Dubédat Screening and Level Two Non-Degeneracy

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imagine the impossible

KIAS KOREA INSTITUTE FOR ADVANCED STUDY

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Kang-Makarov CFT

Kang-Makarov CFT

- Based on the Gaussian Free Field (GFF)
- GFF is a convenient tool for representing **correlation functions**
- Primarily uses tools/ideas of
    - complex differential geometry (coordinate free approach)
    - potential theory on Riemann surfaces
    - Gaussian analysis
- **Penultimate result:** Infinite family of correlation functions that are martingale observables for SLE type processes

GFF and Correlation Functions

- **In this talk:** $D \subset \mathbb{C}$ a simply connected domain
- $\Phi : D \to \mathbb{R}$ a random Gaussian “function” with

$$(\Phi(z) : z \in D) \sim N(0, G_D)$$

  for $G_D : D \times D \to \mathbb{R}$ the (Dirichlet) Green’s function of D
- Correlation function is

$$\mathbb{E}[\mathbb{C}\text{-valued functional of } \Phi],$$

  primarily point evaluations of Φ

GFF and Correlation Functions

- **Examples:**

$$\mathbb{E}[\Phi(z)\Phi(w)] = G_D(z,w)$$

$$\mathbb{E}[\partial\Phi(z)\Phi(w)] = \partial_z G_D(z,w)$$

$$\mathbb{E}[(\partial\Phi(z)\odot\partial\Phi(z))\,(\partial\Phi(w)\odot\partial\Phi(w))] = 2\partial_z\partial_w G_D(z,w)$$

$$\mathbb{E}[(\Phi(z)\odot\Phi(z)\odot\Phi(z))\,(\Phi(w)\odot\Phi(w))] = 0$$

Evaluating Correlation Functions

- **Wick Calculus:** for $Z \sim \mathrm{Normal}(\mu, \Sigma)$ in $\mathbb{R}^n$

$$\mathbb{E}[f(Z+\mu)] = \mathrm{Wick\text{-}Calculus}(\mu, \Sigma, f)$$

  for $f : \mathbb{R}^n \to \mathbb{R}$ analytic
- Power series expansion of f plus Hermite polynomials and combinatorics (Feynman diagrams)

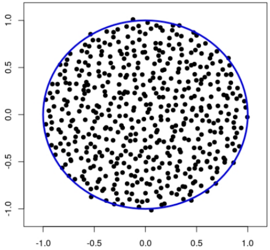
Divisors

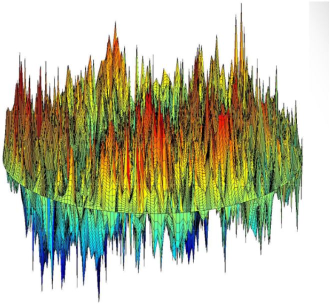
- Add a mean to Φ by introducing a **divisor**

$$\boldsymbol{\beta} = \sum_k \beta_k \cdot q_k$$

- Terminology comes from algebraic geometry
- Divisor specifies location and background of deterministic charges
- Called the **background charge**
- Induces a deterministic mean $\varphi_{\boldsymbol{\beta}} : D \to \mathbb{C}$ for the GFF

CFT and electrostatics





- **Ginibre:** Eigenvalues of $n \times n$ matrix with iid $N_{\mathbb{C}}(0, n^{-1/2})$ entries
- Take two independent copies, give one $+$ charge and the other $-$ charge
- Potential converges as $n \to \infty$ to **Gaussian Free Field**

CFT and electrostatics

- GFF = potential of a background “soup” of $\pm$ particles
- Statistics of potential determined by geometry of surface
- Potential is shifted by the presence of fixed deterministic charges (divisor)
- Observables = expectation of “functionals” of the potential, evaluated at points on the surface

Divisors

Each divisor $\boldsymbol{\beta} = \sum_k \beta_k \cdot q_k$ determines:

