad u > u_{cr}$$

$$2 - \gamma_S = 1 + \alpha$$

$$2 - \gamma_S' = 1 + 1/\alpha$$

$$\downarrow$$

$$\boxed{(1 - \gamma_S)(1 - \gamma_S') = 1}$$

Quadrangulations with weight u per simple block

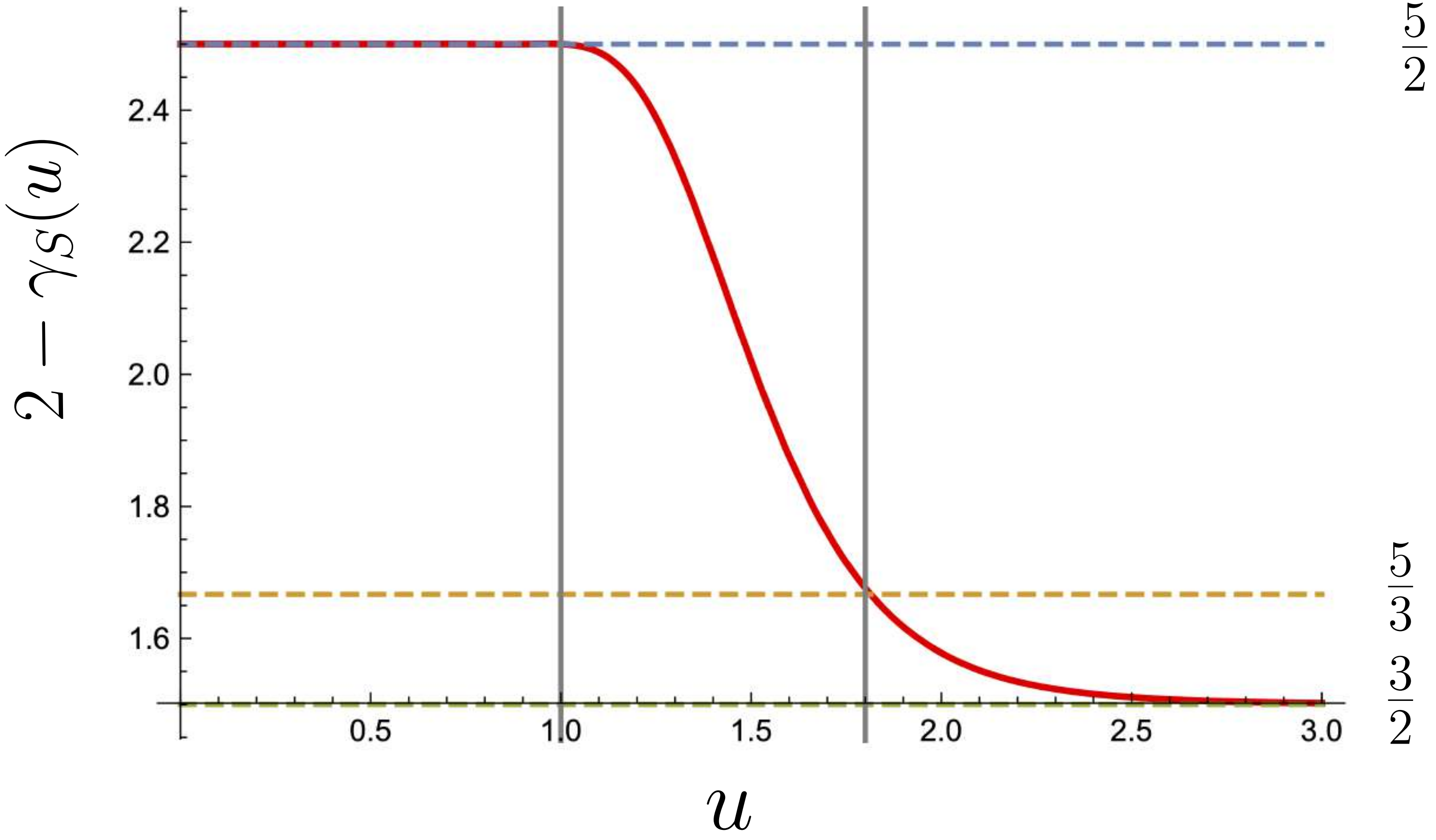
$$M_1(g) = \sum_{n\geq 0} \underbrace{2\frac{3^n}{n+2}\mathrm{Cat}(n)}_{m_n^{(1)}}\, g^n = \frac{18g - 1 + (1-12g)^{3/2}}{54g^2}$$

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$$

$$c = 0$$

$$\gamma = \sqrt{\frac{8}{3}}$$

$$u = 1 \qquad u = u_{\mathrm{cr}} = \frac{9}{5}$$

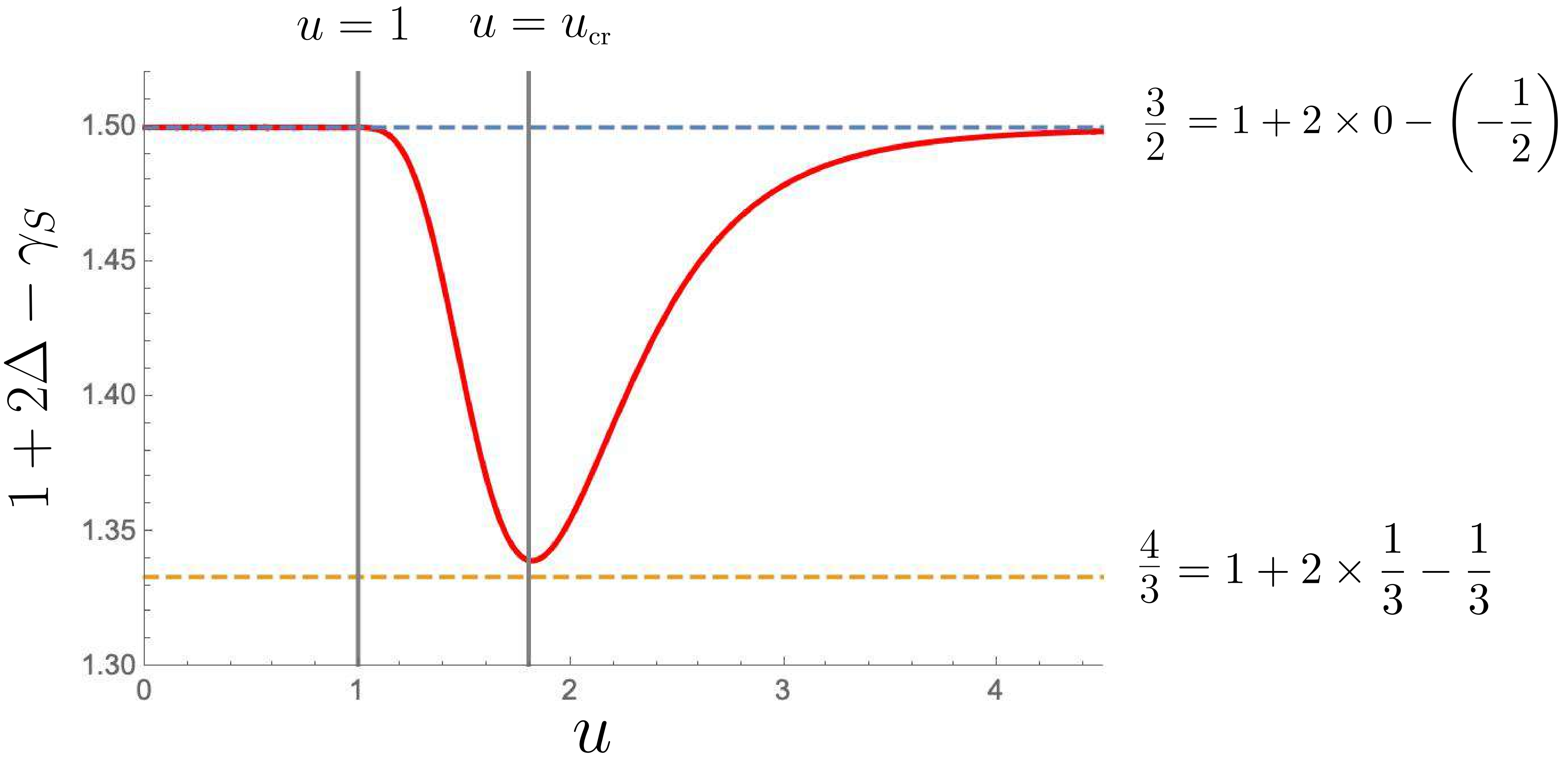


$$\gamma_{\mathrm{S}} = -\frac{1}{2}$$

$$\gamma_{\mathrm{S}}' = \frac{1}{3}$$

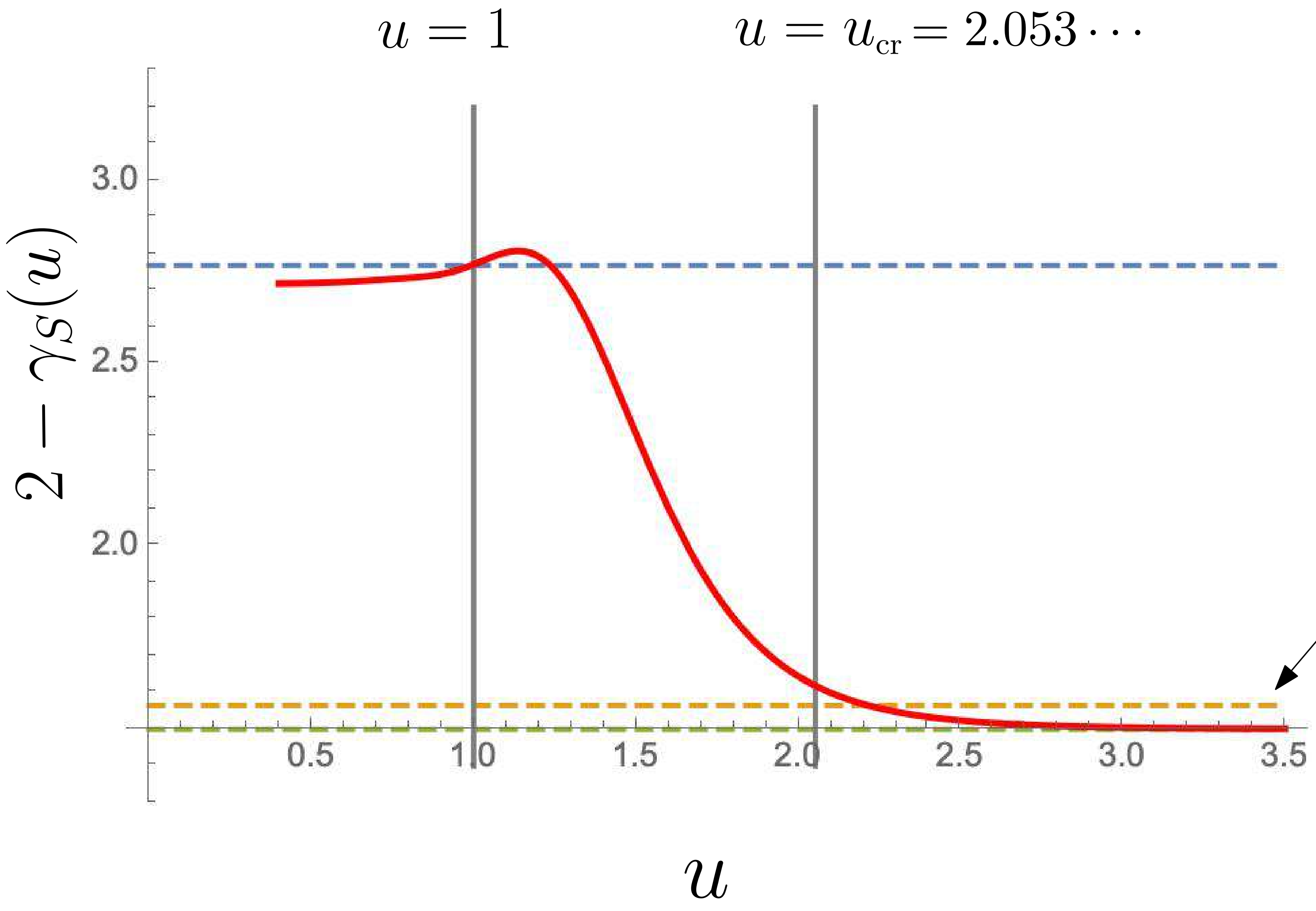
Rooted quadrangulations with a second marked edge

$$\tilde{s}_n^{(1)} = (2n-1)m_n^{(1)} \propto \frac{g_1^{-n}}{n^{1+2\Delta-\gamma_S}}$$ with $\Delta = 0$ hence $\Delta' = \frac{\gamma_S}{\gamma_S - 1} = \gamma_S'$



