Conformal dimension and quasisymmetric rigidity of the Brownian sphere

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(*based on joint work with Yi Tian*)

Cambridge

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Overview

- **Part I**: Conformal dimension. Important quasisymmetric invariant for metric spaces introduced by **Pansu** in 1989 in the context of Gromov-hyperbolic groups.
- **Part II**: Quasisymmetric rigidity. Size of the quasisymmetric automorphism group.

There has been a lot of theory developed in the last decades in this area focused on deterministic objects which are "regular". One motivation for our results is to explore what happens for random objects where these assumptions are broken.

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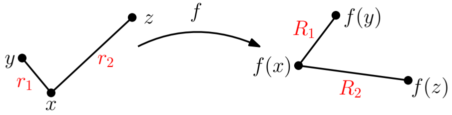

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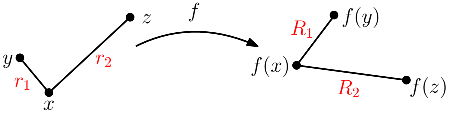

On $\mathbf{R}^d$, **quasiconformal** $\iff$ **quasisymmetric** (**Gehring**, **Väisälä**, **Tukia**).

Conformal dimension

Introduced by **Pansu** (1989) to study boundaries of Gromov hyperbolic groups.

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- The **conformal dimension** of (X, d) is the infimum of the Hausdorff dimensions of all spaces in its gauge:

$$\dim_{\text{conf}}(X, d) := \inf \left\{ \dim_H(Y, d_Y) : (Y, d_Y) \in \mathcal{G}(X, d) \right\}.$$

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- **Important facts:**
  - It is a **quasisymmetric invariant**: if $X \cong_{\mathrm{QS}} Y$, then $\dim_{\mathrm{conf}}(X) = \dim_{\mathrm{conf}}(Y)$.
  - Bounded below by topological dimension and above by Hausdorff dimension:

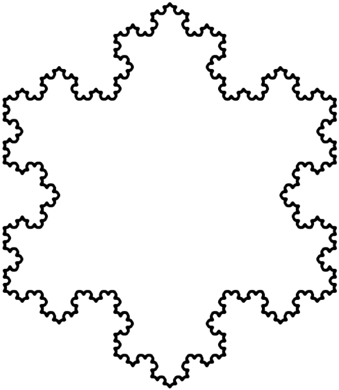
$$\dim_T(X) \leq \dim_{\mathrm{conf}}(X) \leq \dim_H(X).$$

  - The infimum is not always attained. When it is, the minimal metric represents the “most symmetric” or “best” representation.
  - Important variant: Ahlfors regular conformal dimension.

Example I: the Koch snowflake

Consider the **Koch** snowflake K equipped with the Euclidean metric.

- **Hausdorff dimension:**
  $\dim_H(K) = \frac{\log 4}{\log 3} \approx 1.2618.$



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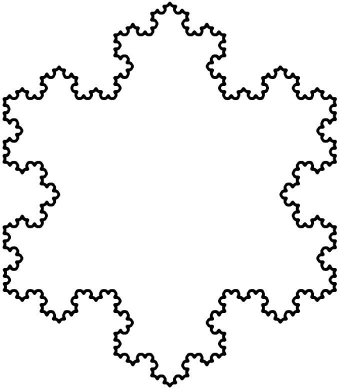

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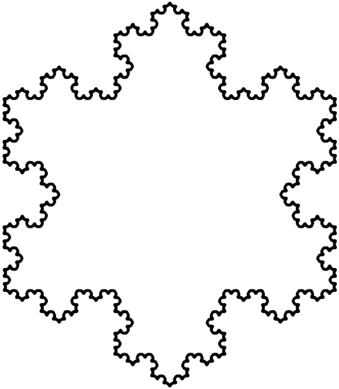
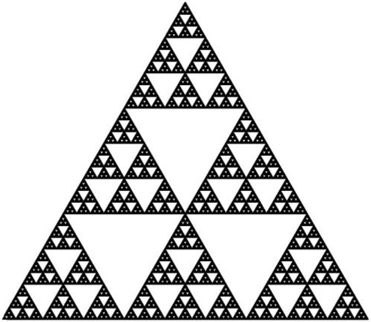
- **Why?**
  - K is a **quasicircle** (quasisymmetrically equivalent to the standard circle $\mathbf{S}^1$).
  - The infimum is attained by the round metric since $\mathbf{S}^1 \in \mathcal{G}(K)$ and $\dim_H(\mathbf{S}^1) = 1$.
  - Thus,
    $\dim_{\mathrm{conf}}(K) = \dim_T(K) = 1 < \dim_H(K)$.



