BALLISTIC AGGREGATION DISPLAYS SELF-ORGANIZED CRITICALITY

Krzysztof Burdzy

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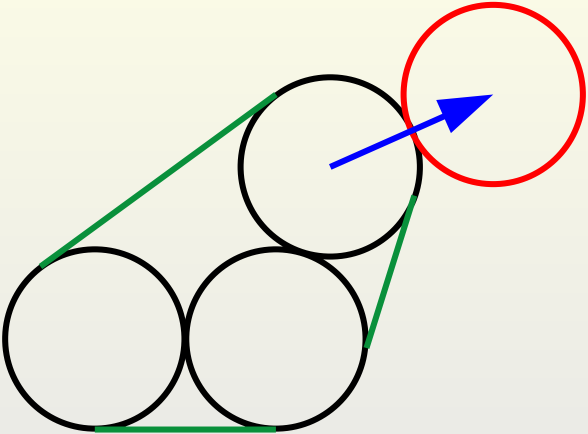
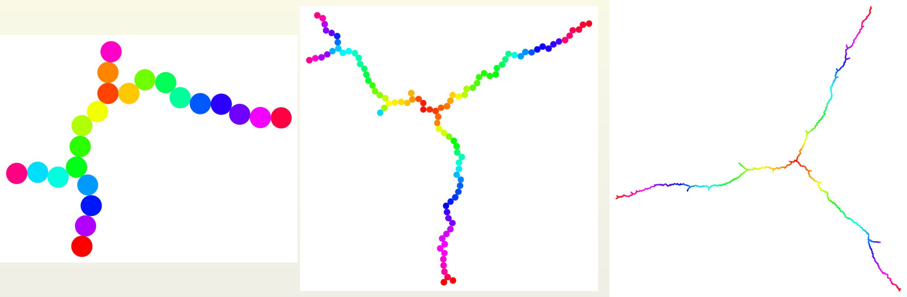
Special thanks to (professor/colleague) Stefan Steinerberger.

Ballistic Aggregation



Ballistic Aggregation: simulations



Color indicates the order of attachment.
The numbers of discs in the clusters are 20, 100 and 1,000.

Steinerberger model

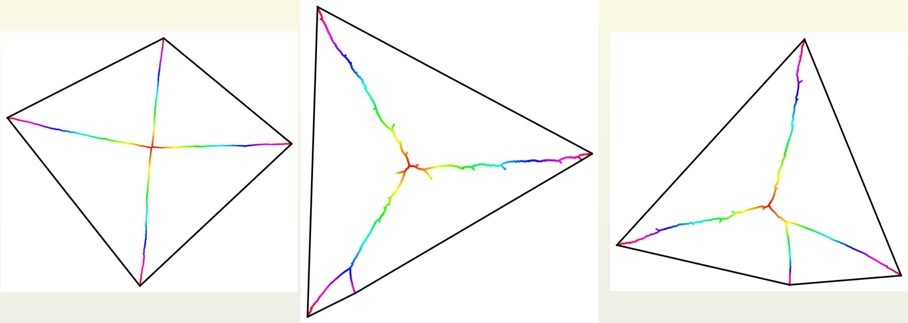
Steinerberger (2025) introduced a "gradient flow aggregation." In his model, a new particle starts "at infinity" and moves in $\mathbb{R}^2$ along the gradient line of

$$E(x) = \sum_{i=1}^{n} \|x - x_i\|^{-\alpha},$$

where $x_i \in \mathbb{R}^2$ are centers of n particles in the current cluster, and $0 < \alpha < \infty$. The main theorems treat finite α, but Section 1.5 in that paper presents simulations and their discussion for the case $\alpha = \infty$.

I call the case $\alpha = \infty$ of the model "ballistic aggregation."

