# Loop ensembles and the Polyakov conjecture

Scott Sheffield

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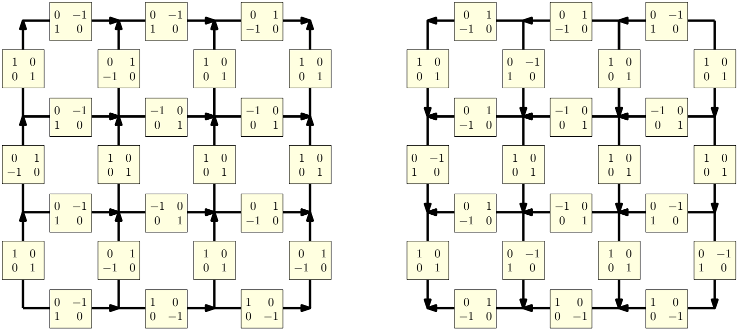
- **Specific to 4D (YM) and 2D (NLSM):** for fixed large $\beta$, little nudge (switching $N$ from 1 to 2, or $n$ from 2 to 3) believed to change phase.
- Today's talk: think about **2D non-linear** $\sigma$ **model** and corresponding loops.

# For comparison: 4D Euclidean Yang-Mills

- Want to assign a matrix to every directed of a grid: $\mathbb{Z}^4$ of particular interest.



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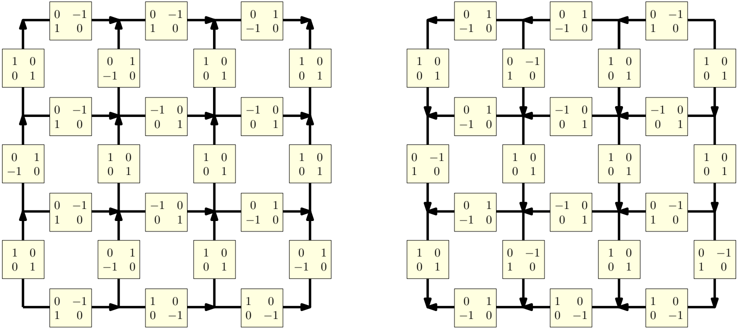

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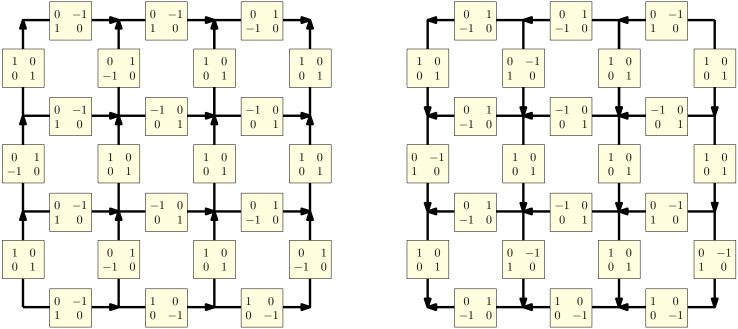

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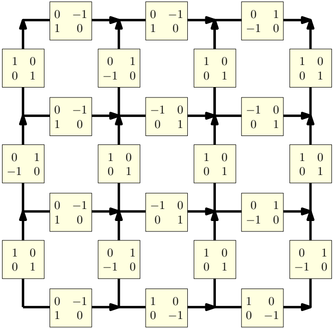


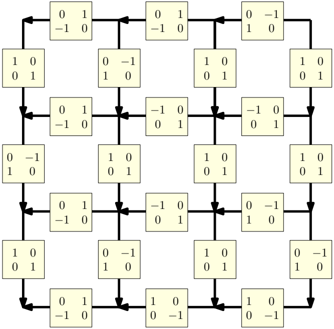


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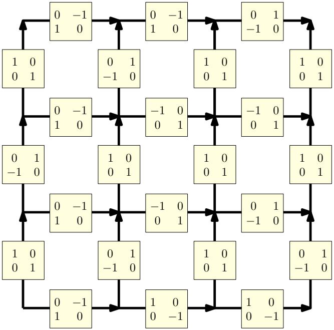

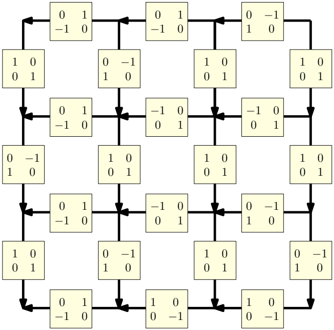


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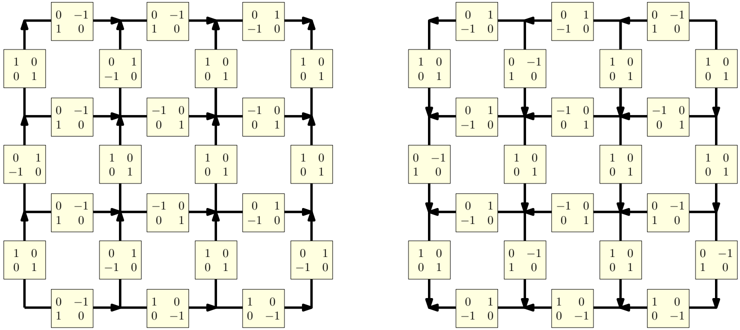

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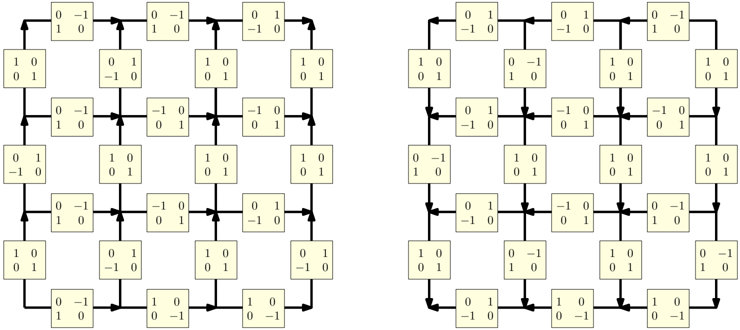

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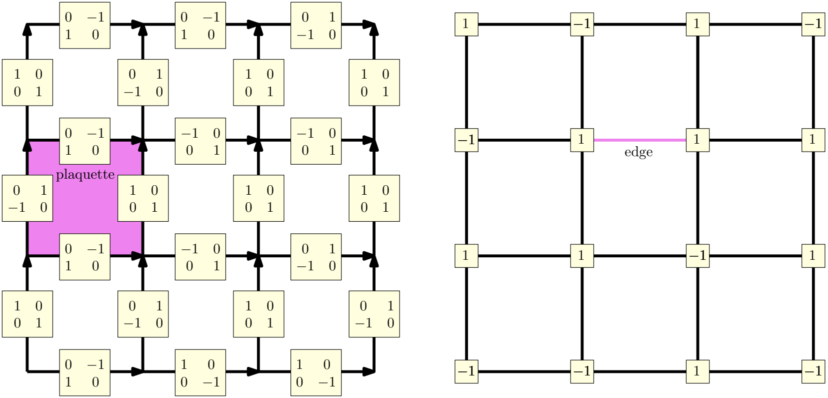
- **Exponential correlation decay** (mass gap)? **Area law** (quark confinement)?