Example II: Sierpiński gasket and carpet



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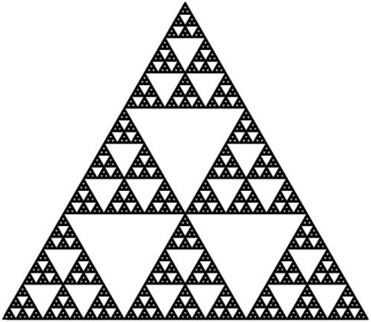

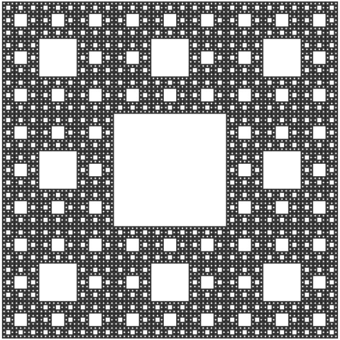

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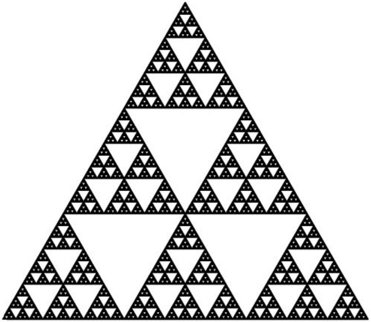

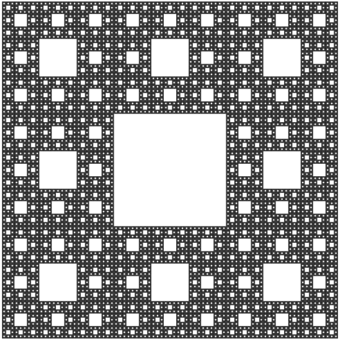

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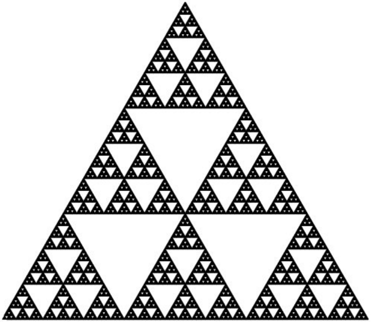

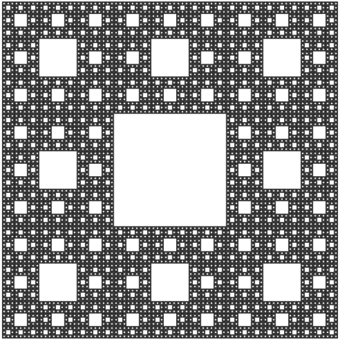

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Exact value is a famous **open problem** as is the question of attainment.

Example III: Graph of Brownian motion

Let $B\colon [0,1] \to \mathbf{R}$ be standard 1D Brownian motion. Consider its graph $X = \{(t, B(t)) \in \mathbf{R}^2 : t \in [0,1]\}$ with the Euclidean metric.

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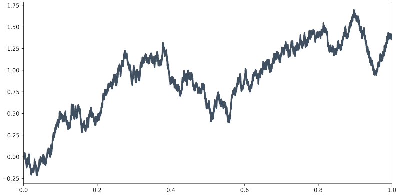

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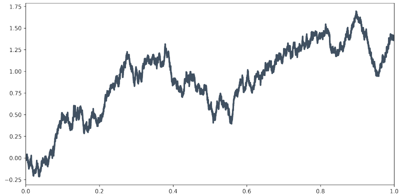

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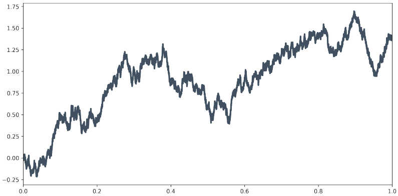
- **Binder-Hakobyan-Li** (2025):
  - X is **minimal** for conformal dimension.
  - Any quasisymmetric image of X must have Hausdorff dimension $\geq 3/2$.