- A number $b \in \mathbb{R}$ through the relation

$$\int \boldsymbol{\beta} = \sum_k \beta_k = 2b$$

- Value $\kappa > 0$ through $b = \sqrt{\kappa/8} - \sqrt{2/\kappa}$
- Deterministic mean $\varphi_{\boldsymbol{\beta}} : D \to \mathbb{C}$
- Deterministic partition function $C[\boldsymbol{\beta}]$ (**Coulomb gas correlation function**)
- Special “growth” points q_k where

$$\beta_k \in \{a, -a-b\} = \{\sqrt{2/\kappa}, -\sqrt{\kappa/8}\}$$

- SLE curve from any growth point with driver

$$dX = \sqrt{\kappa}\, dB + \kappa\, \partial_x \log C[\boldsymbol{\beta}]\, dt$$

- Family $\mathcal{F}_{\boldsymbol{\beta}}$ of correlation functions derived from $\Phi + \varphi_{\boldsymbol{\beta}}$

Martingale property

Theorem

Let $\boldsymbol{\beta}$ be a divisor with $\boldsymbol{\beta}(x) = a$ at some point $x \in \partial D$. Let $\mathbb{E}[F(\Phi + \varphi_{\boldsymbol{\beta}})]$ be a correlation function in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$. Then

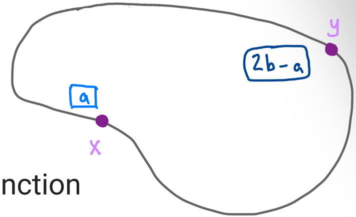
$$t \mapsto \mathbb{E}[F(\Phi + \varphi_{\boldsymbol{\beta}})] \,||\, g_t^{-1}$$

is a martingale for the SLE system (in $\mathbb{H}$)

$$dX = \sqrt{\kappa}\, dB + \kappa\, \partial_x \log C[\boldsymbol{\beta}]\, dt, \quad \partial_t g_t(z) = \frac{2}{g_t(z) - X_t},$$

and all other points in $\boldsymbol{\beta}$ follow the flow of g_t.

Martingale example



- Let $\Phi_{\boldsymbol{\beta}} = \Phi + \varphi_{\boldsymbol{\beta}}$ and take the correlation function

$$M := \mathbb{E}[\Phi_{\boldsymbol{\beta}}(z_1)\bar{\partial}\Phi_{\boldsymbol{\beta}}(z_2)(\partial\Phi_{\boldsymbol{\beta}} * \Phi_{\boldsymbol{\beta}})(z_3)e^{*i(\Phi_{\boldsymbol{\beta}}(z_4)-\Phi_{\boldsymbol{\beta}}(z_5))}]$$

- Conformal transformation rule of $\Phi_{\boldsymbol{\beta}}$ gives conformal transformation rule of M at $z_1, \ldots, z_5$
- $t \mapsto M \,||\, g_t^{-1}$ is evaluation of **deterministic** correlation function on **random** charts
- $M = F(z_1, \ldots, z_5, x, y) \implies$

$$M \,||\, g_t^{-1} = \overline{g_t'(z_2)}^{\alpha_1} g_t'(z_3)^{\alpha_2} g_t'(z_4) g_t'(z_5) F(g_t(z_1), \ldots, g_t(z_5), X_t, g_t(y))$$

- Theorem guarantees it is a martingale for SLE from x to y

Ingredients of Martingale Property

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- Ito formula
- Ward's Identity

$$\mathcal{L}_{k_x}\mathbb{E}F(\Phi + \varphi_{\boldsymbol{\beta}}) = \mathbb{E}[A_{\boldsymbol{\beta}}(x)F(\Phi + \varphi_{\boldsymbol{\beta}})]$$

  for $k_x(z) = (z-x)^{-1}$, $A_{\boldsymbol{\beta}}$ the **stress tensor**
- Cameron-Martin/Girsanov Theorem

$$\mathbb{E}\left[F(\Phi + \varphi_{\boldsymbol{\beta}})\right] = \mathbb{E}\left[e^{\odot i\Phi[\boldsymbol{\beta}-\boldsymbol{\beta}_0]}F(\Phi + \varphi_{\boldsymbol{\beta}_0})\right]$$