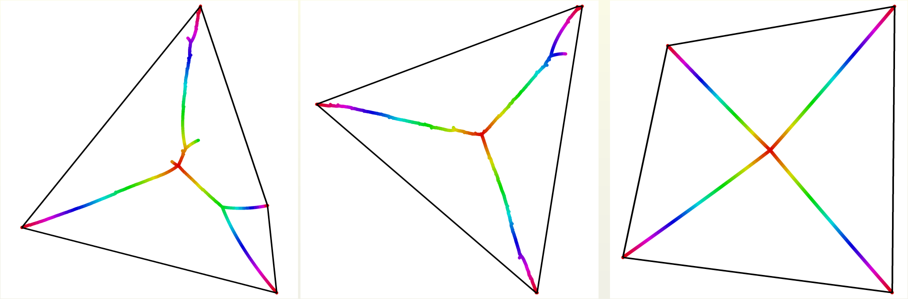
Ballistic Aggregation: simulations



10^4 discs

Three or four large arms. Frequency of four arms: $\approx 5\%$

Ballistic Aggregation: simulations



10^7 discs

Key observations:

(i) Three or four large arms.

(ii) Apparent limit of shape distribution.

Novelty

1. New shape of clusters relative to known models.

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3. New self-organized criticality (SOC) type.

Novelty

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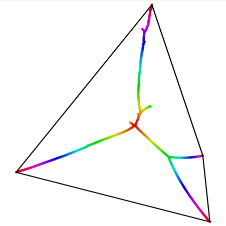
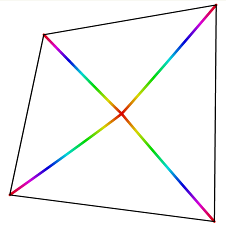
3. New self-organized criticality (SOC) type.

4. The limit of rescaled cluster distributions is a mixture of two configuration types (?).





Cluster size

THEOREM (B and Hoffman (2026+))

For some $c > 0$ and all $n \geq 1$, the diameter of every ballistic aggregation cluster of n discs of radius 1 is greater than $c\, n^{2/3}$.

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CONJECTURE

Diameters of ballistic aggregation clusters grow at a linear rate.

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There is no known non-trivial lower bound for the diameter of DLA clusters. (Trivial: $n^{1/2}$.)

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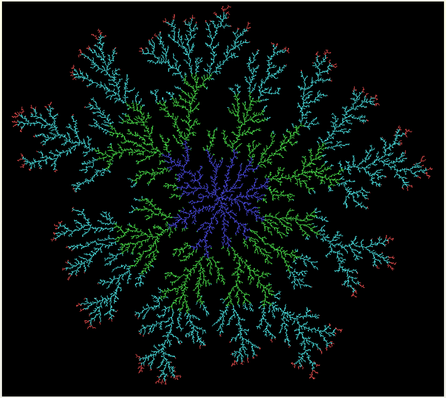
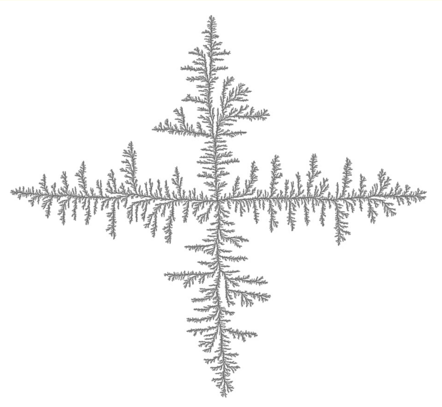
Diameters of ballistic aggregation clusters grow at a linear rate.

There is no known non-trivial lower bound for the diameter of DLA clusters. (Trivial: $n^{1/2}$.)

The theorem is *deterministic*.

Diffusion Limited Aggregation (DLA)



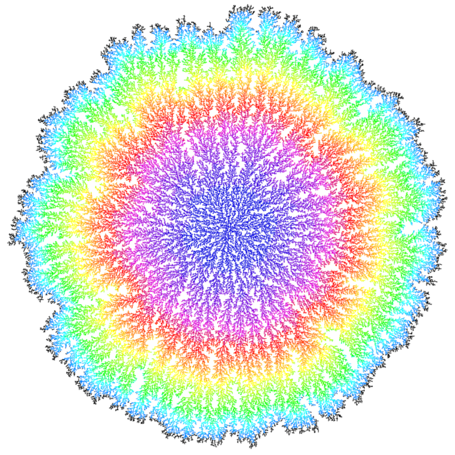


Witten and Sander (1981). Particles arrive along random walk paths.