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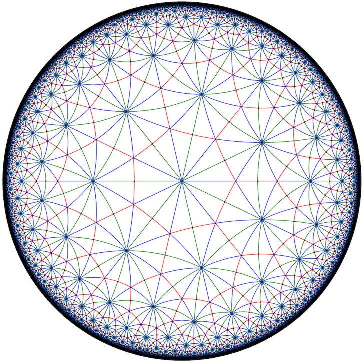

$(2, 3, 7)$ hyperbolic-triangle group;
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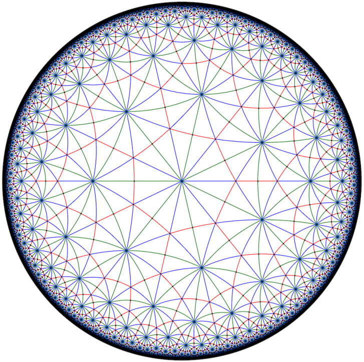

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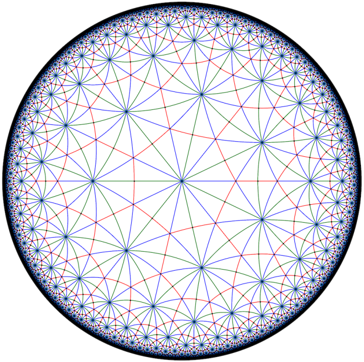

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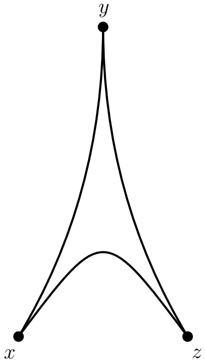

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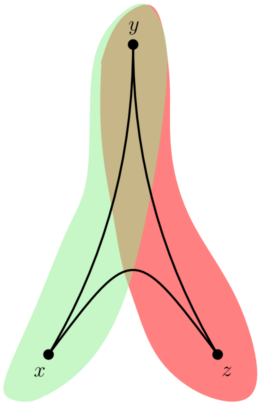

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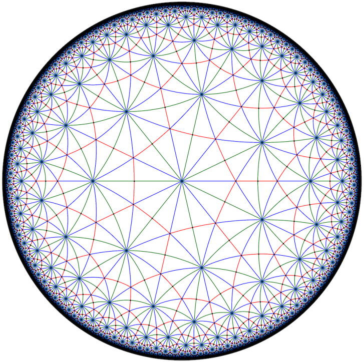

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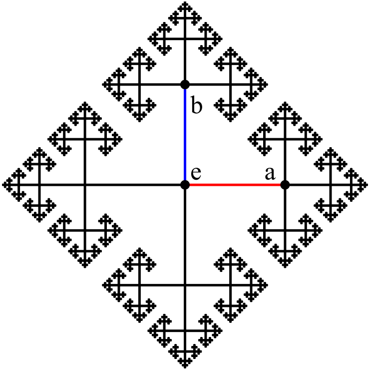

(Credit: **Wikipedia**)
Free group F_2; ∂G is a Cantor set.

Quasi-isometries and the boundary

How does the large-scale geometry of a group relate to its boundary at infinity?

- A map $f\colon X \to Y$ between metric spaces is a **quasi-isometry** if there exist $K \geq 1, C \geq 0$ such that

$$K^{-1}d_X(x,x') - C \leq d_Y(f(x), f(x')) \leq K d_X(x,x') + C$$

for all $x, x' \in X$, and $f(X)$ is C-dense in Y.

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- G_1, G_2 quasi-isometric $\implies \partial G_1, \partial G_2$ quasisymmetrically equivalent $\implies \dim_{\mathrm{conf}}(\partial G_1) = \dim_{\mathrm{conf}}(\partial G_2)$. That is, the **conformal dimension** is a **quasi-isometry invariant** of the group G.