- Operator Product Expansion
- Level Two Degeneracy of Vertex Operators

Ward Identity

- Ward identity

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  for $k_x(z) = (z-x)^{-1}$
- Integration-by-parts formula for Gaussian measure
- Comes from the action that defines the “Gibbs measure” for GFF
- Action takes background charge into account
- $A_{\boldsymbol{\beta}}(z)$ = quadratic function of $\partial\Phi(z)$
- Quadratic has non-random coefficients determined by $\boldsymbol{\beta}$
- **Only involves a first order derivative in x!**

1st Order $\to$ 2nd Order

- Ward equation computes first order derivatives

$$\mathcal{L}_{k_x}\mathbb{E}F(\Phi+\varphi_{\boldsymbol\beta}) = \mathbb{E}[A_{\boldsymbol\beta}(x)F(\Phi+\varphi_{\boldsymbol\beta})]$$

- k_x as singularity at x. Compute Lie derivative for $k_{x+\delta}$ and take $\delta\to 0$:

$$\begin{aligned}\mathcal{L}_{k_{x+\delta}}\mathbb{E}F(\Phi+\varphi_{\boldsymbol\beta}) &= \mathcal{L}_{k_{x+\delta}}\mathbb{E}[e^{\odot ia\Phi(x)}F(\Phi+\varphi_{\boldsymbol\beta-a\cdot x})] \\ &= \mathbb{E}[A_{\boldsymbol\beta-a\cdot x}(x+\delta)e^{\odot ia\Phi(x)}F(\Phi+\varphi_{\boldsymbol\beta-a\cdot x})]\end{aligned}$$

- Laurent expansion of both sides in δ
  - LHS by residue calculus
  - RHS by Wick calculus
- **Result:** δ^{-k} terms match for all k
- Ward forces that δ^0 terms match

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- δ^0 terms:

$$\text{LHS is } \check{\mathcal{L}}_{k_x}\mathbb{E}F(\Phi + \varphi_{\boldsymbol\beta}), \quad \text{RHS is } \mathcal{G}_x\mathbb{E}F(\Phi + \varphi_{\boldsymbol\beta}),$$

  for $\mathcal{G}_x =$ generator of $dX = \sqrt{\kappa}\, dB + \kappa\, \partial_x \log C[\boldsymbol\beta]\, dt$
- Equality means that drift of $M \,||\, g_t^{-1}$ vanishes

1st Order $\to$ 2nd Order

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- $\mathcal{G}_x$ term appears by Wick expansion analysis of

$$A_{\boldsymbol\beta - a\cdot x}(x+\delta)e^{\odot ia\Phi(x)}$$

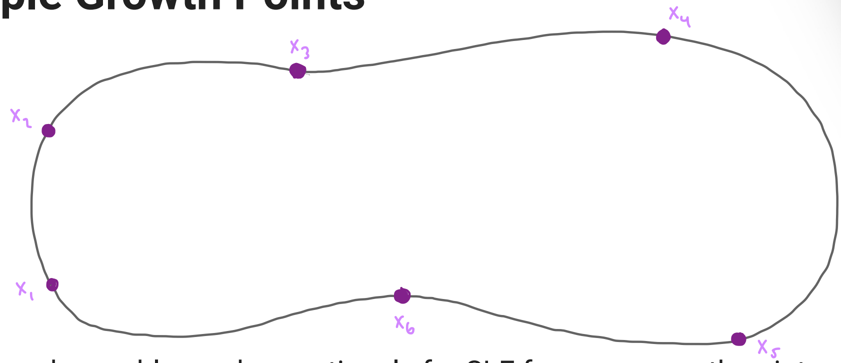
as $\delta \to 0$. δ^0 term is $\mathcal{G}_x e^{\odot ia\Phi(x)}$.