Left figure ($3.3 \cdot 10^4$ particles) by WingkLEE - Own work, Public Domain,
https://commons.wikimedia.org/w/index.php?curid=10522864

Right figure ($1.45 \cdot 10^8$ particles) by Denis S. Grebenkov and Dmitry Beliaev

Alternative (earlier) “ballistic aggregation”



Vold–Sutherland model (1963,1967)

Simulation ($2 \cdot 10^5$ particles) by Tillmann Bosch and Steffen Winter (2024)

Self-organized criticality (SOC)

Per Bak, Chao Tang and Kurt Wiesenfeld (1987)

1. Sandpiles. Avalanche $\to$ subcritical state
   $\to$ avalanches are temporarily inhibited

Picture by Janina Hesse and Thilo Gross (2014)

(Abelian sandpiles)

2. Forest fires. Fire $\to$ subcritical state $\to$ fires are temporarily inhibited

Is ballistic aggregation an example of SOC?

Avalanche analogue: formation of a side branch.

Is ballistic aggregation an example of SOC?

Avalanche analogue: formation of a side branch.

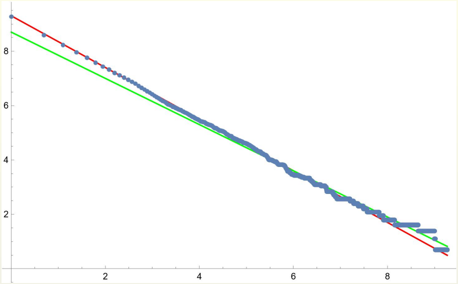
Slowly driven system ✓

No parameter fine-tuning ✓

The size of SOC catastrophes obeys the power law $\boxed{?}$

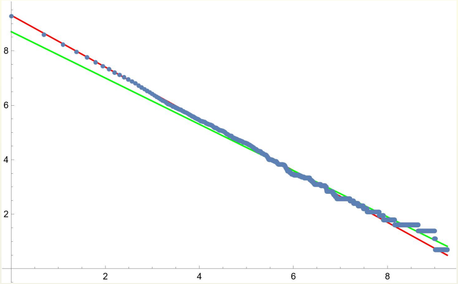
Catastrophe $\rightarrow$ **subcritical state**
$\rightarrow$ **catastrophes are temporarily inhibited** $\boxed{?}$

Branch size in ballistic aggregation



Log-log plot representing the logarithm of the number of side branches longer than n vs $\log n$.
The regression line $y = 8.7 - 0.85x$ is green. The line $y = 9.3 - 0.95x$ (in red) has a better fit than the regression line for most values of n.

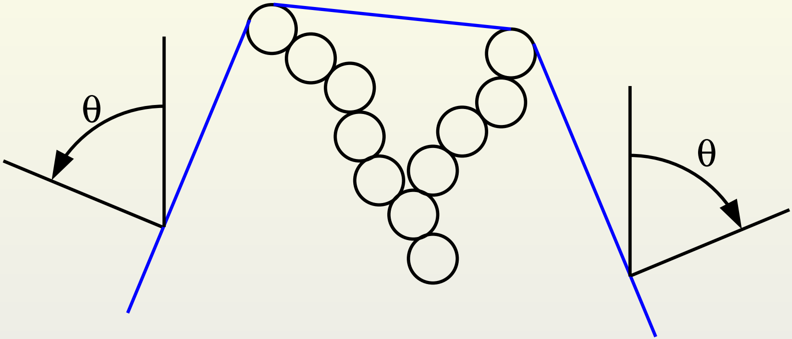
Branch size in ballistic aggregation



Log-log plot representing the logarithm of the number of side branches longer than n vs $\log n$.
The regression line $y = 8.7 - 0.85x$ is green. The line $y = 9.3 - 0.95x$ (in red) has a better fit than the regression line for most values of n.
Frequency of side branches longer than $n \approx \boxed{n^{-\mathbf{0.95}}}$

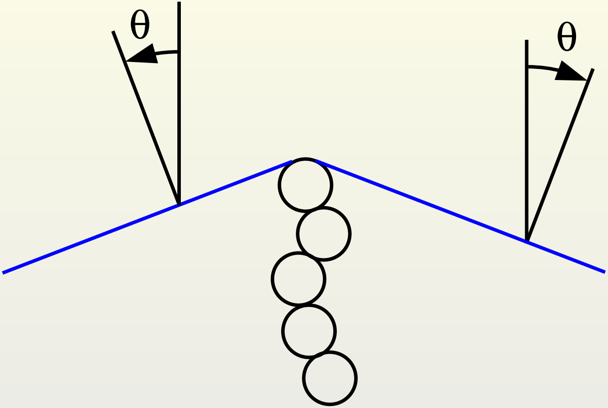
The size of SOC catastrophes obeys the power law ✓