Bicubic maps + Hamiltonian cycle with weight u per irreducible block

Guitter Kristjansen Nielsen '99



$$\frac{13+\sqrt{13}}{6} = 2 - \gamma_{\mathrm{S}}$$

$$\boxed{\gamma = \frac{\sqrt{13}-1}{\sqrt{3}}}$$

$$\frac{13-\sqrt{13}}{6} = 2 - \gamma_{\mathrm{S}}'$$

$$\frac{3}{2}$$

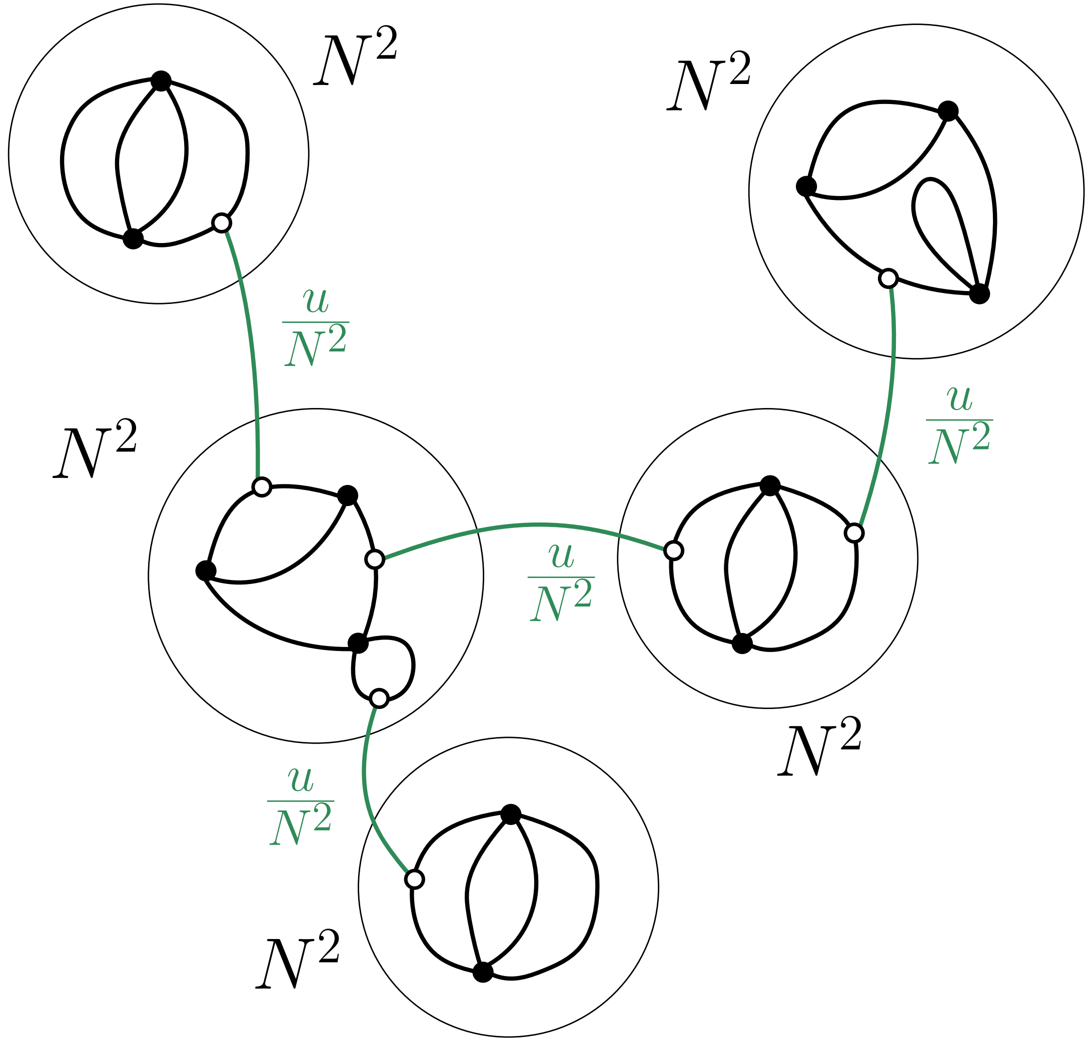
Matrix integral with potential $V(\Phi) = \frac{1}{2}\operatorname{Tr}\Phi^2 - \frac{g}{4N}\operatorname{Tr}\Phi^4 - \frac{u}{4N^2}(\operatorname{Tr}\Phi^2)^2$

Integrate over $N \times N$ Hermitian matrices $\int d\Phi \exp(-V(\Phi)) \underset{N\to\infty}{\sim} e^{N^2 Z(g,u)}$

$Z(g,u)$ g.f. for (rooted) trees

made of 4-regular planar maps

Das Dhar Sengupta Wadia '90



Matrix integral with potential $V(\Phi) = \frac{1}{2}\,\mathrm{Tr}\,\Phi^2 - \frac{g}{4N}\,\mathrm{Tr}\,\Phi^4 - \frac{u}{4N^2}(\mathrm{Tr}\,\Phi^2)^2$

Integrate over $N \times N$ Hermitian matrices $\int d\Phi \exp(-V(\Phi)) \underset{N\to\infty}{\sim} e^{N^2 Z(g,u)}$

Substitution relation for rooted g.f. $Z^\bullet(g,u) = 1 + \left(4g\frac{\partial}{\partial g} + 4u\frac{\partial}{\partial u}\right) Z(g,u)$

$$Z^\bullet(g,u) = \frac{1}{1 - u\,Z^\bullet(g,u)} M\left(\frac{g}{(1 - u\,Z^\bullet(g,u))^2}\right)$$

with $M(g) = \frac{18g - 1 + (1-12g)^{3/2}}{54g^2}$

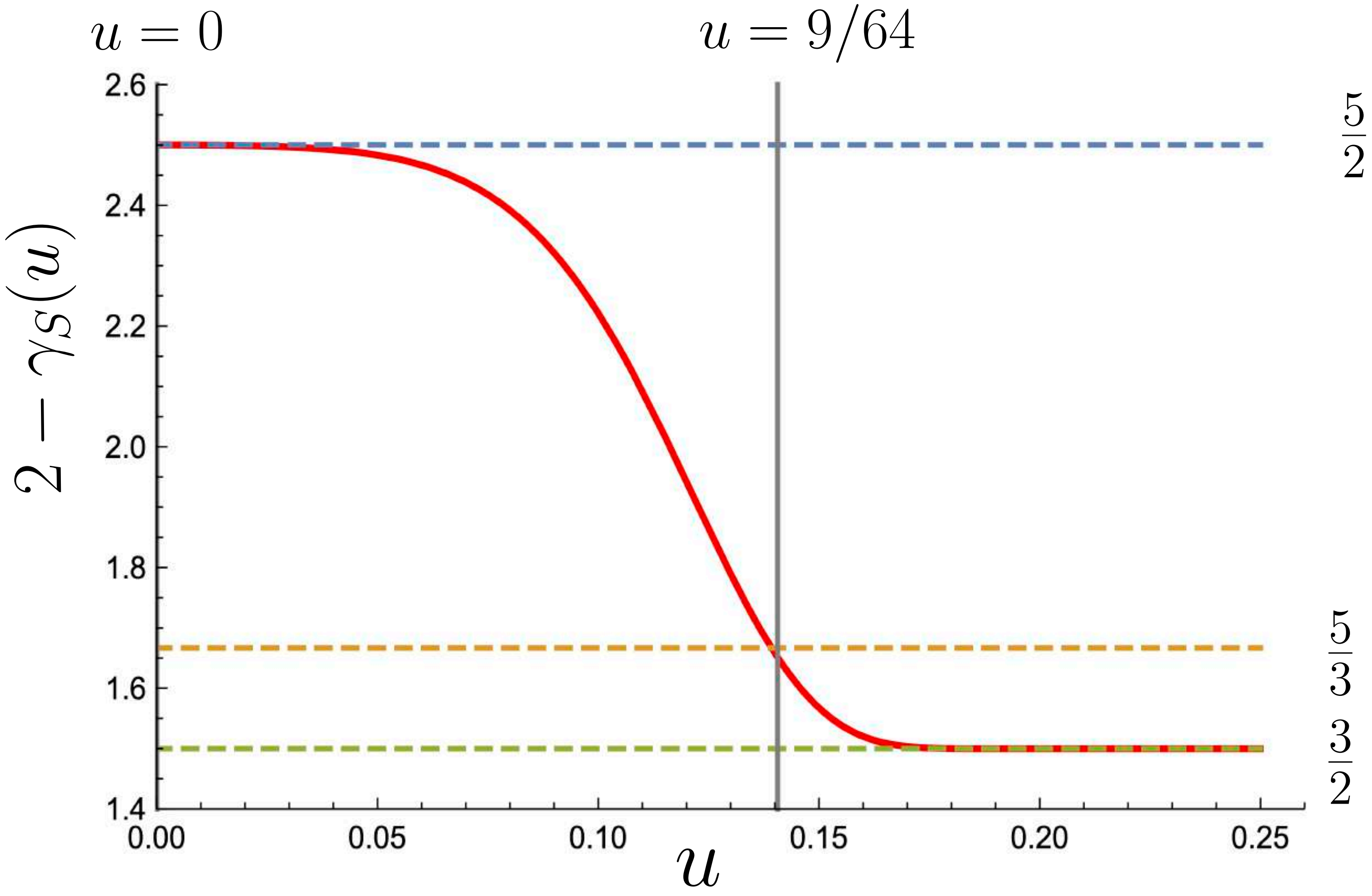
g.f. of rooted planar 4-regular maps

Set $u = \sqrt{g}\,v$, $z = \sqrt{g}$, $y(z) = \frac{z}{1 - u\,Z^\bullet(z^2,u)}$

then $y(z) = z\,\phi(y(z))$ with $\phi(\tau) = \frac{1}{1 - v\,\tau M(\tau^2)}$

$$u_{\mathrm{cr}} = \frac{9}{64}$$

$$D = \frac{1}{2^{2/3}}$$



$u = 0$
$u = 9/64$
2.6
2.4
2.2
2.0
1.8
1.6
1.4
$2 - \gamma_S(u)$
$\frac{5}{2}$
$\frac{5}{3}$
$\frac{3}{2}$
0.00
0.05
0.10
0.15
0.20
0.25
u