Martingale Observable Property

- Program works in more complicated geometries (Ward changes)
- Should work for vector fields with higher order poles (fusion)
- Martingale observable property is robust with respect to **screening**

Multiple Growth Points

Multiple Growth Points



- Same observable can be martingale for SLE from any growth point
- Allowed functionals are $\Phi_{\boldsymbol{\beta}}$, its derivatives, tensor products, and **OPE coefficients**

$$M := \mathbb{E}[\Phi_{\boldsymbol{\beta}}(z_1)\bar{\partial}\Phi_{\boldsymbol{\beta}}(z_2)(\partial\Phi_{\boldsymbol{\beta}} * \Phi_{\boldsymbol{\beta}})(z_3)e^{*i(\Phi_{\boldsymbol{\beta}}(z_4)-\Phi_{\boldsymbol{\beta}}(z_5))}]$$

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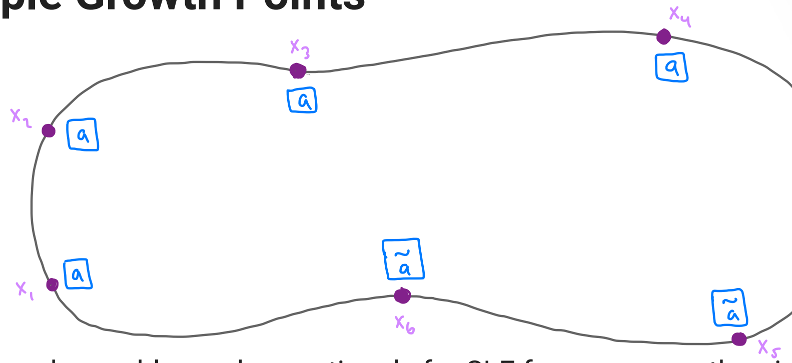

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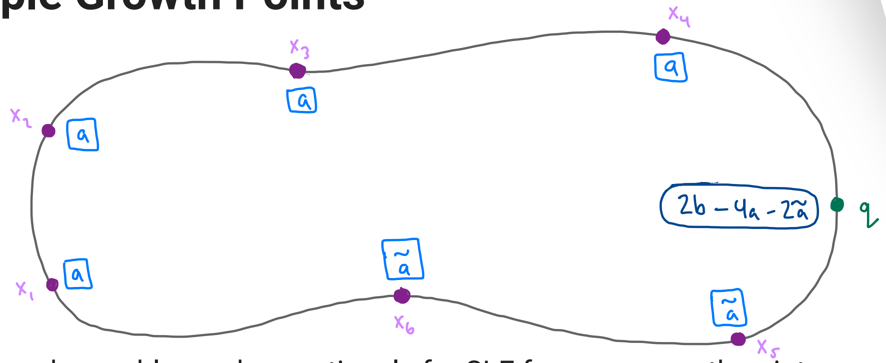

- Same observable can be martingale for SLE from any growth point
- Allowed functionals are Φ_{β}, its derivatives, tensor products, and **OPE coefficients**

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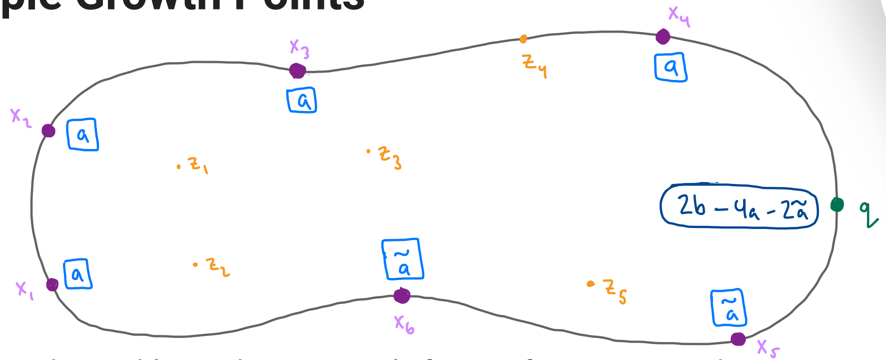

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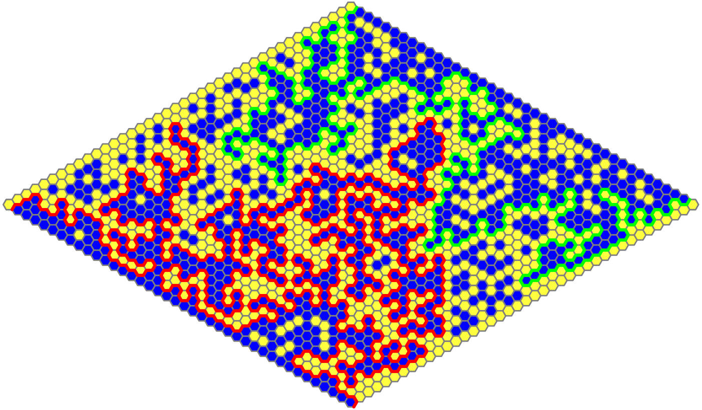
Multiple SLE