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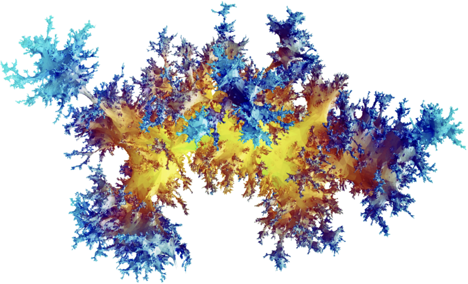
- G_1, G_2 quasi-isometric $\implies$ $\partial G_1, \partial G_2$ quasisymmetrically equivalent $\implies$ $\dim_{\mathrm{conf}}(\partial G_1) = \dim_{\mathrm{conf}}(\partial G_2)$. That is, the **conformal dimension** is a **quasi-isometry invariant** of the group G.
- Famous conjectures related to $\dim_{\mathrm{conf}}(\partial G)$: **Cannon** (1994, $\partial G \cong \mathbf{S}^2$), **Kapovich-Kleiner** (2000, $\partial G \cong$ Sierpiński carpet), etc...

The Brownian sphere



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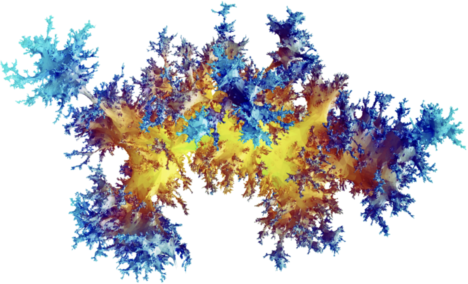

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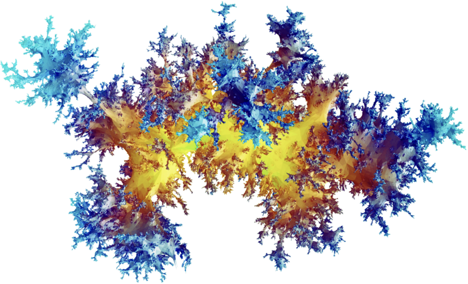

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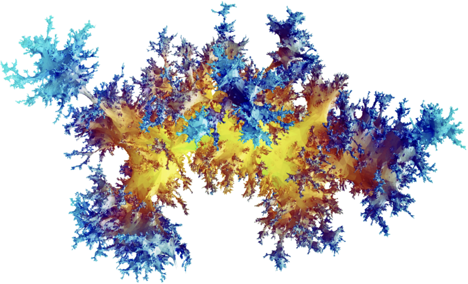

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- Hausdorff dimension $\dim_H(\mathbf{S}_{\mathrm{Br}}) = 4$.
- Equivalent to $\sqrt{8/3}$-Liouville quantum gravity sphere (**M.-Sheffield**).

Conformal dimension of the Brownian sphere

The conformal dimension of the Brownian sphere $(\mathbf{S}_{\mathrm{Br}}, d_{\mathrm{Br}})$ satisfies:

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- **Contrast** with the Brownian graph X, which satisfies $\dim_H(X) = 3/2 = \dim_{\mathrm{conf}}(X)$.

Conformal dimension lower bounds

We spent a lot of time trying to show $\dim_{\mathrm{conf}}(\mathbf{S}_{\mathrm{Br}}) = 4$. How?

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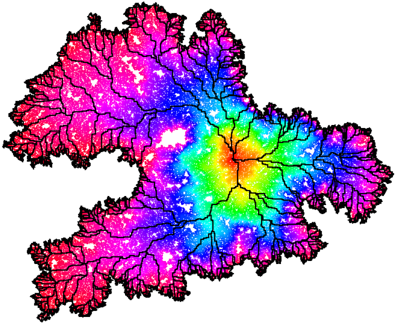
- Theorem requires a sufficiently "rich" family of rectifiable curves.
- Such curve families do **not** exist in the Brownian sphere due to the **confluence of geodesics** phenomenon.