Simply/Doubly rooted maps: Scaling regime

$$\Pr(X_n = \lfloor x\, n^{\frac{1}{\alpha}} \rfloor) \sim \frac{1}{n}\tau(x)\ , \quad \tau(x) = \frac{\alpha^3 \Gamma\left(2 - \frac{1}{\alpha}\right)}{\Gamma(2-\alpha)} \frac{1}{(D\, x)^{1+\alpha}} \mathcal{S}_{\frac{1}{\alpha}}(D\, x)\ .$$

$$n^{\frac{1}{\alpha}} \Pr(X_n^{(2)} = \lfloor x\, n^{\frac{1}{\alpha}} \rfloor) \underset{n\to\infty}{\to} \sigma(x)\ , \quad \sigma(x) = \frac{\alpha \Gamma\left(\frac{1}{\alpha}\right)}{\Gamma(2-\alpha)} \frac{D}{(D\, x)^{\alpha}} \mathcal{S}_{\frac{1}{\alpha}}(D\, x)$$

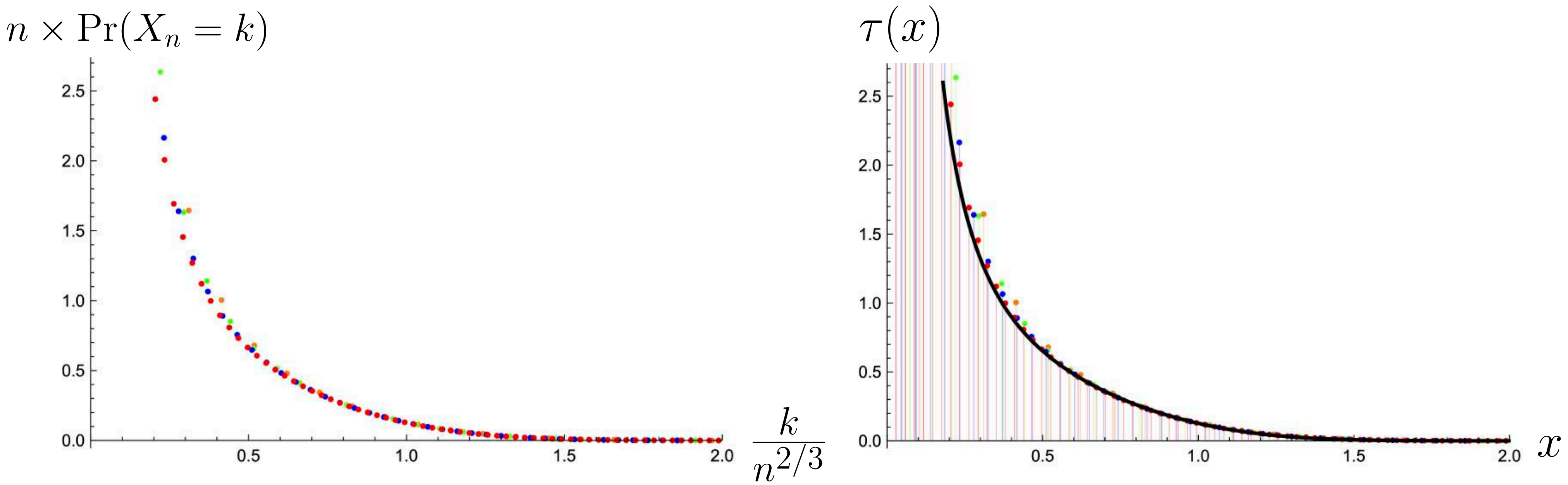
Wright function $W_{\lambda,\mu}$

$$\mathcal{S}_{\frac{1}{\alpha}}(x) = W_{-\frac{1}{\alpha},0}(-x), \quad W_{\lambda,\mu}(z) := \sum_{m\geq 0} \frac{z^m}{m!\,\Gamma(\lambda m + \mu)}\ .$$

$\alpha = 3/2$

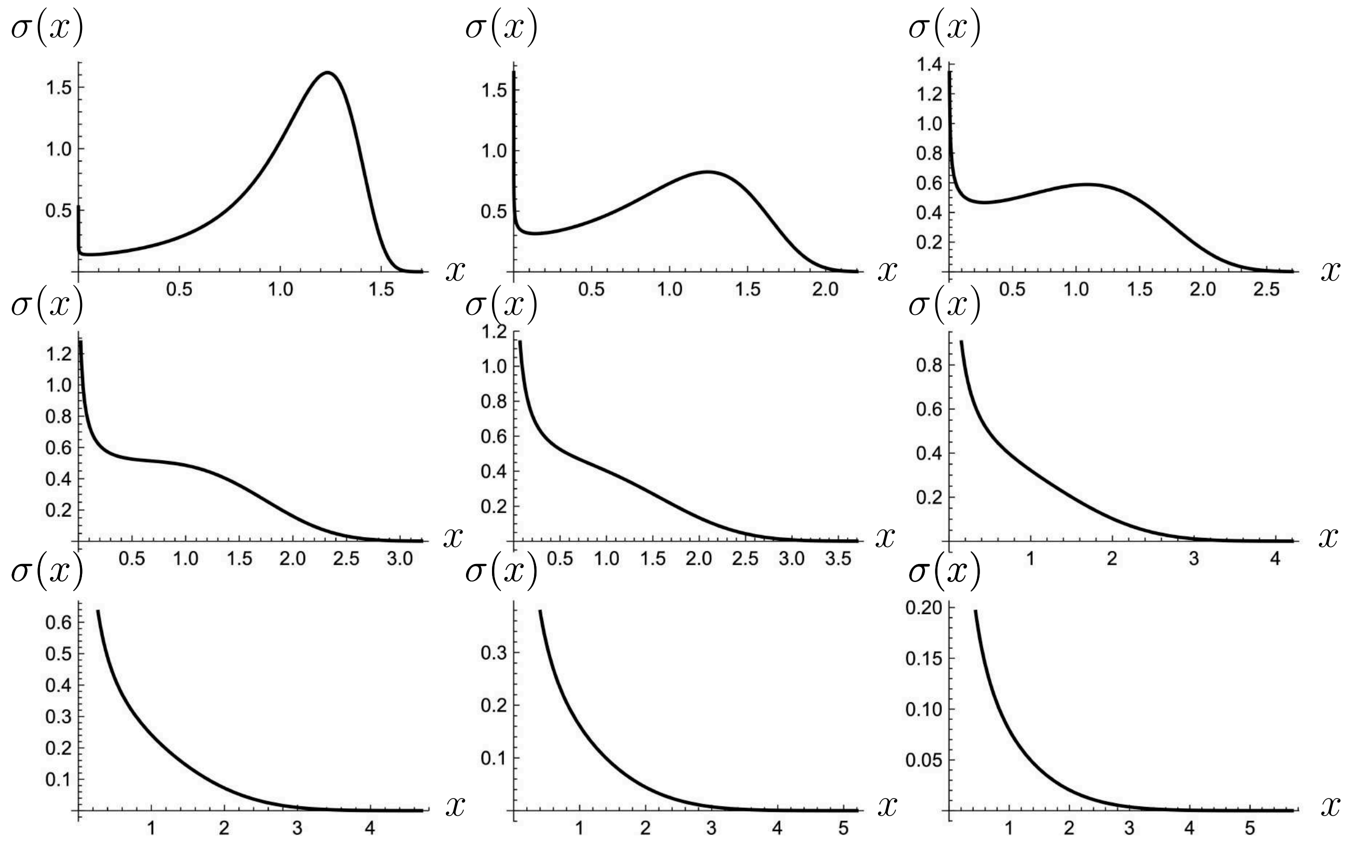
$$\mathcal{S}_{\frac{1}{\alpha}}(x) = 2\frac{x}{3^{2/3}} e^{-\frac{2}{27}x^3} \left(\frac{x}{3^{2/3}} \mathrm{Ai}\left(\frac{x^2}{3^{4/3}}\right) - \mathrm{Ai}'\left(\frac{x^2}{3^{4/3}}\right)\right) = \frac{x}{3^{2/3}} \mathcal{A}\left(\frac{x}{3^{2/3}}\right)$$

Block-weighted quadrangulations



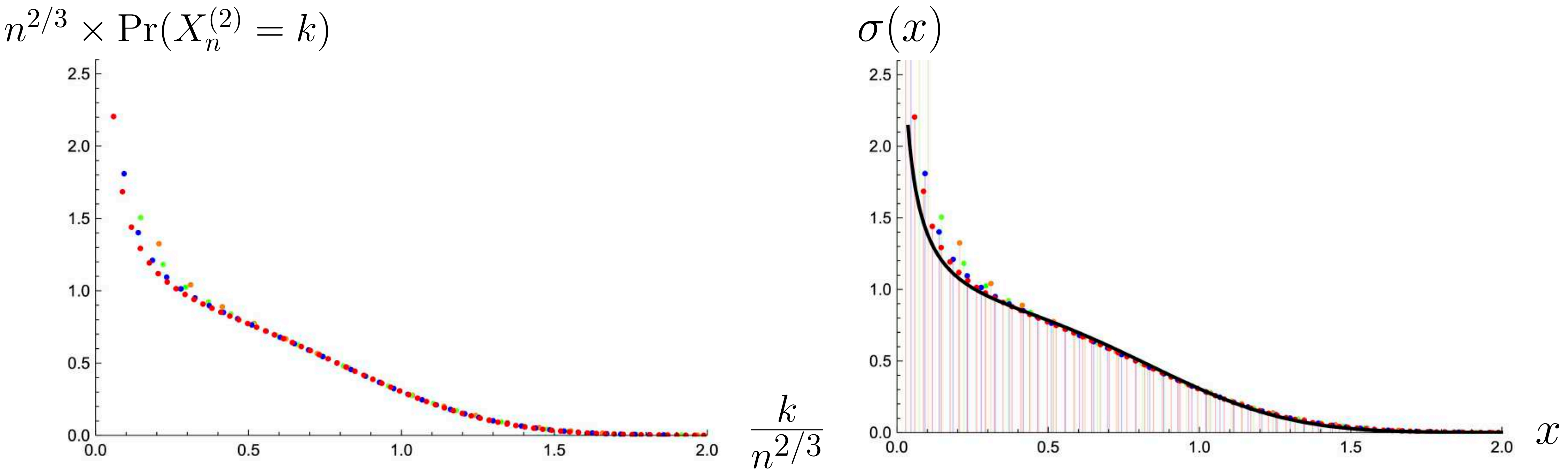
Left: the properly rescaled probability $\Pr(X_n = k)$ for $n = 30, 50, 100, 200$ (orange, green, blue, red) in the case of block-weighted quadrangulations. The points all fall on a well defined curve. Right: the comparaison with the scaling function $\tau(x)$ of (3.15) with $\alpha = 3/2$ and $D = 3/2^{2/3}$. Vertical bars under the discrete points have been added for better visualization.

Doubly rooted maps: Scaling regime



Plot of $\sigma(x)$ versus x for $D = 1$ and $\alpha = 1 + \frac{i}{10}$, $i = 1, 2, \ldots, 9$ (from upper left to bottom right, line by line). The central plot corresponds to the $\alpha = 3/2$ case.