# 2D Euclidean $\sigma$-model

- Assign each **vertex** of $d$-dimensional lattice a **random object** from a *target space* $T$. For example, $T$ can be $\{-1, 1\}$ (**Ising model**) or $S^1$ (**XY-model**) or $\mathbb{R}$ (**Gaussian free field**) or $S^2$ (**Heisenberg model**) or any matrix group.



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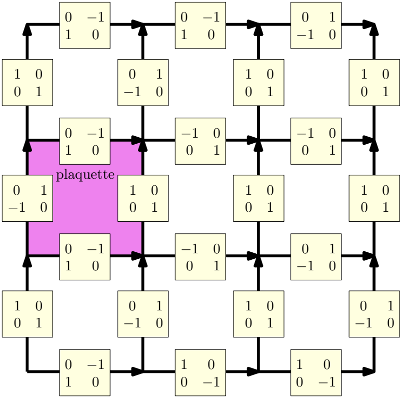


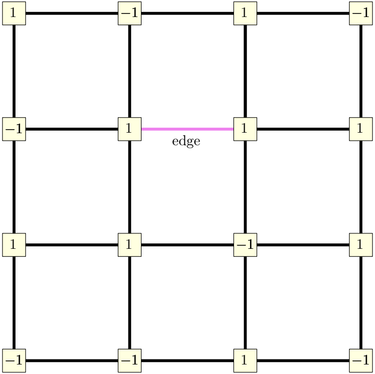


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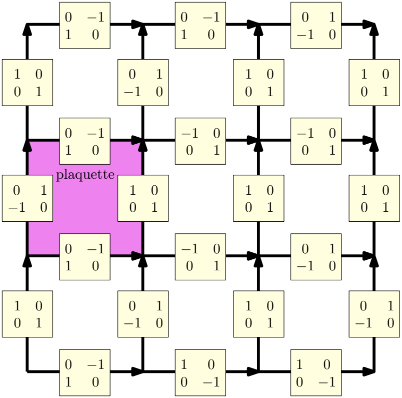


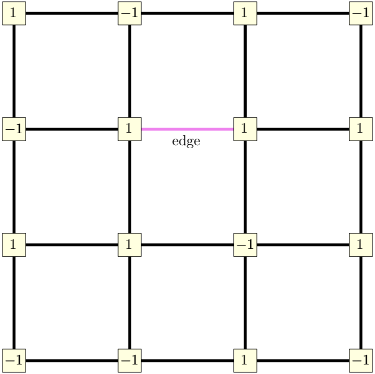


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- Exp. correlation decay at any $\beta$ derived by Polyakov-Wiegmann for spin-$O(N)$ model, where $T = S^{N-1}$. Not mathematically proved.

# Connections to surfaces and loops have a long history

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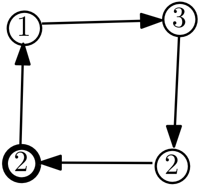
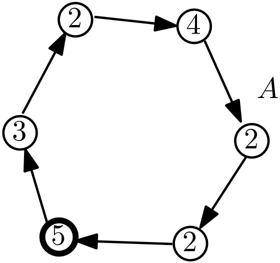
# Graphical representation of a term of $\mathbf{tr}(A^4)\mathbf{tr}(A^6)\mathbf{tr}(A^8)$

$A_{2,1}A_{1,3}A_{3,2}A_{2,2}$





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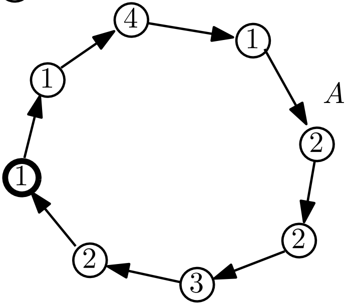

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- A non-zero term in the Wick expansion is an orientation-preserving, label-compatible matching of the edges of the three labeled faces. Each such term contributes 1.

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- Suppose $A$ is sampled from GUE. How do you compute $E[\mathbf{tr}A^4\mathbf{tr}A^6\mathbf{tr}A^8]$?
- Each term of $\mathbf{tr}A^4$ has form $a_{i,j}a_{j,k}a_{k,l}a_{l,i}$. Represent this by a directed square with vertices labeled $i$, $j$, $k$, $l$. One directed edge for each factor.
- Do same thing for $\mathbf{tr}A^6$ and $\mathbf{tr}A^8$. Get a hexagon and octagon.
- **Wick's theorem:** if $X_1, X_2, \ldots, X_{2n}$ are jointly Gaussian, each with mean zero, then what is $E[X_1X_2\ldots X_{2n}]$?
- **Answer:** Consider product like $E[X_1X_2]E[X_3X_4]\ldots E[X_{2n-1}X_{2n}]$. Sum over all $(2n-1)\cdot(2n-3)\cdot\ldots\cdot 1$ such products.
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- So $E[\mathbf{tr}A^4\mathbf{tr}A^6\mathbf{tr}A^8] = \sum_j a_j N^j$ where $a_j$ is number of surfaces with $j$ vertices. Number of faces/edges fixed. Euler's formula: exponent depends only on genus.

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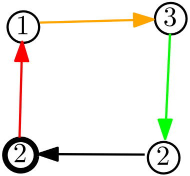
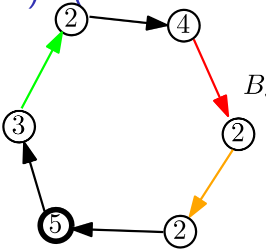
- Similar story for GOE but maps not orientable, weights are signed.