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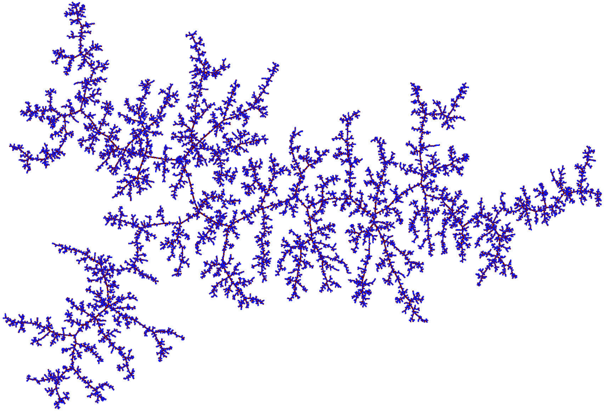
  3. Design an admissible weight achieving $p_c = 2$, using the equivalence between $\sqrt{8/3}$-LQG and $\mathbf{S}_{\mathrm{Br}}$.

What about other random fractals?

(Credit: **Kortchemski**)



Brownian tree (T)

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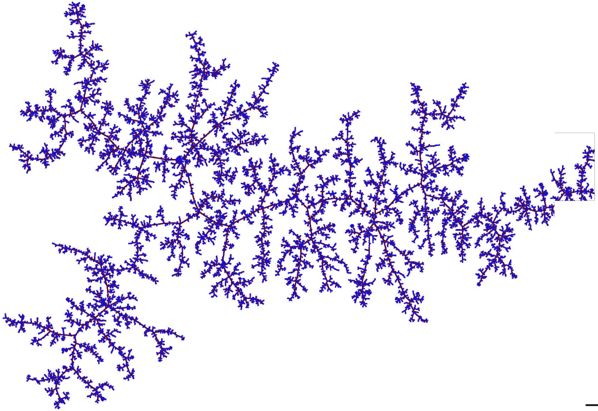

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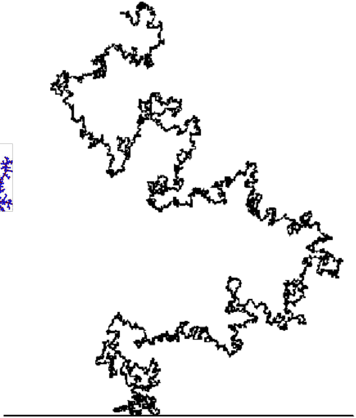


SLE$_\kappa$ trace (η)

Forthcoming work of **Li-Liu**:
$\dim_{\mathrm{conf}}(\eta) = 1$ for $\kappa < 4$

Quasisymmetric rigidity

Let $\mathrm{QS}(X)$ be the group of all quasisymmetric self-homeomorphisms (automorphisms) of a metric space (X, d).

(X, d) is said to be **quasisymmetrically rigid** if its quasisymmetric automorphism group $\mathrm{QS}(X)$ is "small."

- Typically, this means $\mathrm{QS}(X)$ equals the isometry group $\mathrm{Isom}(X, d)$ (modulo scaling).
- For random fractals, it means $\mathrm{QS}(X)$ is trivial (i.e., only contains the identity map).

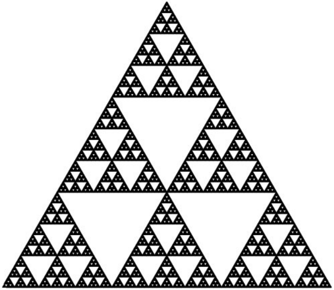
Rigidity results are rooted in classical literature (**Mostow** rigidity for hyperbolic manifolds, **Sullivan** rigidity for limit sets of Kleinian groups, etc...)

Example I: $\mathbf{S}^2$

- The round sphere $\mathbf{S}^2 \cong \widehat{\mathbb{C}}$ is highly quasisymmetrically flexible.
- The group of quasisymmetric automorphisms $\mathrm{QS}(\mathbf{S}^2)$ is **infinite-dimensional**.
- It coincides with the group of all quasiconformal homeomorphisms of the sphere:
$$\mathrm{QS}(\mathbf{S}^2) = \mathrm{QC}(\mathbf{S}^2)$$
- Contrast with the group of conformal automorphisms, which is only 6-dimensional.

Example II: Sierpiński gasket and carpet

Exercise: every quasisymmetric automorphism of the standard Sierpiński gasket is a Euclidean isometry.