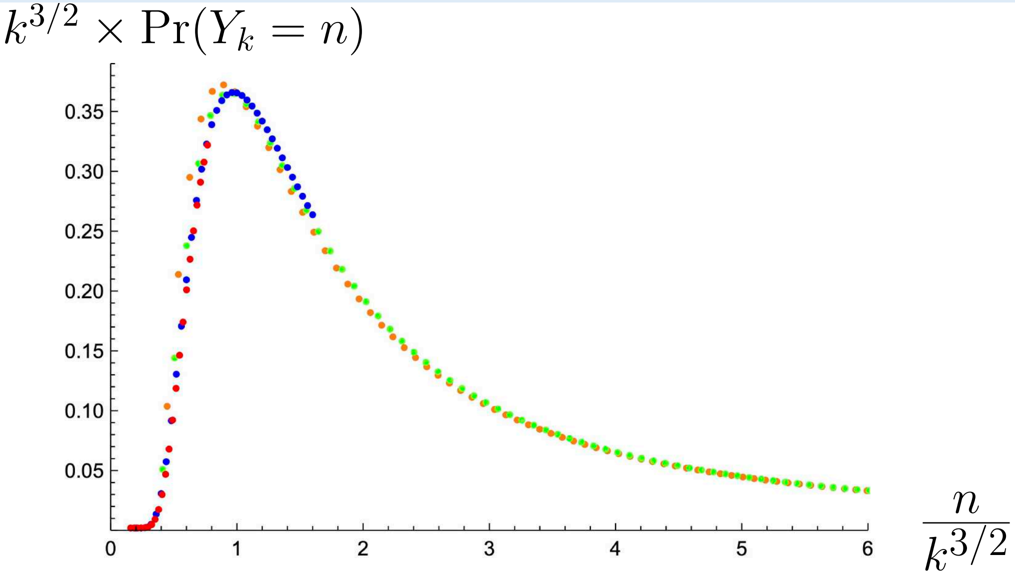
Block-weighted quadrangulations

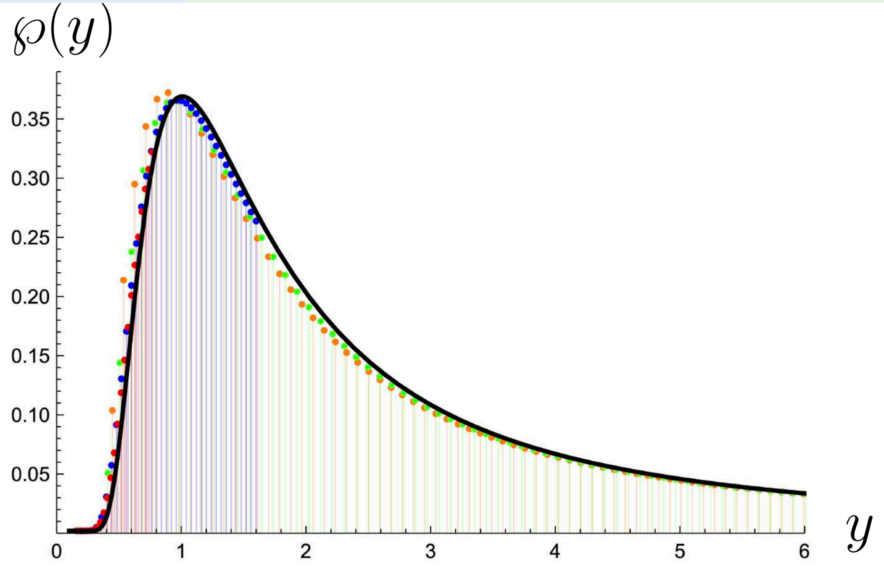


Left: the properly rescaled probability $\Pr(X_n^{(2)} = k)$ for $n = 30, 50, 100, 200$ (orange, green, blue, red) in the case of block-weighted quadrangulations. Right: the comparaison with the probability density $\sigma(x)$ of (3.22) with $\alpha = 3/2$ and $D = 3/2^{2/3}$.

Distribution of total size at *fixed* root block size

$$k^{\alpha}\mathrm{Pr}(Y_k = \lfloor y\, k^{\alpha} \rfloor) \underset{k\to\infty}{\to} \wp(y)\ , \quad \wp(y) = \frac{1}{y}\mathcal{S}_{\frac{1}{\alpha}}\left(\frac{D}{y^{\frac{1}{\alpha}}}\right) \quad \int_0^{\infty} dy\, \wp(y) e^{-\lambda\, y} = e^{-D\, \lambda^{\frac{1}{\alpha}}}$$

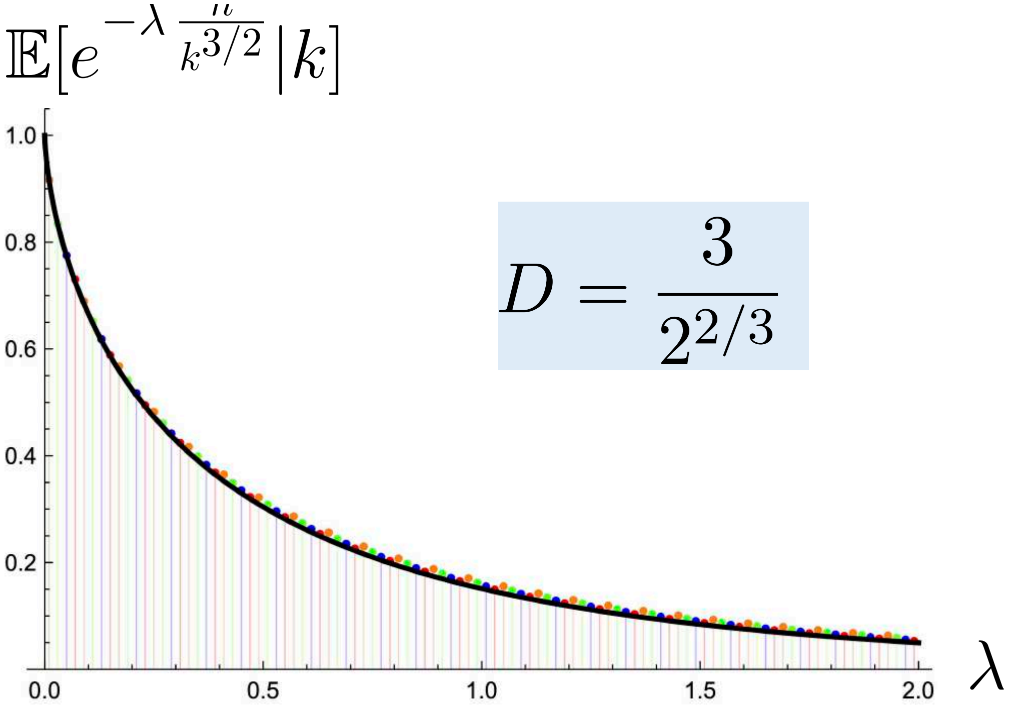




: Left: the properly rescaled probability $\mathrm{Pr}(Y_k = n)$ for $k = 5, 10, 25, 40$ (orange, green, blue, red) and n limited to 200 in the case of block-weighted quadrangulations. These points form a well defined scaling curve. Right: the comparison with the probability density $\wp(y)$ for $\alpha = 3/2$ and $D = 3/2^{2/3}$

Laplace transform

$$\mathbb{E}[e^{-\lambda \frac{n}{k^\alpha}} | k] \underset{k\to\infty}{\longrightarrow} e^{-D\lambda^{\frac{1}{\alpha}}}$$



Dual Liouville quantum measures

$$\mu_{\gamma'}(dz) := \left(\frac{D}{\alpha\Gamma(1-1/\alpha)}\right)^\alpha \int_0^\infty \eta\, \mathcal{N}_{\gamma'}(dz, d\eta), \;\; \gamma' > 2$$

$$\mathbb{E}\left[\exp\left(-u\, \frac{\mu_{\gamma'}(\mathcal{A})}{\mu_\gamma^\alpha(\mathcal{A})}\right) \Big| \mu_\gamma\right] = \exp\left(-D\, u^{1/\alpha}\right)$$

Universal ratio of **dual** p-rooted fixed-size partition functions

Combinatorics

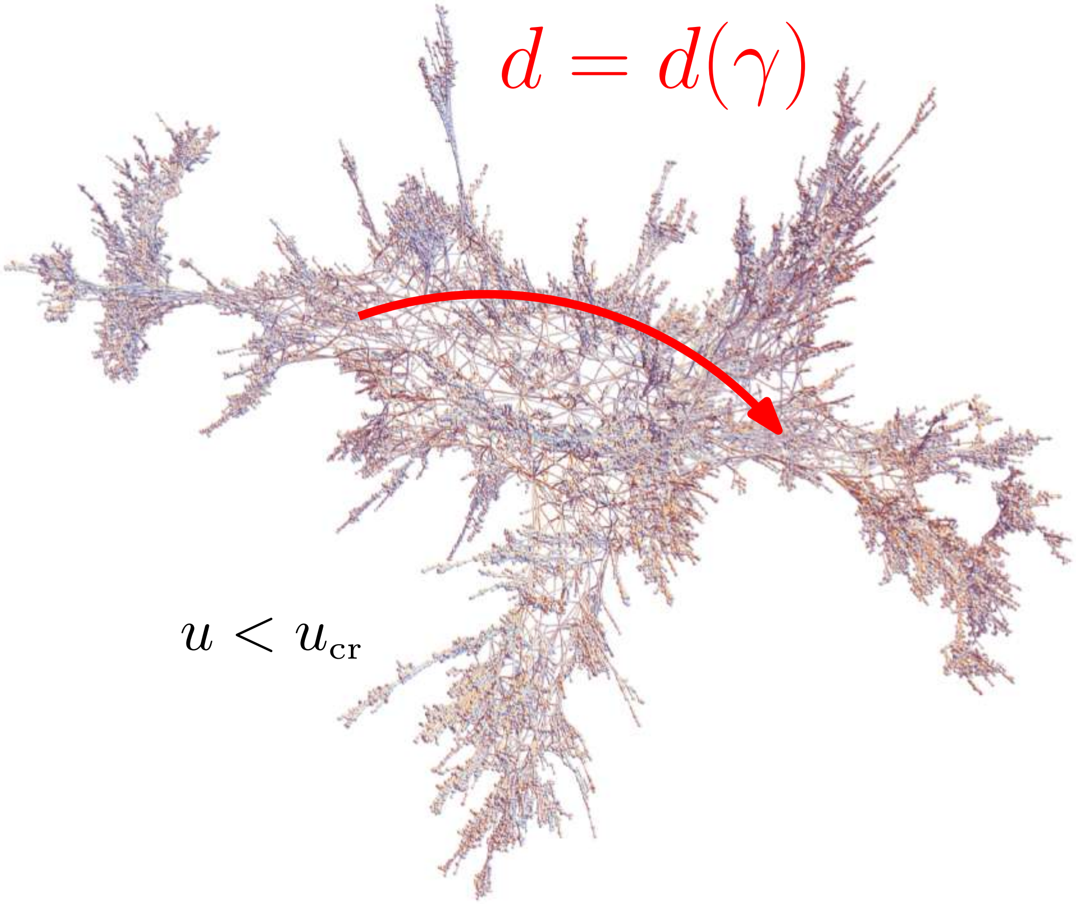
$$\frac{g_{\mathrm{cr}}^n \, \widetilde{Z}_n^{(p)}}{t_{\mathrm{cr}}^k \, Z_k^{(p)}} \underset{n,k\to\infty}{\sim} \mathcal{R}^{(p)}(n,k) := \frac{k}{n}\left(\frac{n^{1/\alpha}}{D\,k}\right)^{p-1-\alpha} \frac{\Gamma(p-1-\alpha)}{\Gamma\left(\frac{p-1-\alpha}{\alpha}\right)}$$