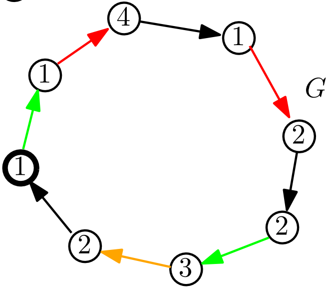
# Graphical representation of a term of $\mathbf{tr}(ROGB)\mathbf{tr}(BGBROB)\mathbf{tr}(GRBRBGOB)$

$R_{2,1}O_{1,3}G_{3,2}B_{2,2}$




$B_{5,3}G_{3,2}B_{2,4}R_{4,2}O_{2,2}B_{2,5}$




$G_{1,1}R_{1,4}B_{4,1}R_{1,2}B_{2,2}G_{2,3}O_{3,2}B_{2,1}$

# Classical matrix-map story in GUE case

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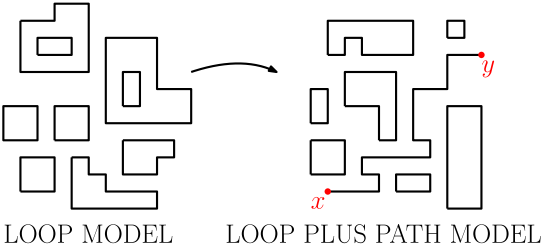
- For any oriented plaquette $P$ can write $\mathbf{tr}P$ for trace of corresponding product of matrices. Now we can formally compute $E[e^{\sum \mathbf{tr}(p)}]$ where sum ranges over all oriented plaquettes. Using Wick's theorem, we get a sum of surfaces built out of oriented plaquettes.

## *Loop ensembles* appear in Ising model, percolation, $O(n)$ model, double dimer, loop soup, etc.

- **CONSIDER:** *some* statistical physics model for a collection of loops. (Maybe loops simple/disjoint, maybe not, maybe oriented, maybe not...) Partition function $Z$ is sum—over all loop configurations—of "weight" associated that loop configuration. **ALSO CONSIDER:** *modified model* with closed loops plus open path from $x$ to $y$. View $x$ and $y$ as "defects." Let $\tilde{Z}$ be associated partition function.



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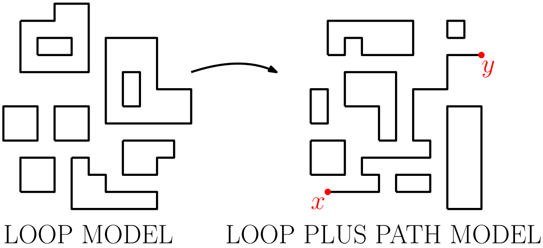


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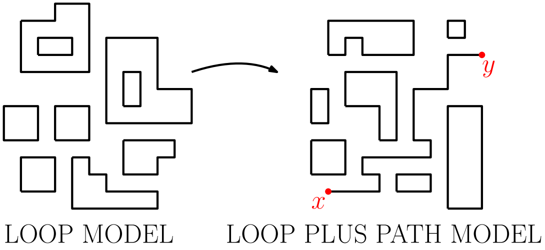


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- **PHASE TRANSITION:** If we tune parameters correctly, can obtain limiting *hybrid model* (Sine-Gordon).

# *Closed surface ensembles* also appear

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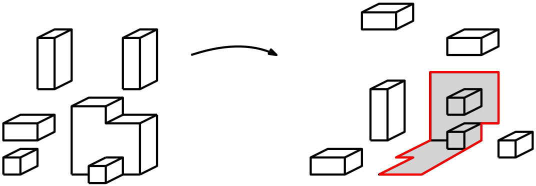

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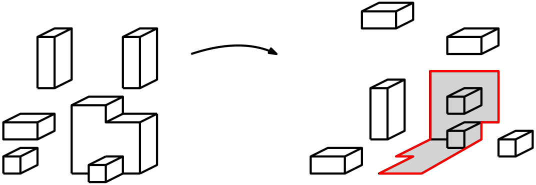

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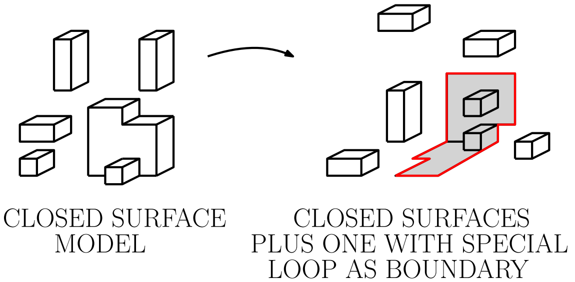


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- **PHASE TRANSITION:** What scaling limits and hybrid limits are possible?

# Spin-$O(n)$ model: start with moments on the sphere

- **Use Gaussian to construct uniform measure on sphere:**
  1. Take $Y_1, Y_2, \ldots, Y_n$ i.i.d. $N(0,1)$.
  2. Write $R^2 = \sum Y_i^2$.
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- This very simple calculation already leads to interesting loop model.

# Note: Yang-Mills has similar simple story in case of $SU(2)$

Write a Haar-random $A \in SU(2)$ as

$$A = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

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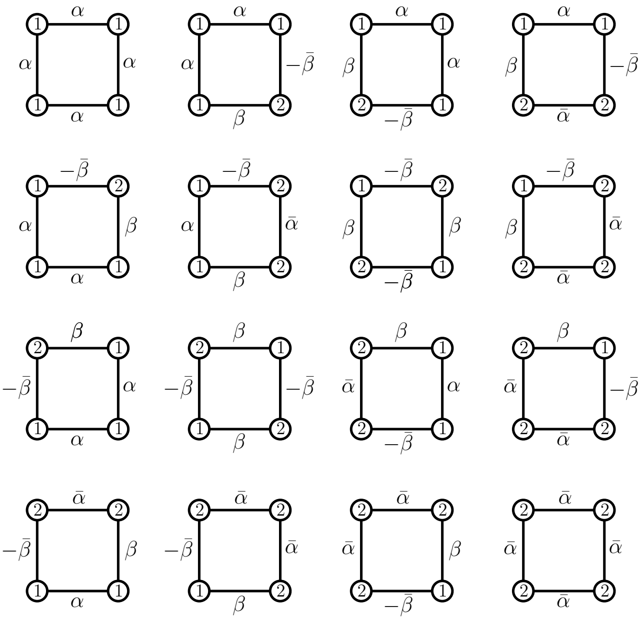
- This is the entry-wise moment rule we will use for $SU(2)$ surface expansions.