Useful in the study of **multiple SLE**



Scaling limit of interface curves with alternating boundary conditions

Multiple SLE

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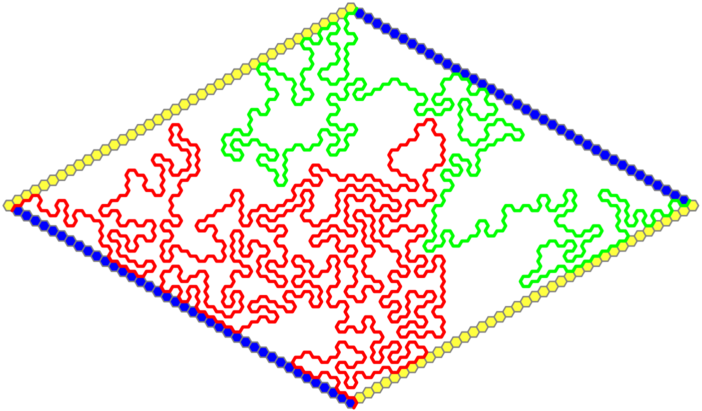
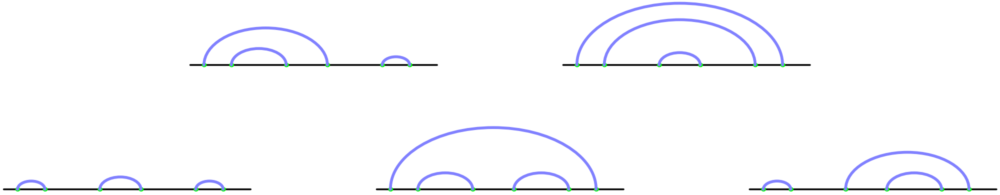


Scaling limit of interface curves with alternating boundary conditions

Loewner Construction of Multiple SLE

- **Goal:** In $\mathbb{H}$ with $x_1 < x_2 < \ldots < x_{2n}$, $x_i \in \mathbb{R}$, use Loewner flow to construct n SLE curves that connect x_i in a non-crossing way
  - *Pure law:* n curves almost surely connect according to fixed pattern
  - *Mixture:* Positive linear combination of pure laws



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- **Goal:** In $\mathbb{H}$ with $x_1 < x_2 < \ldots < x_{2n}$, $x_i \in \mathbb{R}$, use Loewner flow to construct n SLE curves that connect x_i in a non-crossing way
  - *Pure law:* n curves almost surely connect according to fixed pattern
  - *Mixture:* Positive linear combination of pure laws
- **Ansatz:** There exists a function $Z = Z(x_1, \ldots, x_{2n})$ such that curve from x_i is SLE with driver

$$dX = \sqrt{\kappa}\, dB + \kappa \partial_{x_i} \log Z(x_1, \ldots, x_{2n})\, dt$$

Multiple SLE Literature

Global constructions

Weight the product measure of individual SLE curves by an appropriate factor. Iterated re-sampling methods.
[Lawler-Kozdron, Lawler, Peltola-Wu, Beffara-Peltola-Wu, . . .]

Local construction via Loewner flows

Grow multiple curves by adding singularities to the Loewner equation, vector of singularities is a multi-dimensional diffusion.
[Dubédat, Zhan, Bernard-Bauer-Kytölä, Kytölä-Peltola, . . .]