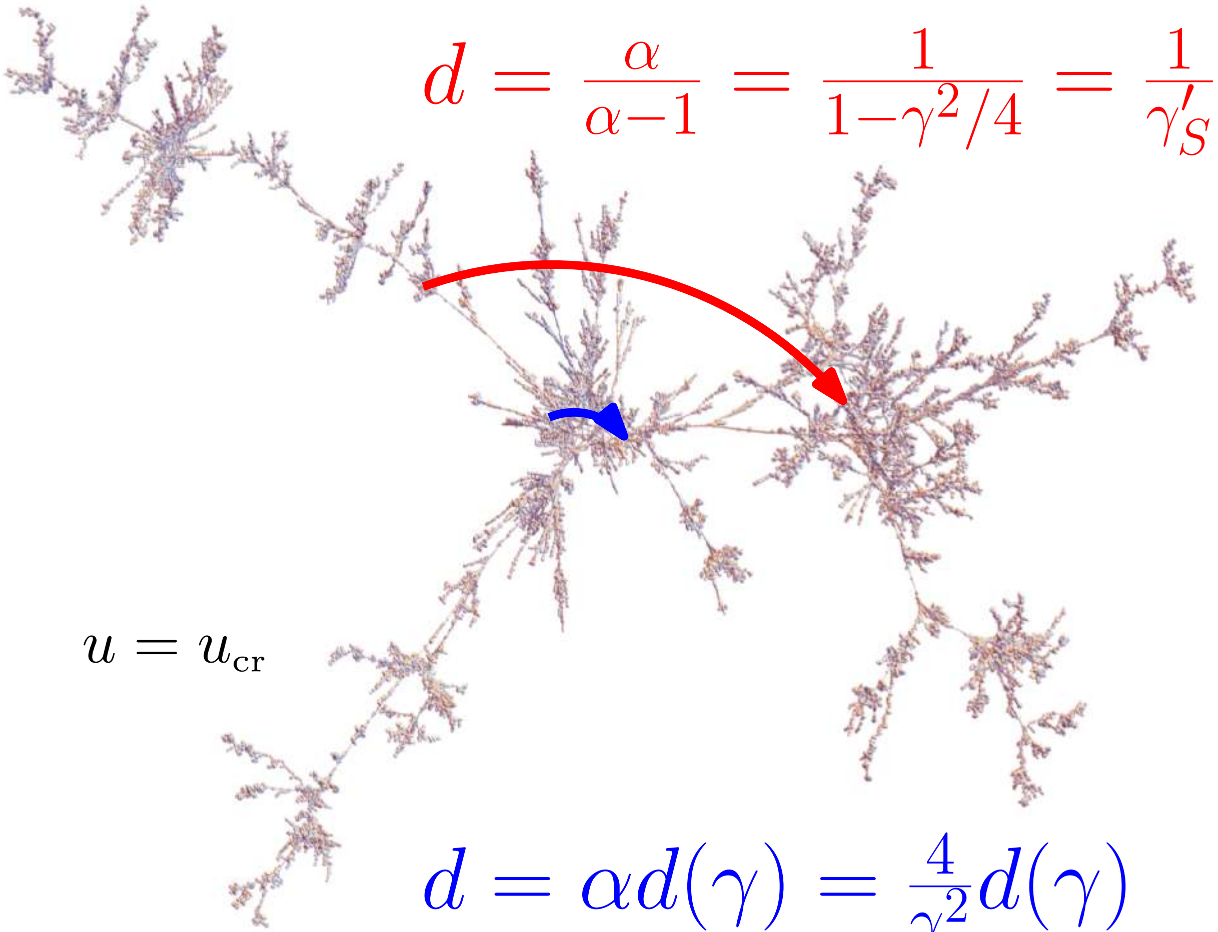
Dual Liouville Partition Functions *(David Kupiainen Rhodes Vargas '16)*

$$\frac{\widetilde{\mathcal{Z}}_{\gamma',\hat{g}}^{(z_i\gamma)_i}(A')}{\mathcal{Z}_{\gamma,\hat{g}}^{(z_i\gamma)_i}(A)} = \frac{\Gamma(p-1-\alpha)}{\Gamma((p-1-\alpha)/\alpha)} \frac{A}{A'} \left(\frac{A'^{\frac{1}{\alpha}}}{DA}\right)^{p-1-\alpha}$$

Distances

$$\text{distance} \sim n^{1/d}$$

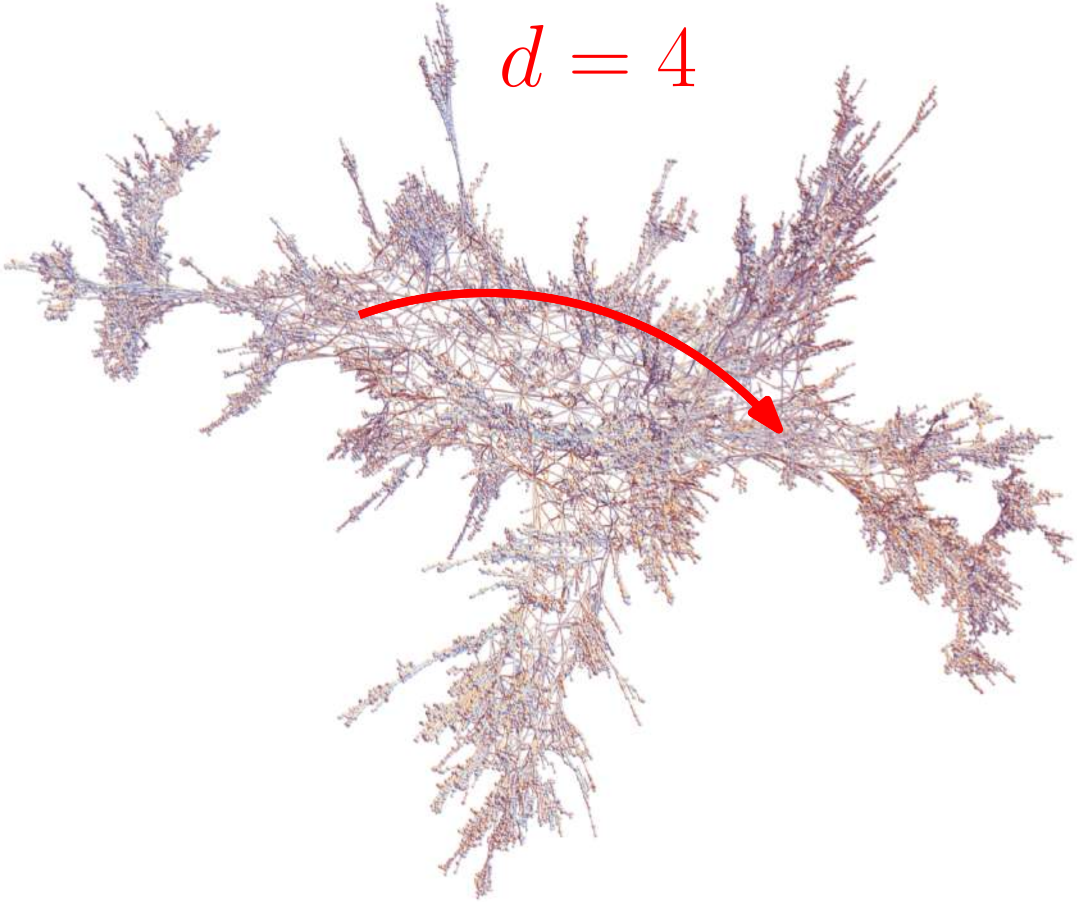


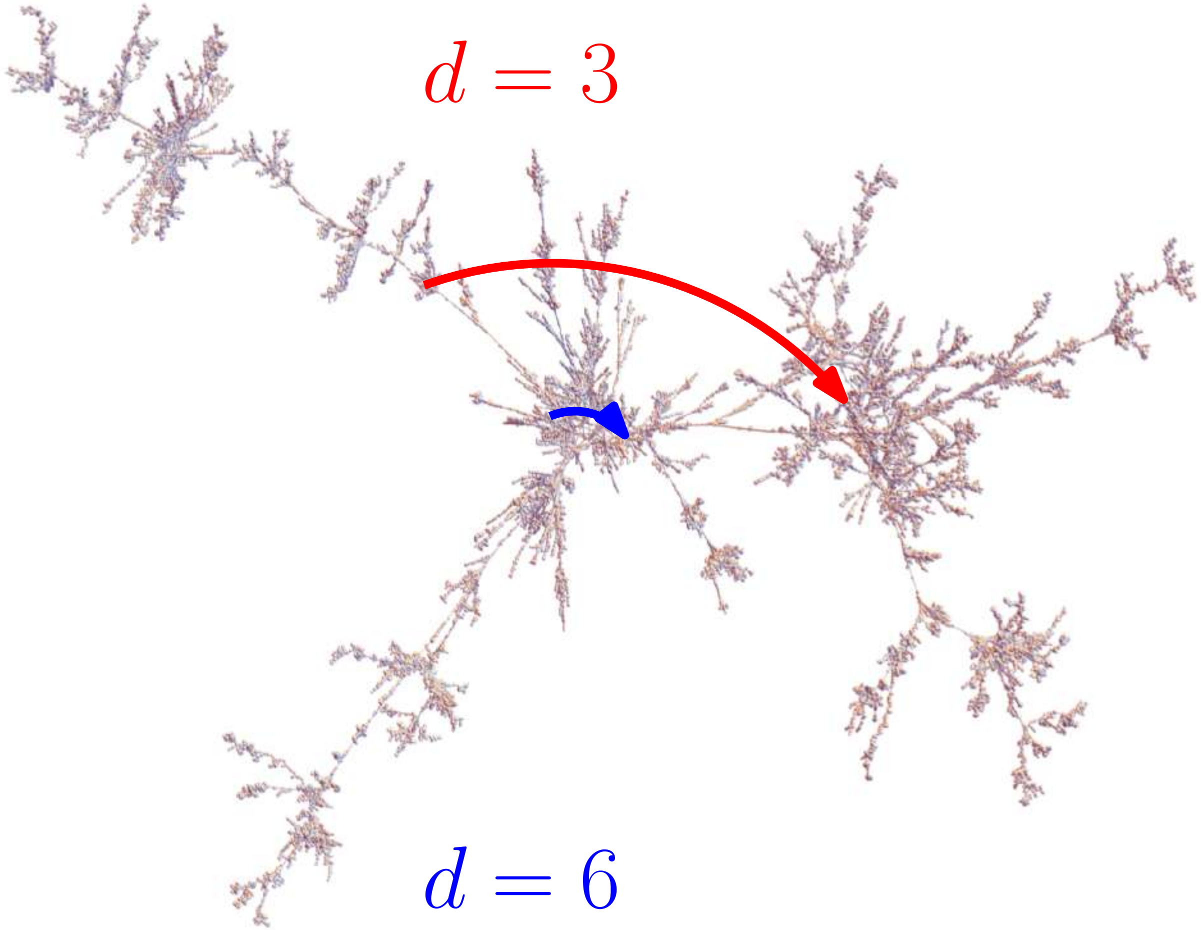


$\gamma = \sqrt{\frac{8}{3}}$

$$\text{distance} \sim n^{1/d}$$

Fleurat Salvy '24





$$\gamma = \sqrt{\frac{8}{3}}$$

Distance profile in the same block at $u = u_{\mathrm{cr}}$

D – Guitter '25
Bouttier Guitter Manet '26+

$$r = \mathrm{distance}/n^{1/6}$$

$$\rho(r) = \frac{3^2 2^{7/3}}{\Gamma\left(\frac{4}{3}\right)} \int_0^\infty x^{7/2} dx \, \frac{e^{-x^3/r^6}}{r^9} \mathrm{sh}(x) \frac{\mathrm{c}(x)\left(\mathrm{c}(x) + \mathrm{ch}(x)\right) - 2}{\left(\mathrm{c}(x) - \mathrm{ch}(x)\right)^3}$$

$$\mathrm{c}(x) = \cos\left(\frac{\sqrt{3x}}{2^{2/3}}\right) \quad , \quad \mathrm{ch}(x) = \cosh\left(\frac{3\sqrt{x}}{2^{2/3}}\right)$$