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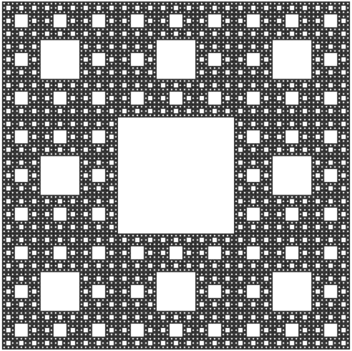
Bonk-Merenkov (2013):

- Every quasisymmetric automorphism of the standard square Sierpiński carpet S_3 is a Euclidean isometry.
- S_p is QS equivalent to S_q iff $p = q$.



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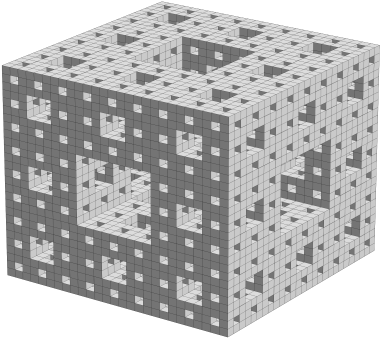
Open question: is the **Menger** sponge QS rigid?



Menger sponge

Rigidity of the Brownian Sphere

Theorem (M.-Tian)

Let $(\mathbf{S}_{Br}, d_{Br})$ *be the Brownian sphere.*

1. *Almost surely,* $(\mathbf{S}_{Br}, d_{Br})$ *admits no non-trivial quasisymmetric automorphisms:*

$$\mathrm{QS}(\mathbf{S}_{Br}) = \{\mathrm{Id}\} \quad \textit{almost surely.}$$

2. *If* $(\mathbf{S}_{Br}^{(1)}, d_1)$ *and* $(\mathbf{S}_{Br}^{(2)}, d_2)$ *are two independent Brownian spheres, they are almost surely not quasisymmetrically equivalent.*

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Some consequences:

- The Brownian sphere cannot arise as the boundary of a random group type construction (e.g., Gromov hyperbolic groups) since conjugation by a generator induces a non-trivial quasisymmetry of the boundary.
- Another proof that the conformal structure on the Brownian sphere is uniquely determined by its metric structure (no Efron-Stein, removability, or random walk in random environment).

Relation to uniformization

The rigidity and dimension results reveal a fundamental difference between classical/metric uniformization and random surfaces:

- **Classical conformal uniformization:** Every simply connected Riemann surface is conformally equivalent to $\mathbb{C}$, $\mathbb{H}$, or $\widehat{\mathbb{C}}$. All topological spheres with conformal structures map to the round sphere $\mathbf{S}^2$.

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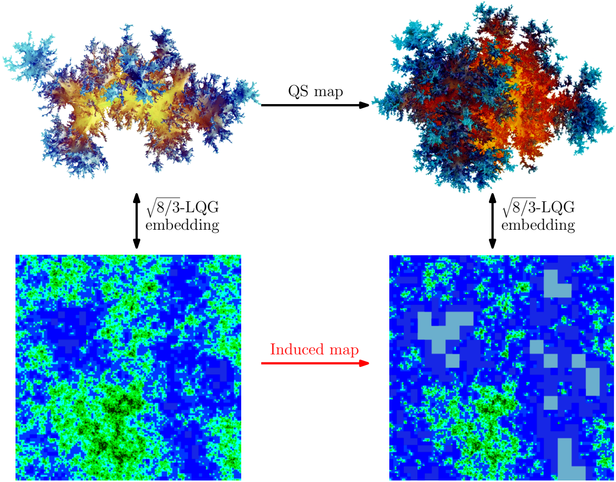
- **The random setting: no model space.**
  - Two independent Brownian spheres are almost surely not quasisymmetrically equivalent.
  - There is no single canonical "model space" (such as $\mathbf{S}^2$) that can quasisymmetrically uniformize all Brownian spheres.