# Edge (clockwise-oriented) edge $e$ has $(\alpha(e), \beta(e))$ pair



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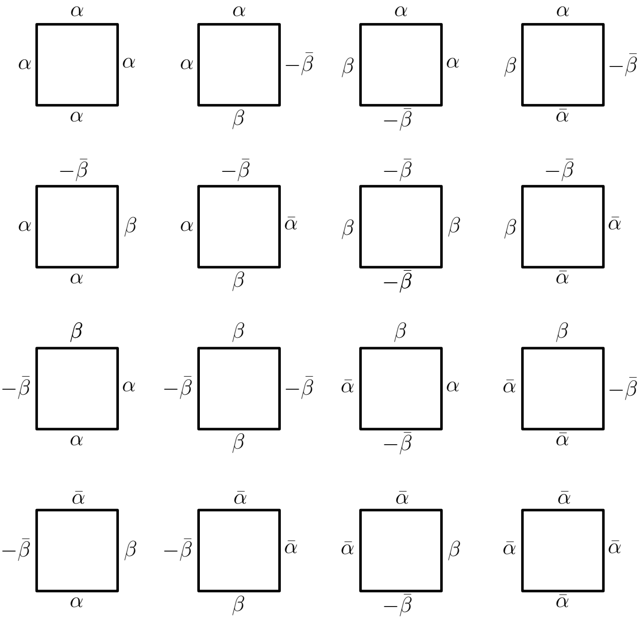

# The 2D spin $O(n)$ model

Let $\Lambda \subset \mathbf{Z}^2$ be a finite graph. Put a spin $\sigma_x \in S^{n-1}$ on each vertex and define

$$\mu(d\sigma) \propto \prod_{\{x,y\}\in E(\Lambda)} P(\sigma_x \cdot \sigma_y) \prod_{x\in V(\Lambda)} d\omega(\sigma_x), \qquad P(t) = t^k(t+1).$$

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- On the hexagonal lattice, a degree-3 vertex makes the bookkeeping especially clean.

# Loop expansion: edges become strands

For the scalar spin product,

$$\sigma_x \cdot \sigma_y = \sum_{i=1}^{n} \sigma_x^{(i)} \sigma_y^{(i)}.$$

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- The two-point function becomes a ratio:

$$\langle \sigma_x \cdot \sigma_y \rangle = \frac{Z_{x,y}}{Z_\varnothing},$$

where $Z_{x,y}$ has one open path from $x$ to $y$ and $Z_\varnothing$ has only closed loops.

# A useful simplification on the hexagonal lattice

On the hexagonal lattice each vertex has degree 3.

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- With $P(t) = t^k(t+1)$, one can view the model as the fully packed background plus a parity defect layer.
- Parity contours are not the same thing as the macroscopic joined loops.

# The special case $n = 2$: the XY model and a height field

For $n = 2$, write $\sigma_x = e^{i\theta_x}$.

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- The loop-side height has a different meaning: height differences count signed loops separating points.

# Electric versus magnetic insertions

**XY side**

$$\mathbf{E}\left[e^{i(\theta(x)-\theta(y))}\right]$$

Defect in the spin phase.

- correlation functions are computable in the Gaussian approximation;
- large-scale behavior is governed by a rough field.

**Loop side**

$$h(x)-h(y)=\sum_{\gamma}\operatorname{sgn}(\gamma)\, I_{x,y}(\gamma)$$

Vortex/branch defect.

- height differences count signed nesting;
- the variance is an expected sum of squared winding numbers.

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- Loops are weighted by a fugacity $n$.
- Increasing $n$ favors having more loops, penalizes long connections.
- In a critical/rough regime, the number of loops separating two points grows like log of distance.
- Intuitively, tiny increase in $n$ makes it much more expensive to waste a lot of space on big loops.

# What is proved and what is conjectural in 2D

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# What is proved and what is conjectural in 2D

- Rigorous for XY: BKT transition, exponential decay on one side, power-law lower bounds on the other.
- Rigorous for the loop/height dictionary: the loop expansion gives exact correlations as ratios of partition functions.
- Question: whare are macroscopic joined loops for $n = 2$? Any vague relation to $\mathrm{CLE}_4$?

# 4D Euclidean Yang–Mills: the basic setup

Put a matrix on each oriented edge of $\mathbf{Z}^4$.

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$$P(x) = x^k(x+1)$$
plays the same role as in the 2D spin model.
- The analog of a loop configuration is now a surface configuration: plaquettes are glued along edges.

# U(1): exact random surfaces

For $U(1)$, write $U_e = e^{i\theta_e}$ and $Q_p = e^{i\Theta_p}$.

- The clean choice is often

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- Thus

$$\langle W(C) \rangle = \frac{Z_C}{Z_\varnothing},$$

  where $Z_C$ sums over surfaces with boundary $C$.

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- In the weak-coupling (large-$\beta$) Coulomb phase, the 2-form field is expected to look approximately Gaussian.
- But the surface geometry is a separate question: how many connected components? How much branching? How much percolation?
- A Wilson loop probes the free-energy cost of forcing a spanning surface.

## SU(2): the same expansion, but signed

Write a Haar-random matrix as

$$A = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

- The entry moments are elementary:

$$\mathbf{E}\left[A_{11}^{a} A_{12}^{b} A_{21}^{c} A_{22}^{d}\right] = 0 \quad \text{unless } a = d,\ b = c,$$

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- The only intrinsic sign comes from the $-\overline{\beta}$ entry.

## Surface expansion lemma for SU(2)

After expanding all plaquettes and integrating all edges:

- admissible configurations are labeled plaquette sheets glued along edges;

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- the weight is constant up to the local sign $(-1)^{N_{12}}$ from the $A_{12}$ term;
- the factorial factor $a!b!/(a+b+1)!$ counts the compatible pairings at an edge;
- in the fully packed case $P(x) = x^k$, the denominator depends only on the local occupancy, so after normalization all admissible surfaces have the same magnitude.

So the SU(2) model is a **signed random-surface ensemble**.

# Wilson loops in 4D: spanning surfaces

Insert a large Wilson loop

$$W(C) = \mathsf{tr}_{\mathrm{norm}}\left(\prod_{e\in C} U_e\right).$$

Then the modified expansion forces a surface with boundary $C$.

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- Area law means $-\log\langle W(C)\rangle$ grows like minimal spanning area.

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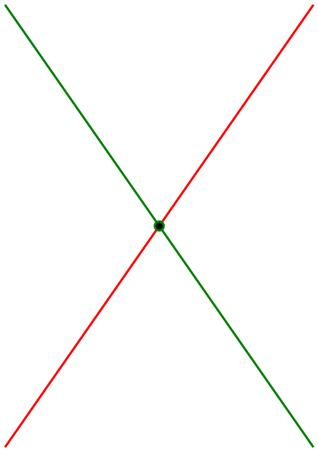
- Because $2 = 1 + 1$. This leads to many special properties, more to come.
- Why is 4 a special dimension for Yang-Mills?
- Because $4 = 2 + 2$. This leads to a related set of special properties.

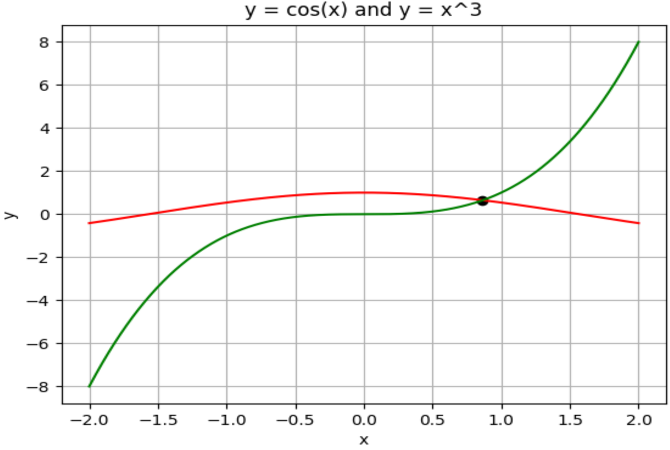
# codimension 1 + codimension 1 = codimension 2

Non-parallel lines in plane meet at single point.