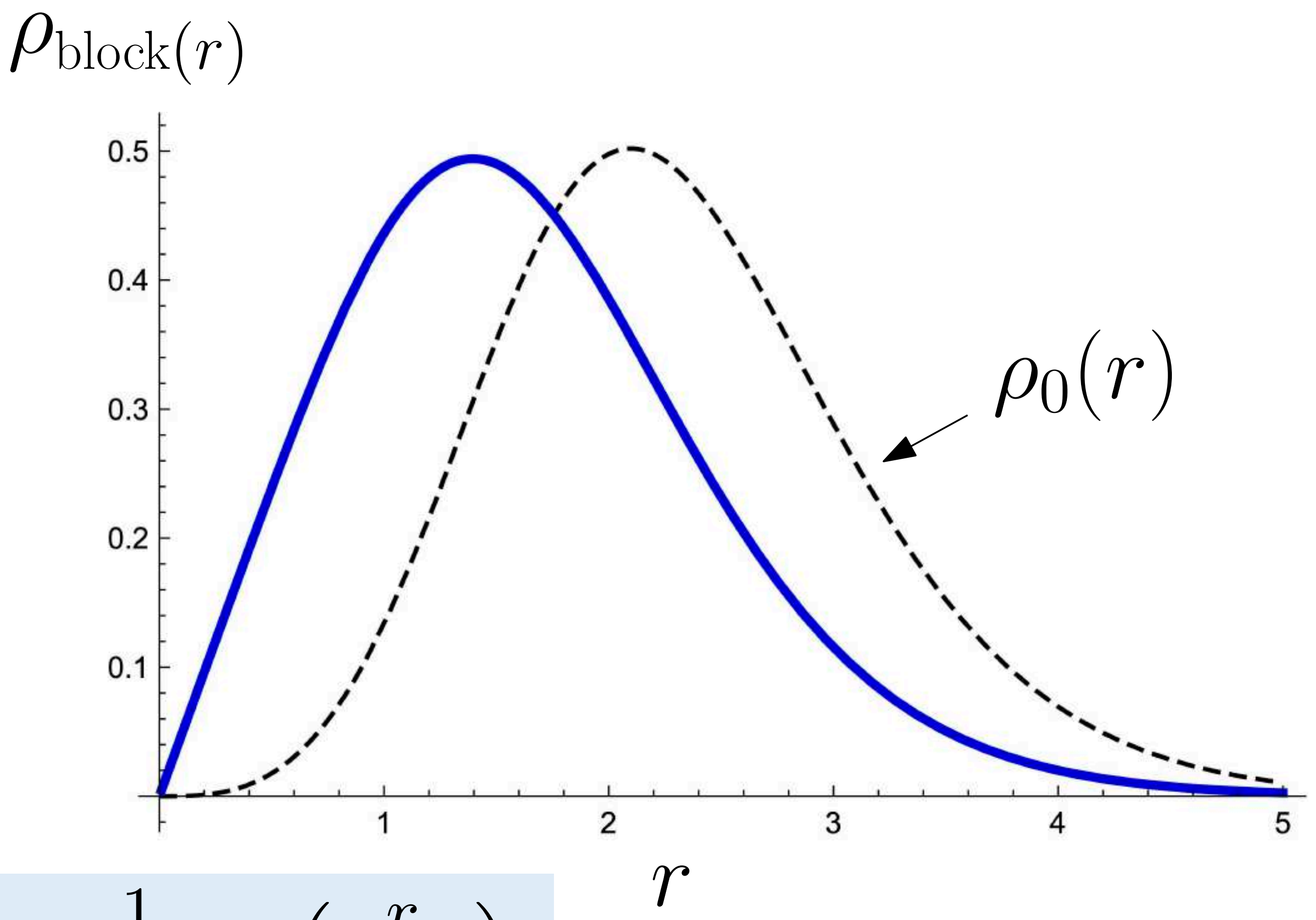
$$\mathrm{s}(x) = \sin\left(\frac{\sqrt{3x}}{2^{2/3}}\right) \quad , \quad \mathrm{sh}(x) = \sinh\left(\frac{3\sqrt{x}}{2^{2/3}}\right)$$

$$\gamma = \sqrt{\frac{8}{3}}$$

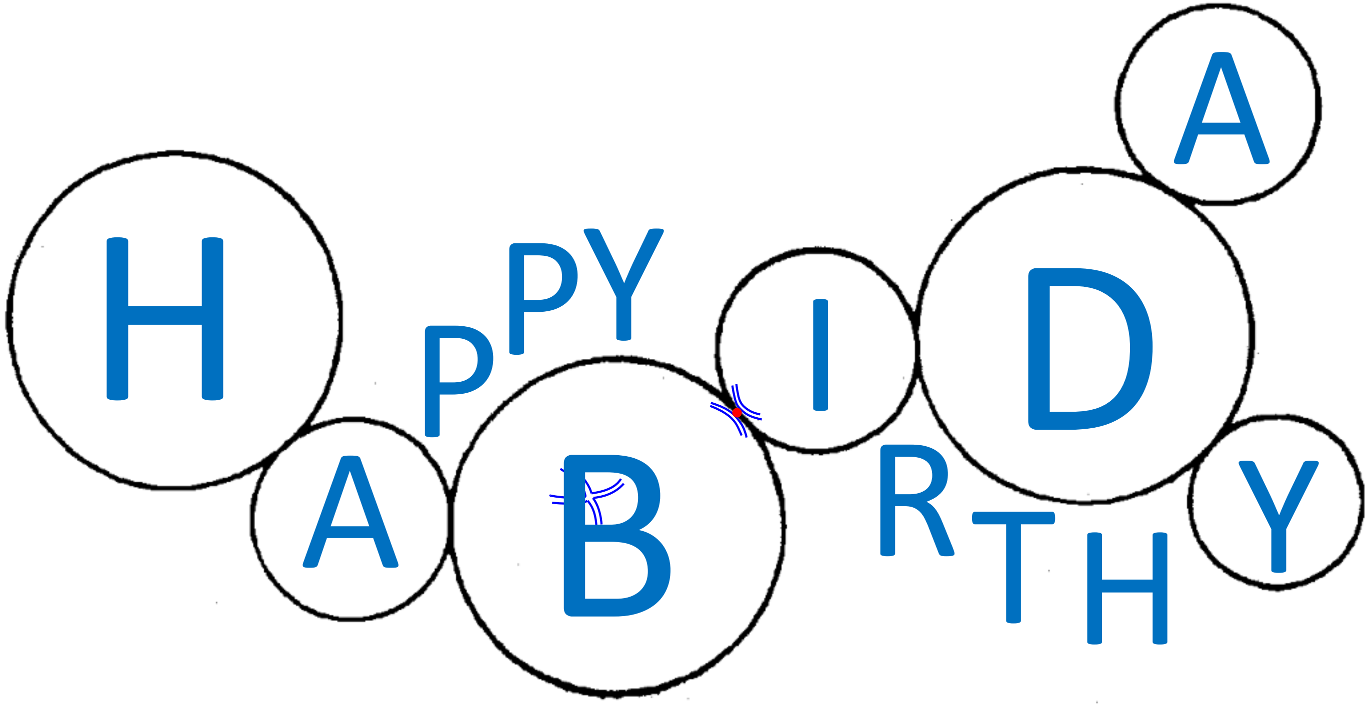
Distance profile in the same block at $u = u_{\mathrm{cr}}$

D - Guitter '26
Bouttier Guitter Manet '26+

$$r = \mathrm{distance}/n^{1/6}$$

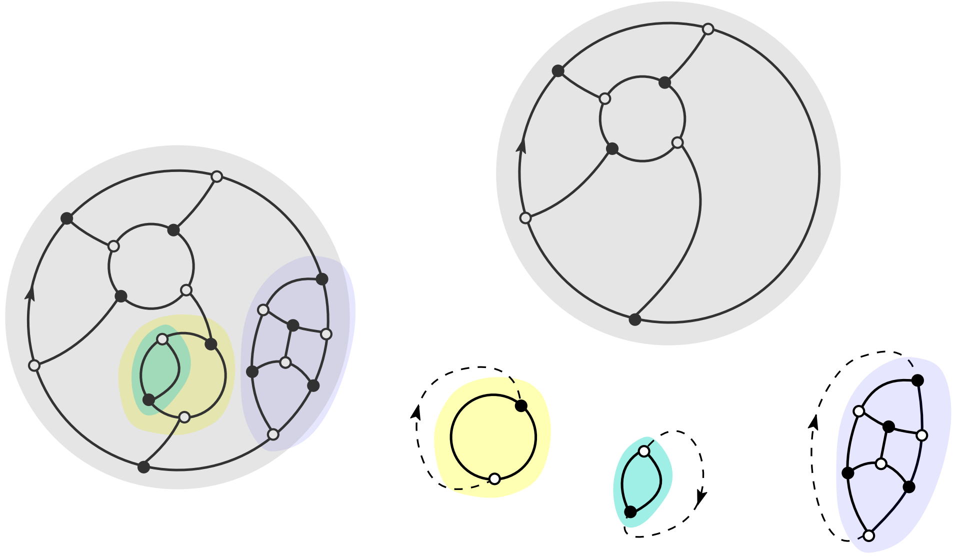


$$\rho_{\mathrm{block}}(r) = \int_0^\infty dx\, \sigma(x)\, \frac{1}{x^{1/4}} \rho_0\left(\frac{r}{x^{1/4}}\right)$$



GREG!