Conformal weldings of LQG surfaces

[Ang-Holden-Pu-Sun]

Dubédat Screening

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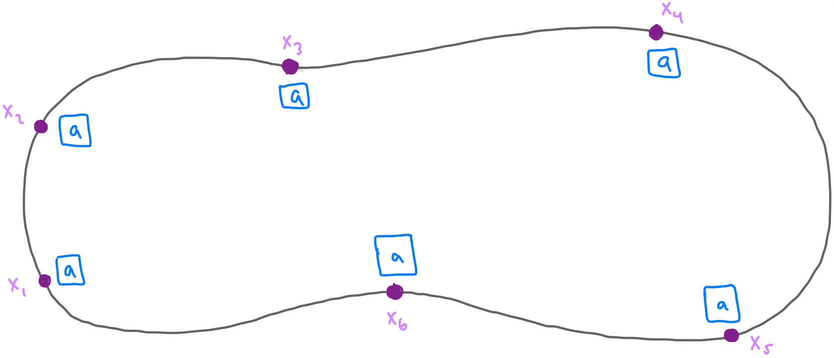
- Dubédat proposal:
  - Introduce a specific divisor β with additional **screening variables**
  - Integrate out screening variables from $C[\beta]$ to produce a partition function Z that satisfies the requirements of multiple SLE



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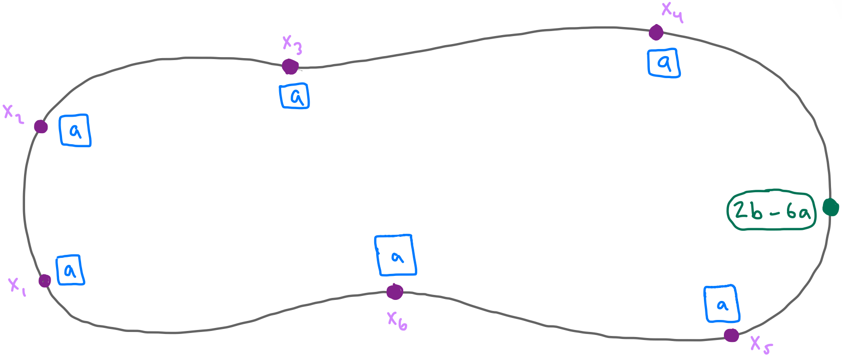

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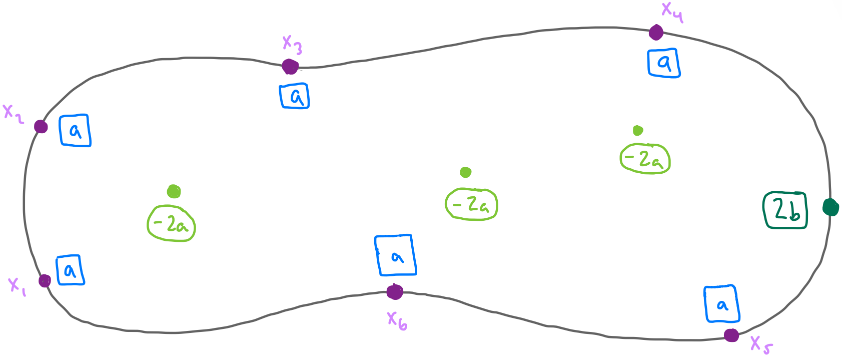


Screening Variables

- Charges in divisor β with

$$\beta_k \in \{-2a, 2a + 2b\}$$

- $C[\beta]$ is a $[1,0]$-differential at screening variables

$$\beta_k \in \{-2a, 2a + 2b\} \implies \lambda_b(\beta_k) = 1$$

- **Geometric facts:**
  - $[1,0]$-differentials can be integrated out along contours
  - output is still **coordinate free**
  - Lie derivative of $[1,0]$-differential is a total derivative

$$\mathcal{L}_v f = v\partial f + 1 \cdot v' f = \partial(vf)$$

  - Implies $\oint_{\mathcal{C}} \mathcal{L}_v f$ vanishes for closed contours

Screened Coulomb Gas Correlation Function

- If divisor β has screening variables $\zeta_1, \ldots, \zeta_k$ plus other variables $z_1, \ldots, z_j$ then

$$\begin{aligned}\mathcal{L}_v C[\beta] &= \sum_i \mathcal{L}_v(\zeta_i) C[\beta] + \sum_i \mathcal{L}_v(z_i) C[\beta] \\ &= \sum_i \partial_{\zeta_i}(v(\zeta_i) C[\beta]) + \sum_i \mathcal{L}_v(z_i) C[\beta]\end{aligned}$$