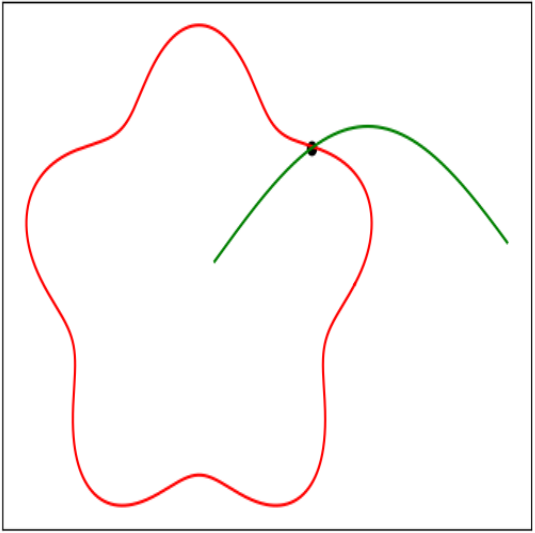
# non-linear continuous functions (intermediate value thm)

If **red function higher on left**, **green higher on right**, must be point of intersection. Car on narrow one-lane road cannot "pass" another without collision.

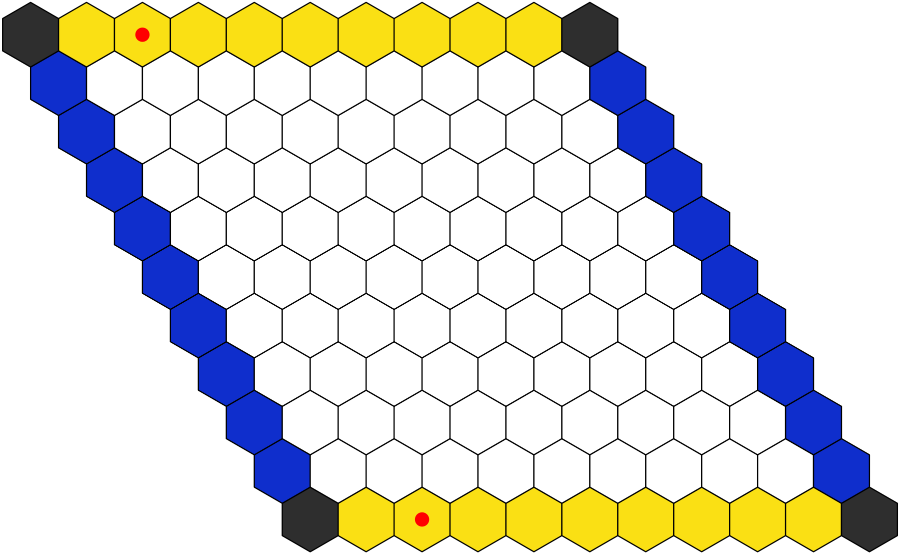


# Jordan curve theorem

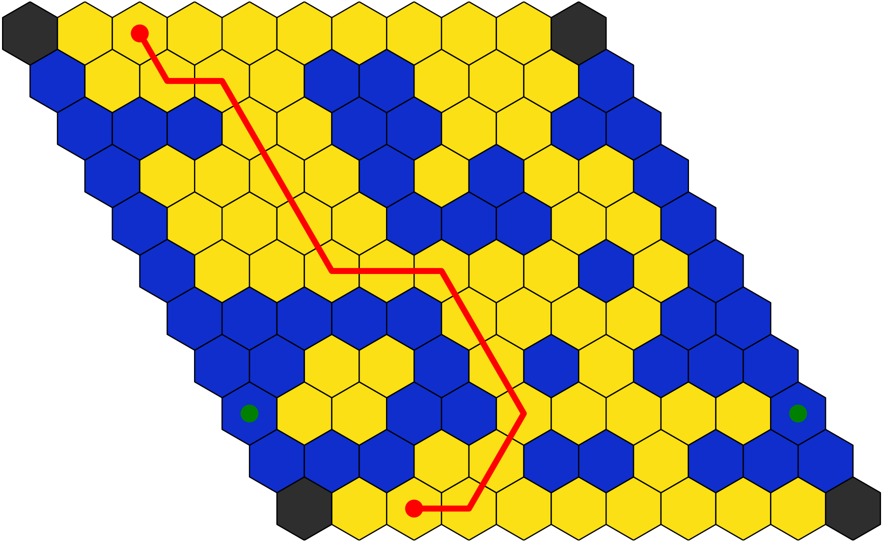
Non-self-intersecting closed loop (red) divides plane into inside and outside. Green path from inside to outside must intersect the red curve.