Bicubic maps *Tutte '63*



$$m_n^{(1)} = \frac{3}{2}\frac{2^n}{(n+1)(n+2)}\binom{2n}{n}\,, \quad n \geq 1\,.$$

$$M_1(g) := \sum_{n\geq 0} m_n^{(1)} g^n = 1 + \frac{1}{32g^2}\left(-1 + 12g - 24g^2 + (1-8g)^{3/2}\right)$$

$$M_1(g) = B\left(gM_1(g)^3\right)$$

$$M_u(g) = 1 + u\left[B\left(gM_u(g)^3\right) - 1\right]$$

$$D = \frac{17}{5\,2^{2/3}}$$

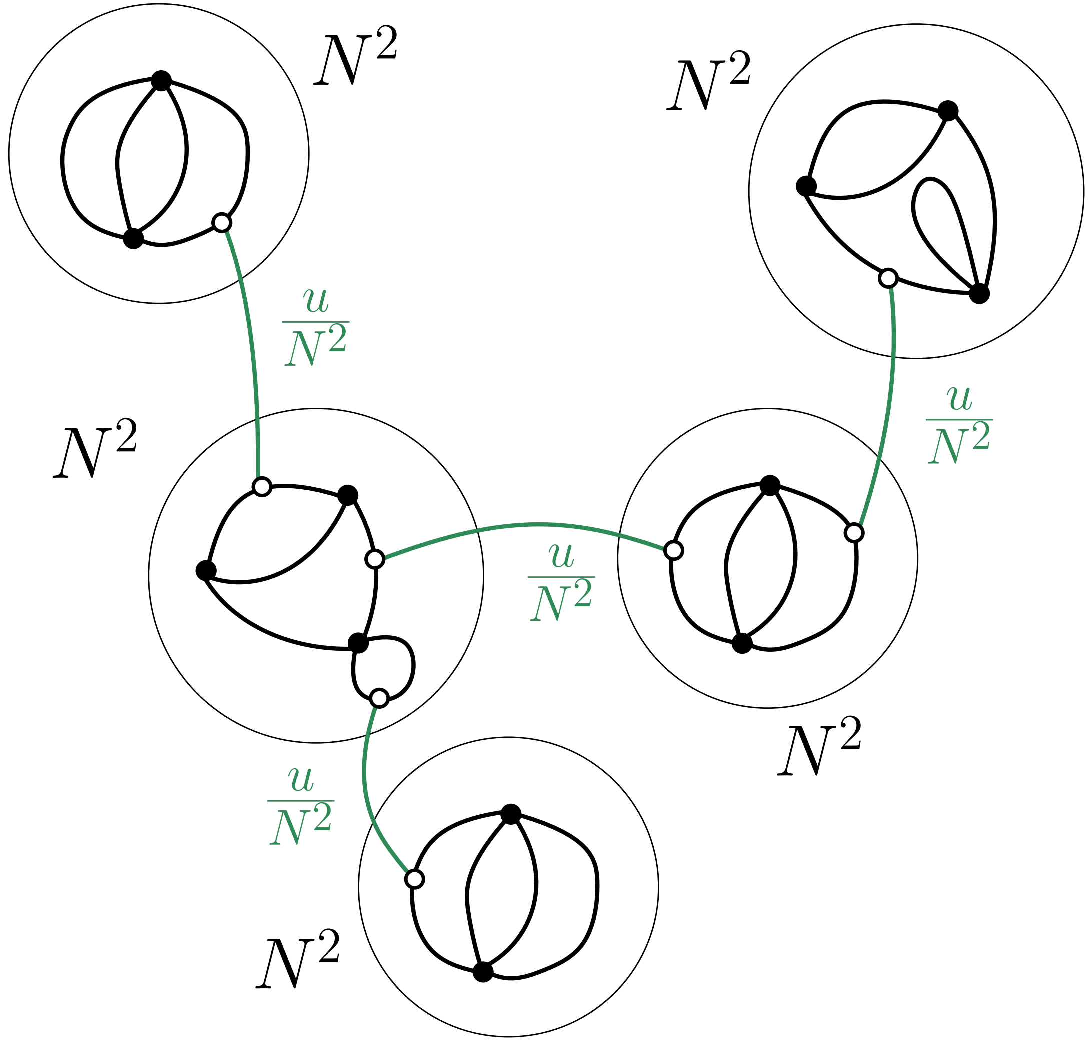
Matrix integral with potential $V(\Phi) = \frac{1}{2}\operatorname{Tr}\Phi^2 - \frac{g}{4N}\operatorname{Tr}\Phi^4 - \frac{u}{4N^2}(\operatorname{Tr}\Phi^2)^2$

Integrate over $N \times N$ Hermitian matrices $\int d\Phi \exp(-V(\Phi)) \underset{N\to\infty}{\sim} e^{N^2 Z(g,u)}$

$Z(g,u)$ g.f. for (rooted) trees

made of 4-regular planar maps

Das Dhar Sengupta Wadia '90



Matrix integral with potential $V(\Phi) = \frac{1}{2}\,\mathrm{Tr}\,\Phi^2 - \frac{g}{4N}\,\mathrm{Tr}\,\Phi^4 - \frac{u}{4N^2}(\mathrm{Tr}\,\Phi^2)^2$

Integrate over $N \times N$ Hermitian matrices $\int d\Phi \exp(-V(\Phi)) \underset{N\to\infty}{\sim} e^{N^2 Z(g,u)}$

Substitution relation for rooted g.f. $Z^\bullet(g,u) = 1 + \left(4g\frac{\partial}{\partial g} + 4u\frac{\partial}{\partial u}\right) Z(g,u)$

$$Z^\bullet(g,u) = \frac{1}{1 - u\,Z^\bullet(g,u)} M\left(\frac{g}{(1-u\,Z^\bullet(g,u))^2}\right)$$ with $M(g) = \frac{18g - 1 + (1-12g)^{3/2}}{54g^2}$

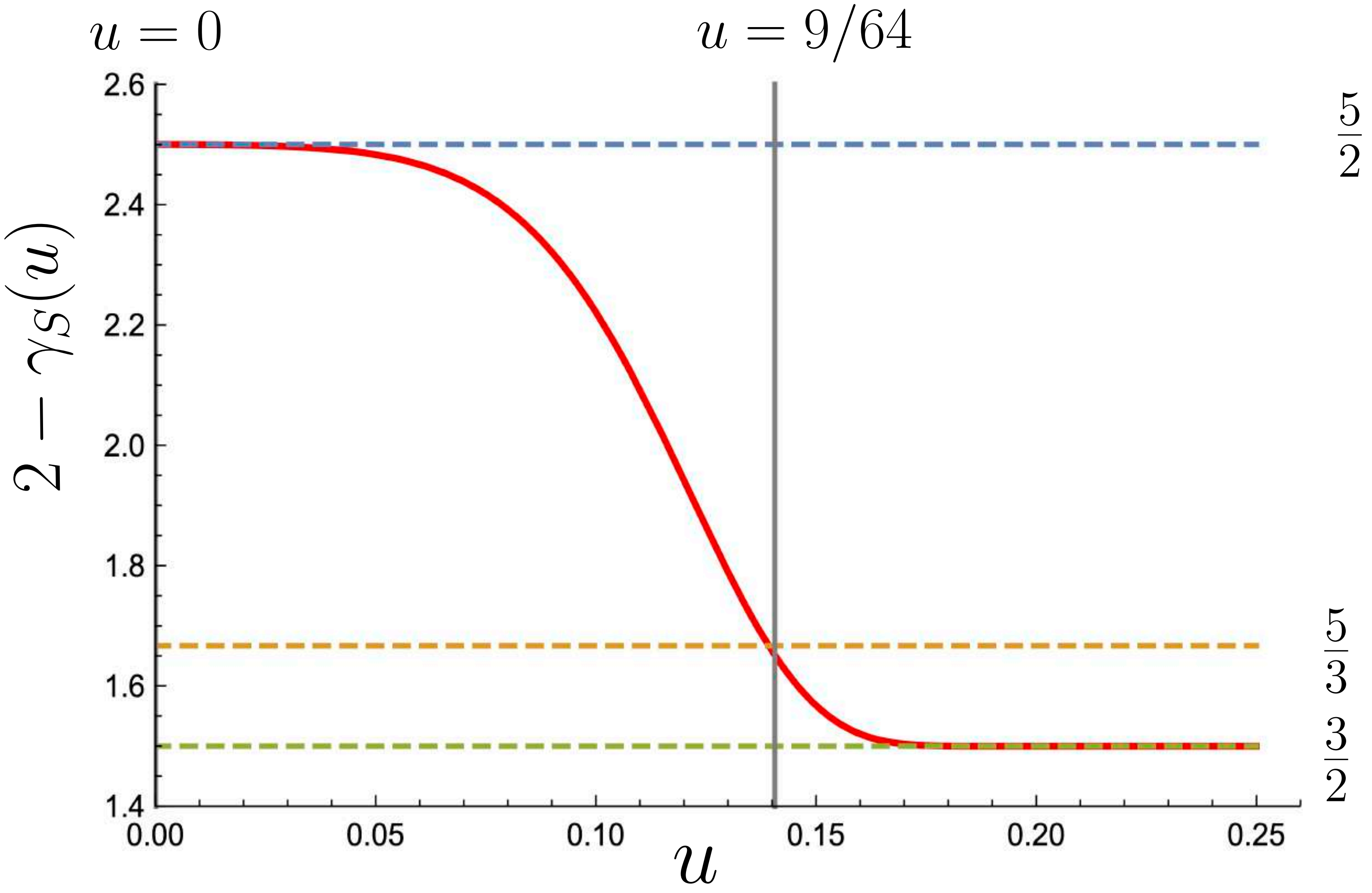
g.f. of rooted planar 4-regular maps

Set $u = \sqrt{g}\,v\ ,\ z = \sqrt{g}\ ,\ y(z) = \dfrac{z}{1 - u\,Z^\bullet(z^2,u)}$

then $y(z) = z\,\phi(y(z))$ with $\phi(\tau) = \dfrac{1}{1 - v\,\tau M(\tau^2)}$

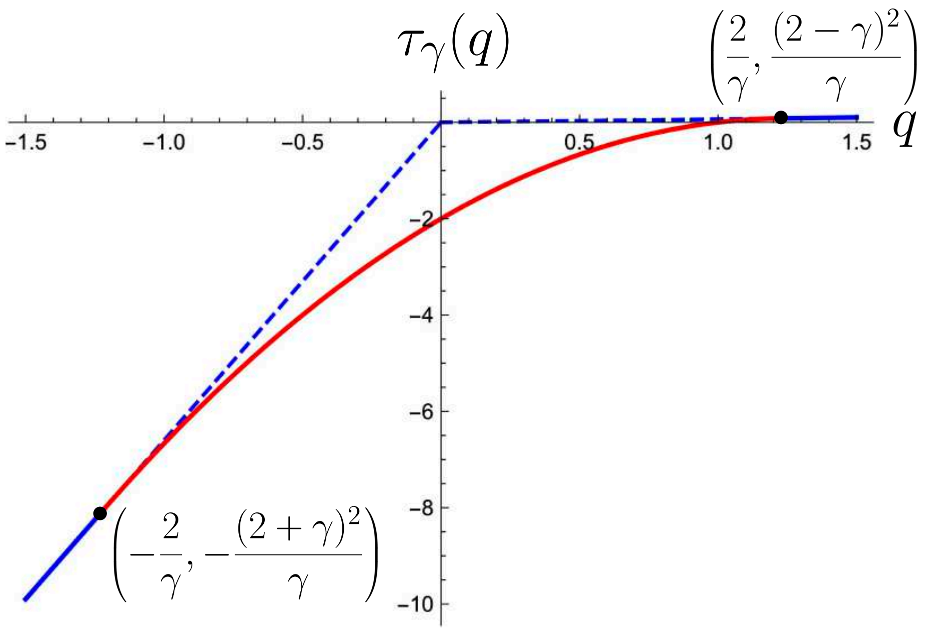
$$u_{\mathrm{cr}} = \frac{9}{64}$$

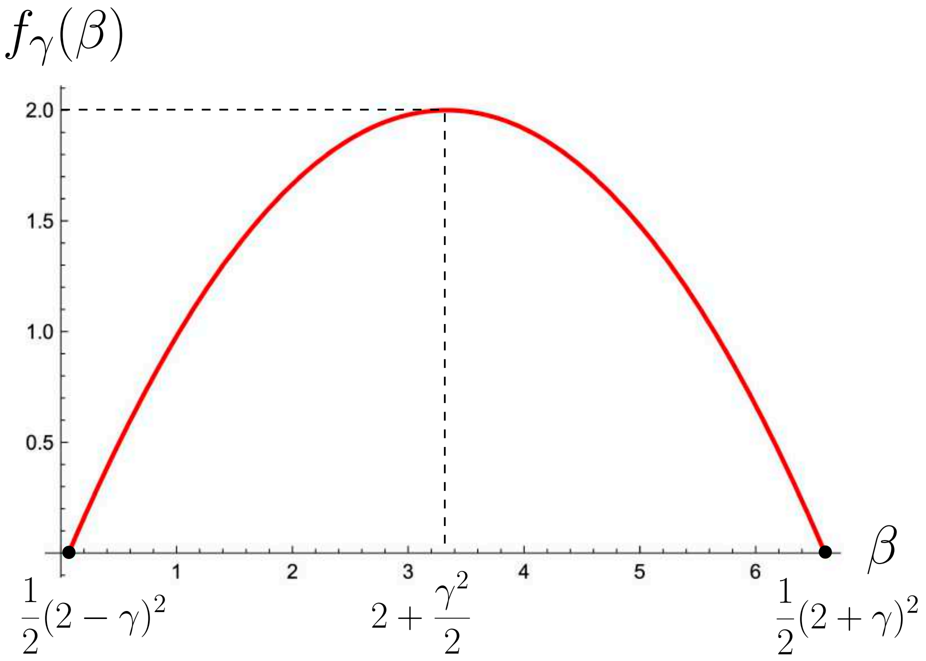
$$D = \frac{1}{2^{2/3}}$$



$u = 0$
$u = 9/64$
2.6
2.4
2.2
2.0
1.8
1.6
1.4
$2 - \gamma_S(u)$
$\frac{5}{2}$
$\frac{5}{3}$
$\frac{3}{2}$
0.00
0.05
0.10
0.15
0.20
0.25
u