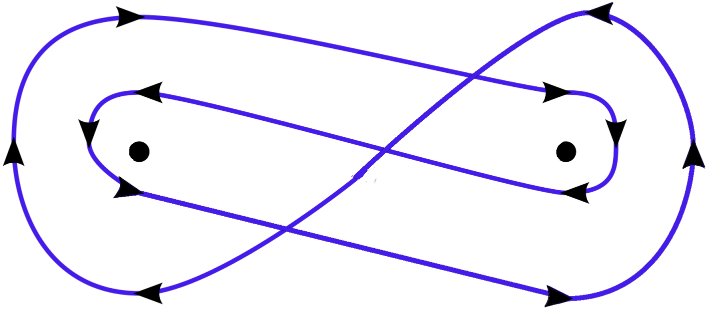
- Integrate ζ_j along **closed** contours $\mathcal{C}_1, \ldots, \mathcal{C}_k$

$$\mathcal{L}_v \oint_{\mathcal{C}_1} \ldots \oint_{\mathcal{C}_k} C[\beta]\, d\zeta_1 \ldots d\zeta_k = \sum_i \mathcal{L}_v(z_i) \oint_{\mathcal{C}_1} \ldots \oint_{\mathcal{C}_k} C[\beta]\, d\zeta_1 \ldots d\zeta_k$$

- Contours must deal with monodromy and not make everything zero

Screened Coulomb Gas Correlation Functions

- Common choice of contour:



Pochhammer

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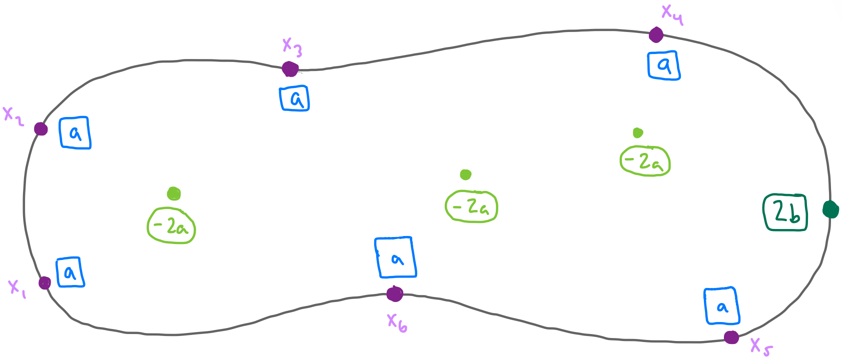

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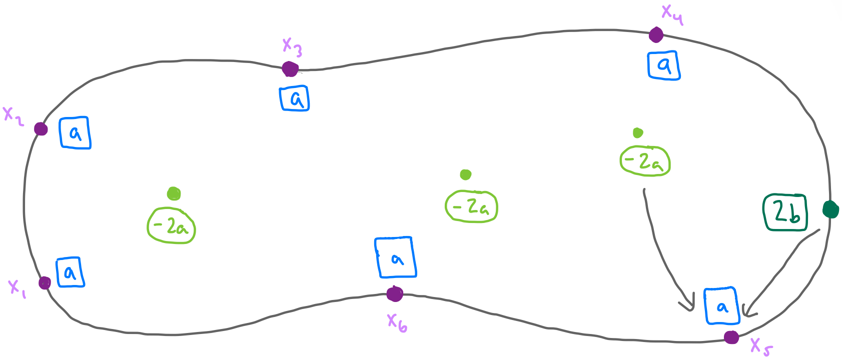