## If inner hexagons randomly colored blue/yellow, then either



# path through yellow connects dots on yellow sides or...



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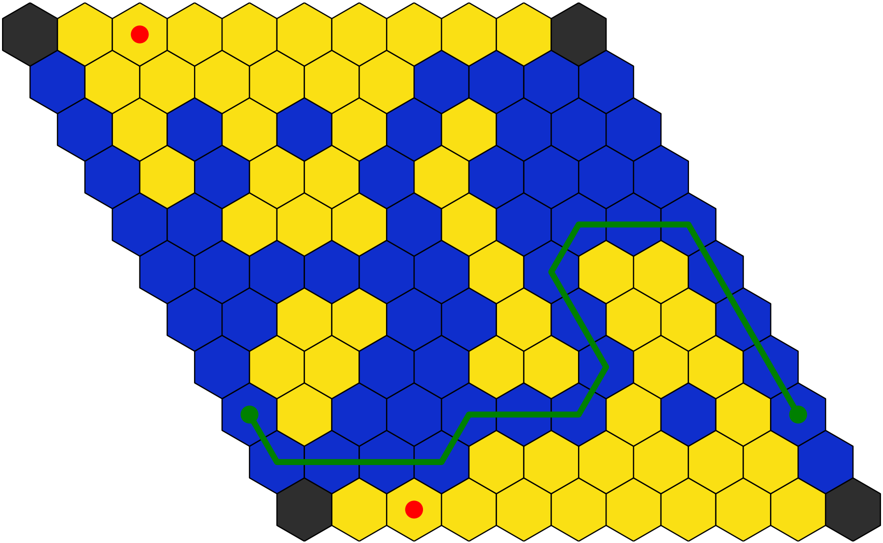
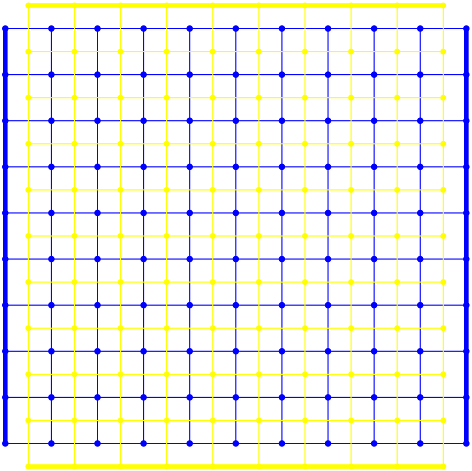
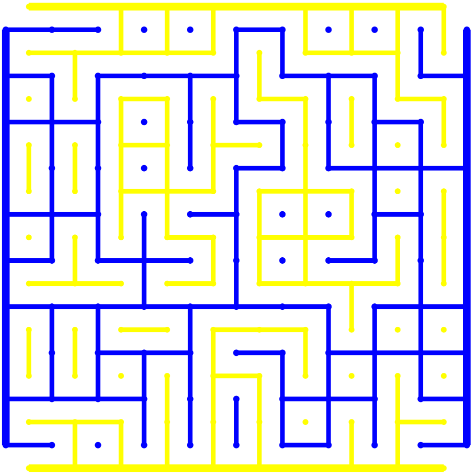


If we toss a coin at each blue/yellow crossing to decide which edge to keep...



# result either has yellow top-bottom or blue left-right path



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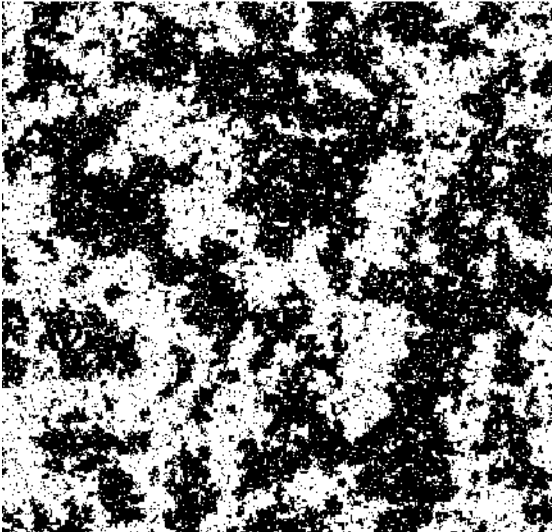
- Similar story for bond percolation, or if colors chosen from e.g. 2D Ising model (toy magnetism model).

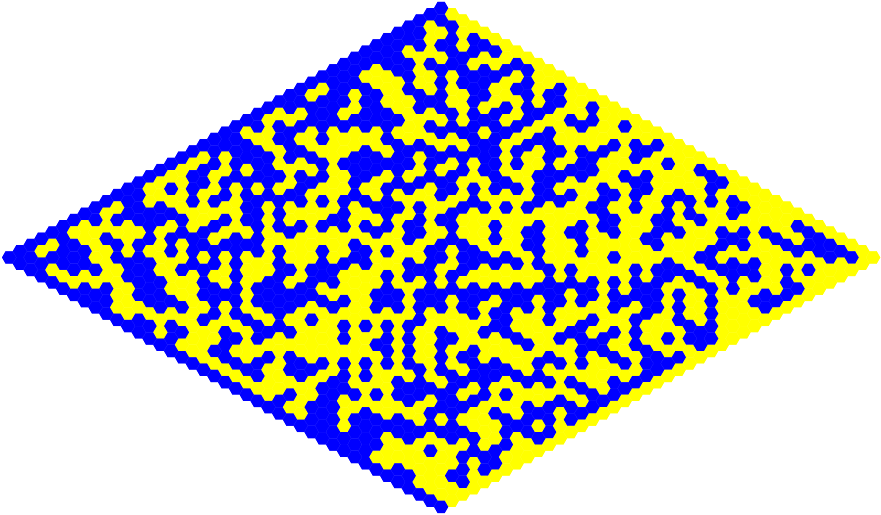
# Critical 2D Ising on fine grid

Scaling limit of 2D **Ising Model** is well understood (see Schramm, Smirnov, Chelkak, Hongler, Werner, Duminil-Copin, many others in physics and math).



# Fix boundary to obtain distinguished blue-yellow interface

Limiting curves exist in many settings. Schramm-Loewner evolution with $\kappa = 2$ (LERW), $\kappa = 3$ (Ising interface), $\kappa = 4$ (GFF level line), $\kappa = 6$ (percolation interface), $\kappa = 8$ (UST boundary).



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- Conclude that Dirichlet energy of $f$ equals Dirichlet energy of $g$, i.e., Dirichlet energy is invariant under domain rescaling (as well as rotation/translation).
- A function $\phi$ that "looks locally" like a rotation/translation/rescaling is called *conformal*.

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- Let us now consider a "domain-rescaled" version of $f$.
- Write $g(x) = f(x/c)$. What is the Dirichlet energy of $g : cD \to \mathbb{R}$?
- $$\nabla g(x) = \frac{1}{c}\nabla f(x/c) \quad \text{so} \quad |\nabla g(x)|^2 = \frac{1}{c^2}|\nabla f(x/c)|^2$$
- Switching from $D$ to $cD$ stretches area by $c = c^1$ in one coordinate direction and $c = c^1$ in the other, hence $c^1 c^1 = c^{1+1} = c^2$ overall.
- Conclude that Dirichlet energy of $f$ equals Dirichlet energy of $g$, i.e., Dirichlet energy is invariant under domain rescaling (as well as rotation/translation).
- A function $\phi$ that "looks locally" like a rotation/translation/rescaling is called *conformal*.
- If we write $h = f(\phi(x))$ where $\phi$ is conformal map from a domain $\tilde{D}$ to $D$, then similar argument shows that Dirichlet energy of $h$ is same as Dirichlet energy of $f$.