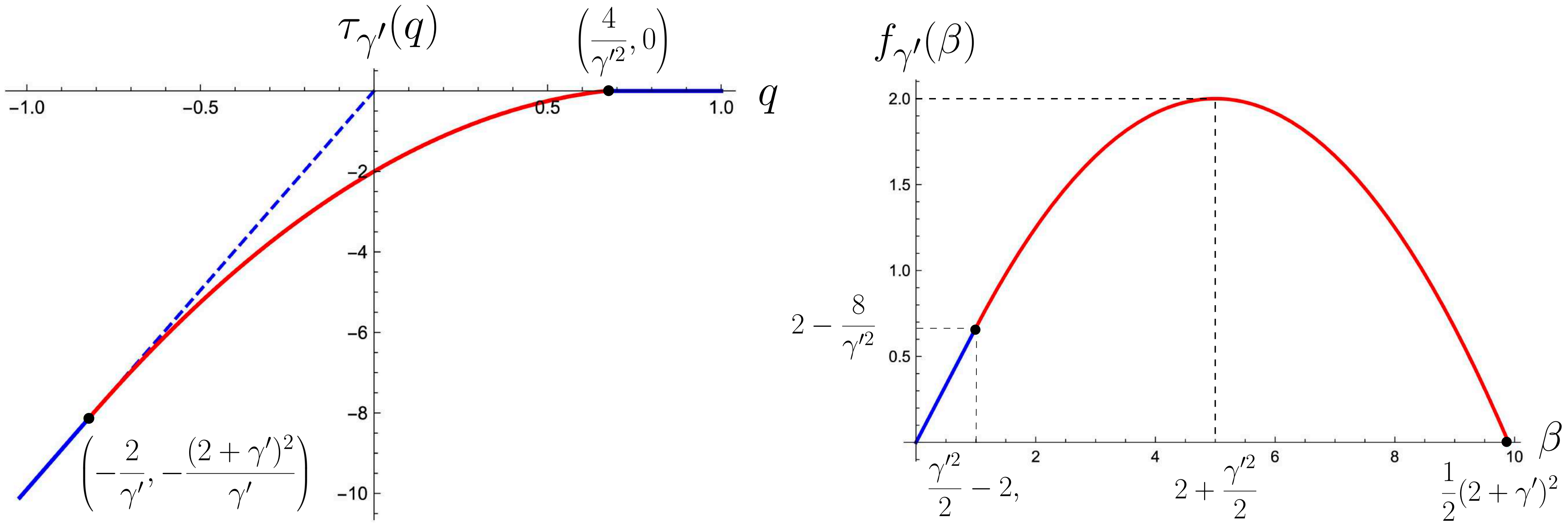
Multifractal spectra - *Liouville measure (Bertacco '23)*





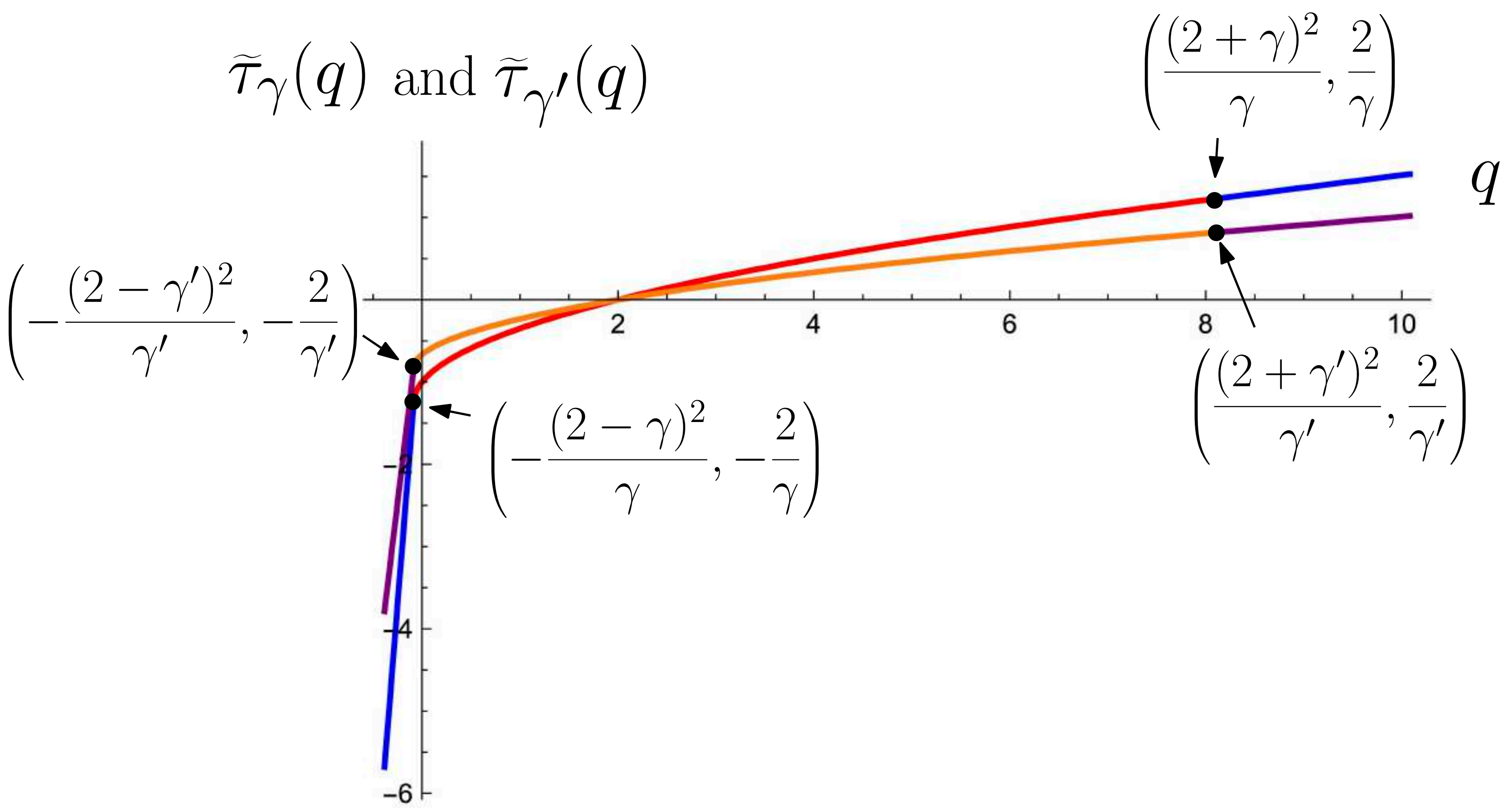
Left: *a.s.* LQG L^q−spectrum τ_γ. Right: *a.s.* LQG multifractal dimension spectrum f_γ. Both curves numerically correspond to the $\gamma = \sqrt{8/3}$ case.

Multifractal spectra - *Dual Liouville measure*

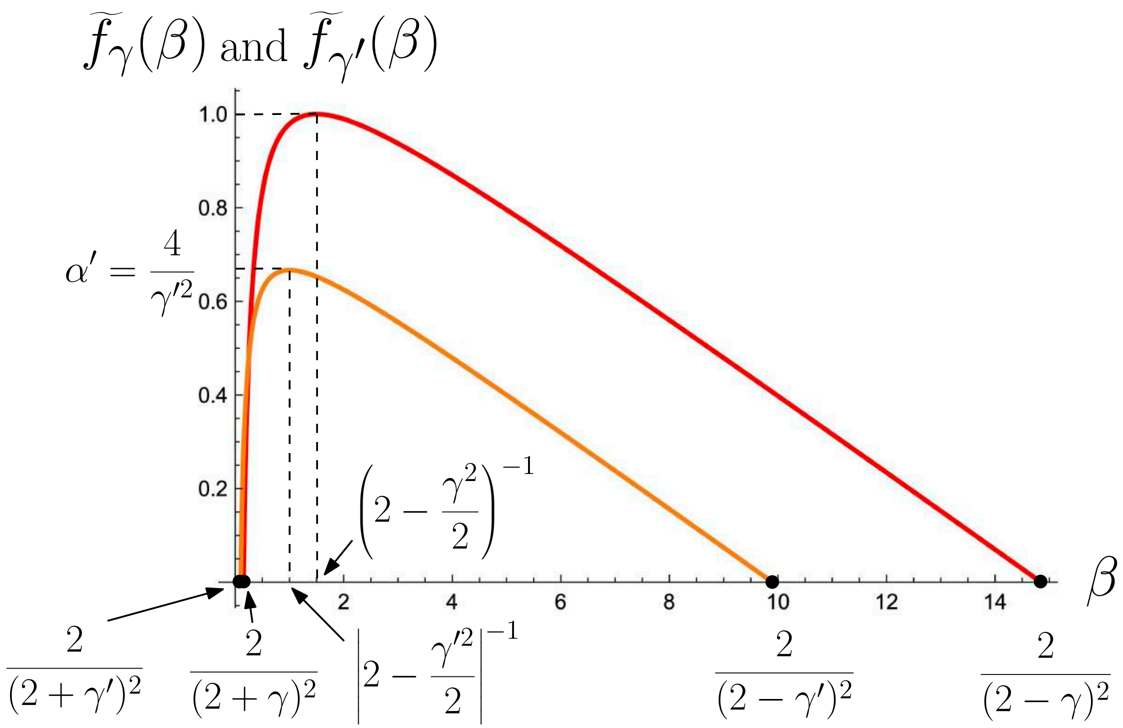


Left: *a.s.* LQG L^q−spectrum $\tau_{\gamma'}$. Right: *a.s.* LQG multifractal dimension spectrum $f_{\gamma'}$. Both curves numerically correspond to the dual $\gamma' = \sqrt{6}$ case.

Dual multifractal L^q-spectra – *Euclidean random ball measure*



Dual dimension spectra – *Euclidean random ball measure*



: Red: *a.s.* dimension spectrum $\widetilde{f}_\gamma$. Its maximum value is 1, the dimension of an LQG surface in quantum ball units. Orange: dual *a.s.* dimension spectrum $\widetilde{f}_{\gamma'}$. Its maximal value, $4/\gamma'^2 = \alpha' < 1$, corresponds to the part of the dual LQG surface without localized area, *i.e.*, to the principal bubble [26, 25], or root block [30]. The $(\gamma = \sqrt{8/3}, \gamma' = \sqrt{6})$ dual cases are depicted here.