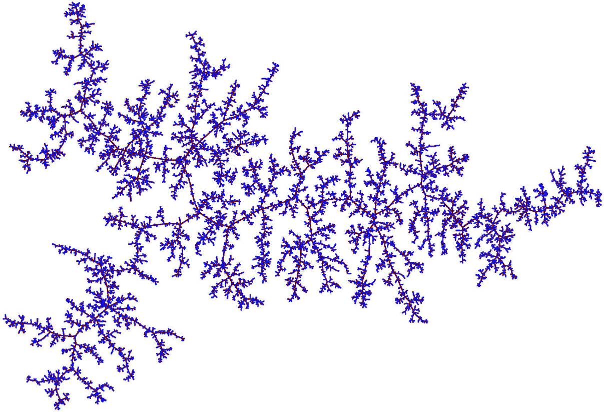
Proof sketch



- QS map between independent Brownian sphere instances.
- Each has an embedding into **C** from $\sqrt{8/3}$-LQG equivalence.
- We show the **induced map** is **Euclidean** QC, hence has an a.e. non-singular derivative, so locally linear. Contradiction.

Rigidity of other random fractals?

(Credit: **Kortchemski**)



Brownian tree

Bonk-Meyer: Are two independent CRTs QS equivalent? Is the Brownian tree QS rigid?

Rigidity of other random fractals?

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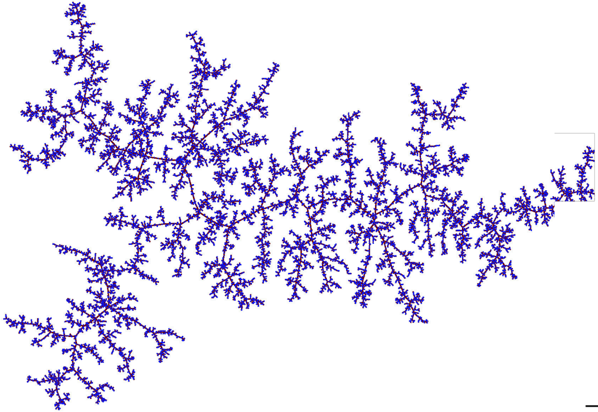

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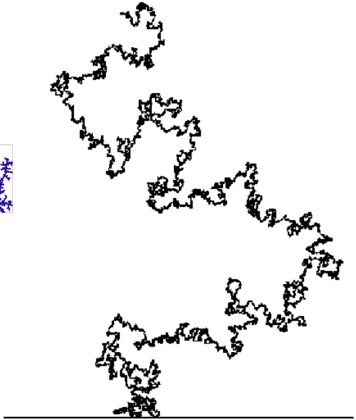

SLE$_\kappa$ trace

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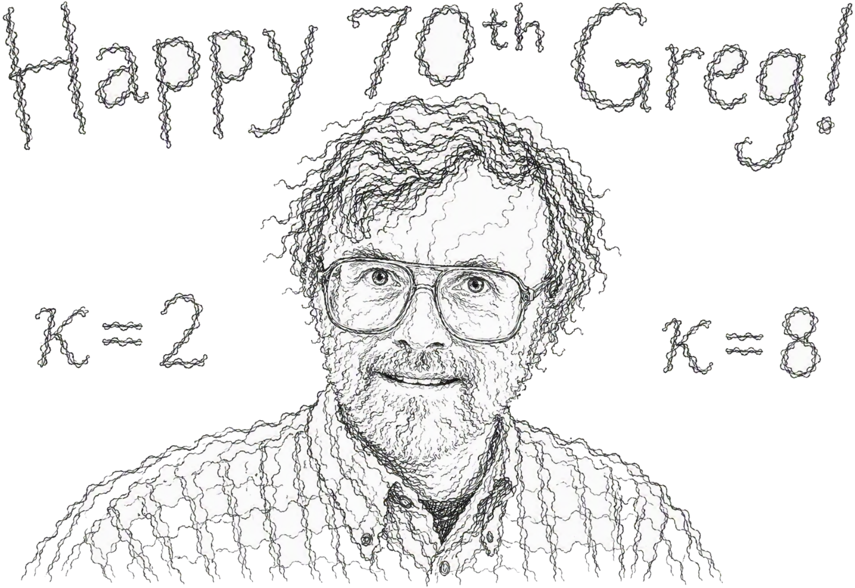
McMullen: Are two independent SLEs QS equivalent? Is SLE QS rigid?


Happy 70th Greg!
$\kappa = 2$
$\kappa = 8$