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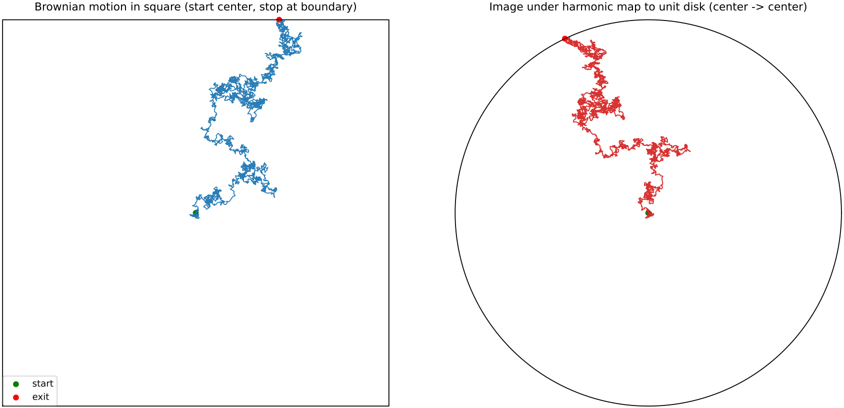
- Exit probabilities for random path called “2D Brownian motion” are governed by harmonic functions. Implies Brownian motion is conformally invariant up to time change.

# Brownian motion in square and conformal image

Ultra high-resolution: Left: Brownian path in square. Right: mapped image in disk (harmonic extension approx).



## “Gaussian free field” is “conformally invariant”

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- Makes rigorous sense if restrict $f$ to finite dimensional space of functions. Take limit as space gets larger, define $f$ as random generalized function.
- Conformal invariance of Dirichlet energy implies conformal invariance of Gaussian free field.

# Uniform spanning tree

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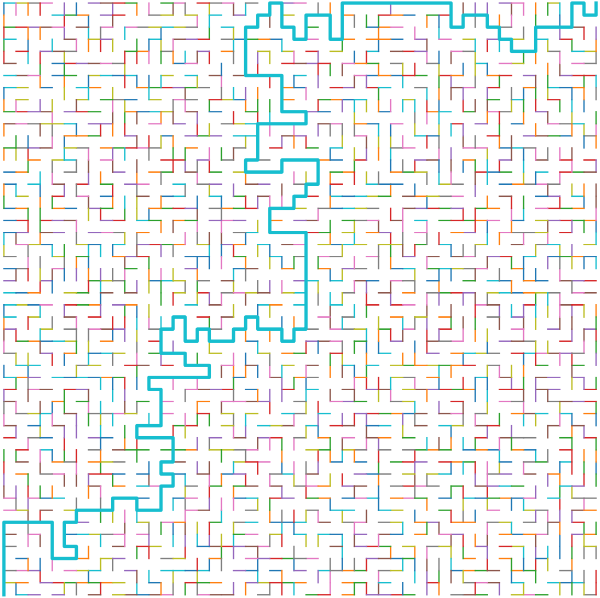
- Correlations in uniform spanning tree related to those of Gaussian free field.

# Uniform spanning tree on grid with corner-corner path



# Duality of forms

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- "Gradient vector fields" (a.k.a. "curl–free vector fields") can be transformed into "divergence free vector fields" by rotating all vectors 90 degrees.
- One-form white noise can be orthogonally projected onto curl-free and divergence-free components. The former is the gradient of a Gaussian free field instance.

## Perspective on $1 + 1 = 2$

- Math physics results in 2 (and higher) dimensions have generated excitement in math community (see e.g. 5 recent Fields medals: Okounkov, Werner, Smirnov, Duminil-Copin, Hairer).

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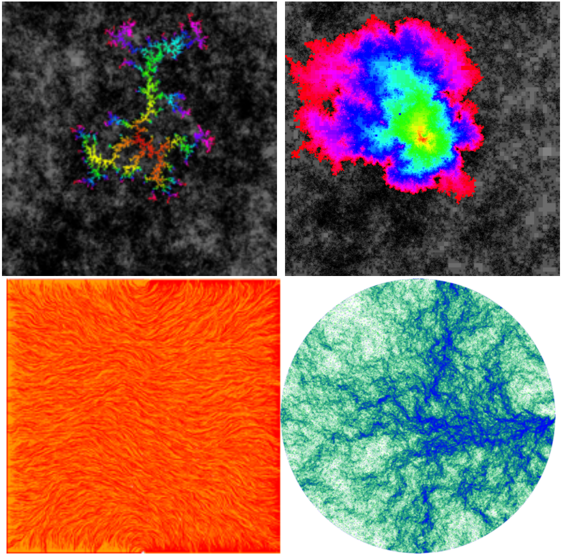
- Sizable branch of mathematical physics called *conformal field theory* has explored this area for several decades. A related theory of *random surfaces* has emerged. Many experts in our collaboration.
- In some cases we can rigorously construct "off-critical scaling limits" that look like the models above at small scales but differ at large scales: off-critical percolation, off-critical Ising model, off-critical uniform spanning tree (Sine-Gordon model), etc.

# DLA, Brownian sphere, imaginary geometry, mating trees...



Simulations by Jason Miller. Nice pictures in 2D. Exciting work by many people.

## 2+1=3 and spanning trees/dual plaquette trees

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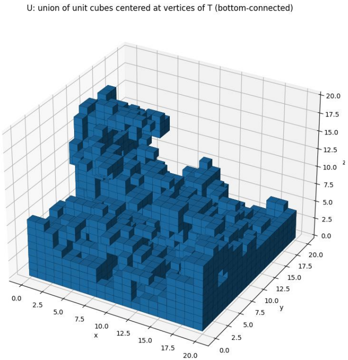

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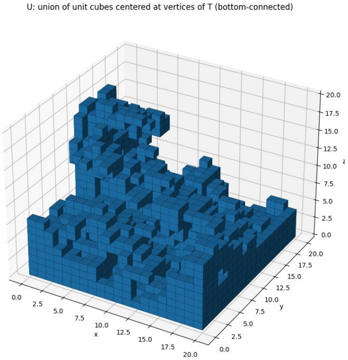
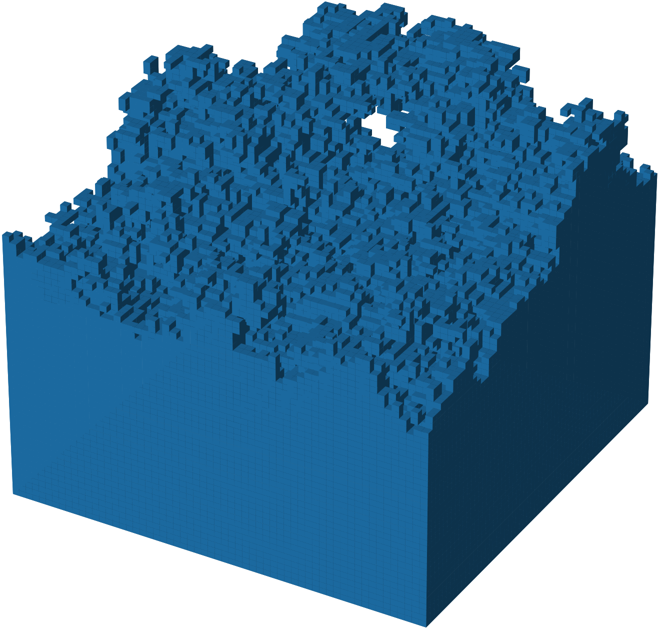
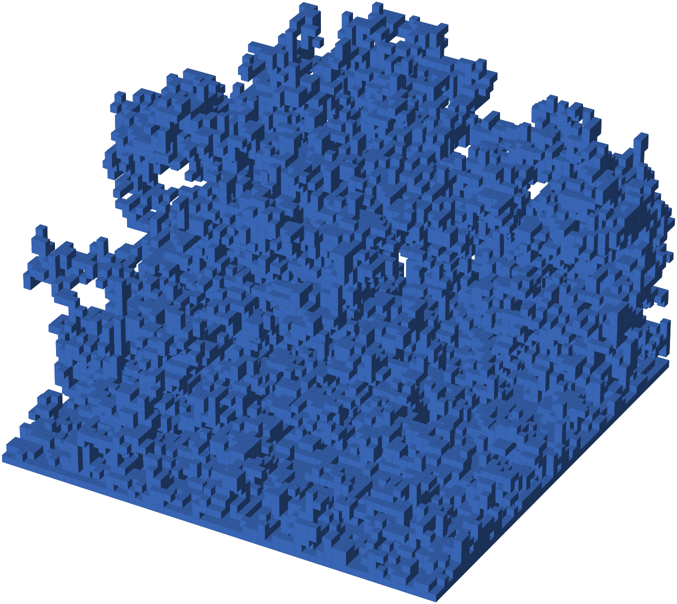