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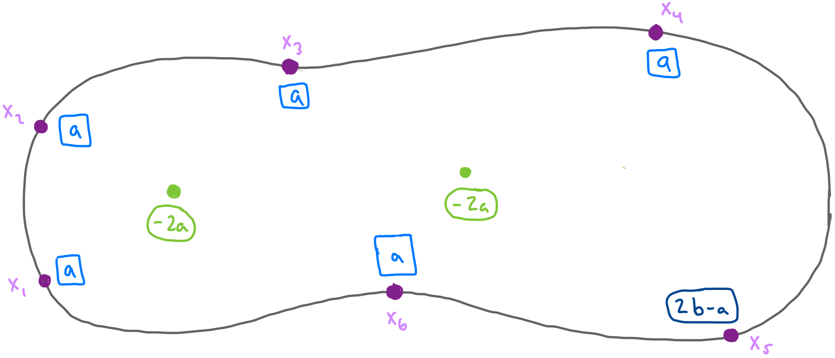

Screened Coulomb Gas Correlation Function

Theorem

Let β be the Dubédat divisor with screening variables $\zeta_1, \ldots, \zeta_{n-1}$. Let

$$Z = \oint_{\mathcal{C}_1} \ldots \oint_{\mathcal{C}_{n-1}} C[\boldsymbol{\beta}]\, d\zeta_1 \ldots d\zeta_{n-1}.$$

Then $Z = Z(x_1, \ldots, x_{2n})$ will generate multiple SLE curves. For multiple SLE grown from $\beta(x_i) = a$ the screened correlation function

$$\oint_{\mathcal{C}_1} \ldots \oint_{\mathcal{C}_{n-1}} \mathbb{E}[F(\Phi + \varphi_{\boldsymbol{\beta}})]\, d\zeta_1 \ldots d\zeta_{n-1}$$

is a martingale observable.

Screened Martingale Observables

- **Takeaway:** Screening preserves martingale observable property, *at the a charges*
- What about at the special $2b - a$ charge?

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- **Answer:** No! Extra drift terms appear. But they can be computed via CFT.

Screened Martingale Observables

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- **Takeaway:** Screening preserves martingale observable property, *at the* a *charges*
- What about at the special $2b - a$ charge?
- **Answer:** No! Extra drift terms appear. But they can be computed via CFT.
- **More nuance:**
  - BPZ equation still holds at $2b - a$ charge
  - Martingale observable property fails at the $2b - a$ charge

Level Two Non-Degeneracy

- **Level two degeneracy:** If $\boldsymbol{\beta}(x) = a$ then

$$\mathcal{L}_{k_{x+\delta}} \mathbb{E} F(\Phi + \varphi_{\boldsymbol{\beta}}) = \mathbb{E}[A_{\boldsymbol{\beta}-a\cdot x}(x+\delta) e^{\odot i a \Phi(x)} F(\Phi + \varphi_{\boldsymbol{\beta}-a\cdot x})]$$

- δ^0 terms:

$$\text{LHS is } \check{\mathcal{L}}_{k_x} E[F(\Phi + \varphi_{\boldsymbol{\beta}})], \quad \text{RHS is } \mathcal{G}_x E[F(\Phi + \varphi_{\boldsymbol{\beta}})]$$

- Wick calculus on $A_{\boldsymbol{\beta}-a\cdot x}(x+\delta) e^{\odot i a \Phi(x)}$ as $\delta \to 0$ has δ^0 term

$$\mathcal{G}_x e^{\odot i a \Phi(x)}$$

Level Two Non-Degeneracy

Level Two Non-Degeneracy

- **Level two non-degeneracy:** If $\beta(x) = 2b - a$ then

$$A_{\boldsymbol{\beta}-2b-a\cdot x}(x+\delta)e^{\odot i(2b-a)\Phi(x)}$$

- δ^0 terms:

$$\text{LHS is } \check{\mathcal{L}}_{k_x} E[F(\Phi + \varphi_{\boldsymbol{\beta}})], \quad \text{RHS is more complicated}$$

- Via much Wick calculus the δ^0 term of $A_{\boldsymbol{\beta}-2b-a\cdot x}(x+\delta)e^{\odot i(2b-a)\Phi(x)}$ is

$$\mathcal{G}_x e^{\odot i(2b-a)\Phi(x)} + \text{explicit extra terms}$$

- CFT can be used to compute the explicit extra terms
- Dimension of δ^0 term at x = $2+$ dimension of $C[\beta]$ at x (**level two**)