- Rooted critical bond percolation cluster also has dual boundary surface. Compare large tree surface and (rougher looking) percolation surface.





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- **4D spanning plaquette trees** are self dual. Just like **2D spanning (edge) trees**. (Duval, Klivans, Martin, many others) Similar relationships to Gaussian fields and discrete Laplacian determinants.

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- **UPSHOT:** In the lattice version of the famous Yang-Mill problem, the so-called *Wilson loop expectation* of a loop $L$ is equivalent to the *insertion cost* of $L$ in a related random surface model. Some parts of this theory are classical but there have been many recent developments.

## Refine big goal: related goals, starting with $U(N)$ say

- **BIG DISCRETE TASK ONE:** Expand set of $(\beta, N, d)$ for which we prove area law and exponential correlation decay in lattice models. Use surface sums to give intuition in cases where formal proofs remain out of reach. Fix $d$, and consider $(N, \beta) \in \mathbb{Z}_+ \times \mathbb{R}_+$. Area law easy to prove near $\beta$-axis (big $N$, small $\beta$). Depending on $d$, area law may be false or true but hard to prove near $N$-axis (big $\beta$, small $N$).

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- **BIG CONTINUUM TASK TWO:** Understand scaling limit of model obtained when $\beta, N \to \infty$ together and lattice size tends to zero. Make/strengthen connection to supercritical Liouville quantum gravity. Start with $d = 2$ and express some of the known Wilson loop expectation formulas in terms of LQG limits.

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- At "very low temperature" lattice Yang-Mills looks like a 1-form Lie-algebra-valued GFF and the "low-temperature" fine-mesh limit is a continuum Lie-algebra-valued GFF. Can define Wilson-loop holonomy for "averages of loops" or certain (infinitely long) "big loops." If temperature scales to zero more slowly as mesh size tends to zero... is there a non-Gaussian "off critical" limit?

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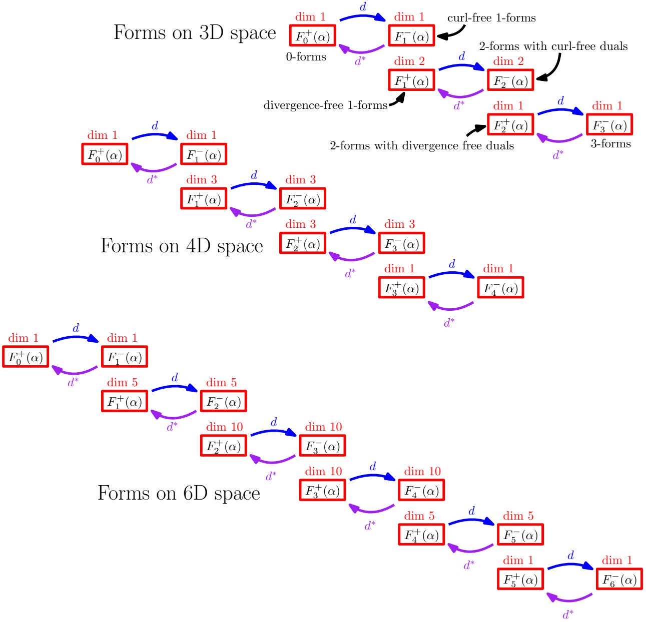
- A "connection" can be understood as a Lie-algebra-valued 1-form. In discrete case, this is matrix-valued function on edges.
- A "Wilson loop expectation" in $U(N)$ lattice Yang-Mills is equivalent to the "partition function" of a "discrete closed 2-form model" in the presence of a "loop-shaped defect."
- Define $d$ and $d^*$ operators on space of forms. If $f$ is a form, can define $(f, f)_d$ to be $L^2$ norm of $df$. If $f$ is a 0-form (an ordinary function) then $(f, f)_\nabla = (f, f)_d$. Note that $(d + d^*)^2 = -\Delta$ and $(f, f)_{d+d^*} = (f, -\Delta f)$. Can define GFF, white noise, fractional Gaussian field, etc. on space of $k$-forms in straightforward way.
- At "very low temperature" lattice Yang-Mills looks like a 1-form Lie-algebra-valued GFF and the "low-temperature" fine-mesh limit is a continuum Lie-algebra-valued GFF. Can define Wilson-loop holonomy for "averages of loops" or certain (infinitely long) "big loops." If temperature scales to zero more slowly as mesh size tends to zero... is there a non-Gaussian "off critical" limit?
- For analogy, consider dimer model: 2-forms replaced by div-free 1-forms, *loop shaped defects* become *pair-of-point defects* and *spanning surfaces* become *spanning paths*. (If $U(N)$ is replaced by $O(N)$ or $SU(N)$, may replace dimer with e.g. Ising or Potts when making analogy.)

# Discrete and continuous Gaussian forms: $(d + d^*)^2 = -\Delta$



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8. In both cases, can ask what happens when we "discretize" by *conditioning* on value at each edge (resp. plaquette) being fixed multiple of $\epsilon$. Interesting phase transition?

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- $\alpha$ is also product of all non-zero eigenvalues of matrix $M$.
- The DGFF partition function can be be written $\int_R (2\pi)^{-|V-1|/2} e^{-(f,-\Delta f)/2} df = \alpha^{-1/2}$.