Convex hull corner magnification



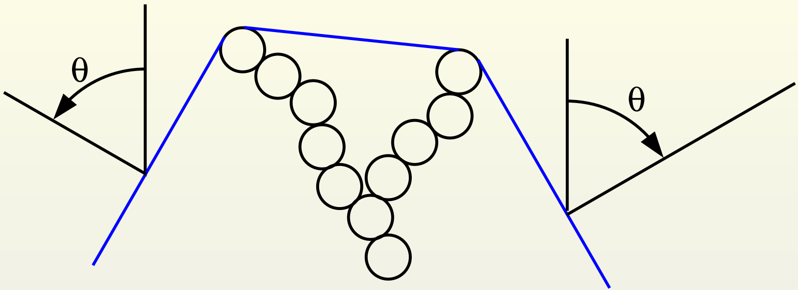
Assumption: θ is constant.

Obtuse corner



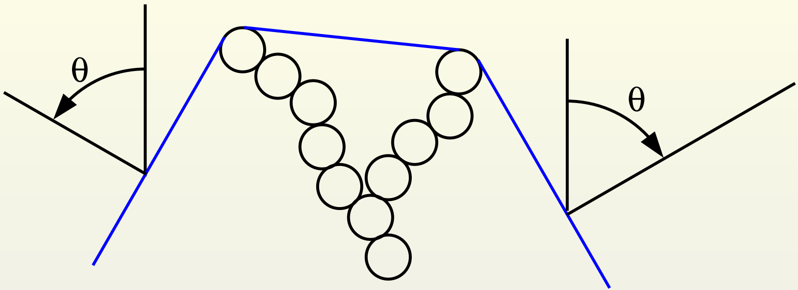
$\theta < \pi/4 \;\rightarrow$ obtuse corner $\rightarrow$ no branching

Equilateral triangle



convex hull ≈ equilateral triangle ⇒ $\theta = \pi/3$

Equilateral triangle



convex hull $\approx$ equilateral triangle $\Rightarrow$ $\theta = \pi/3$

If $\pi/4 < \theta < \pi/2$ and branches are more or less equally long then the branches will not branch.

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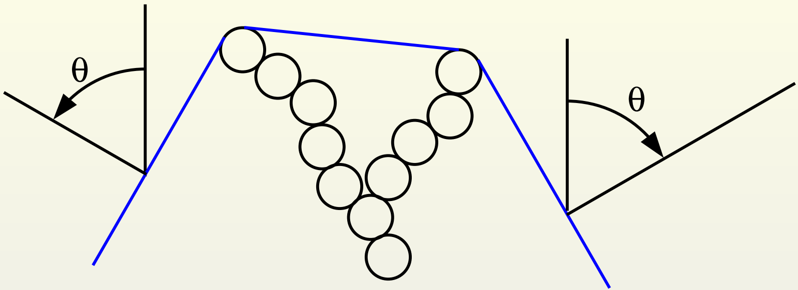

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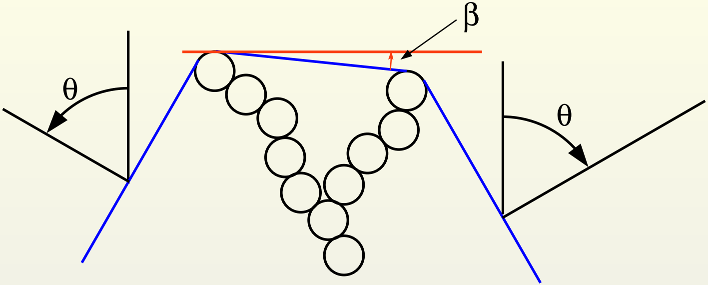
Catastrophe $\to$ subcritical state $\to$ catastrophes are temporarily inhibited $\checkmark$

Branch lifetime



$|\beta|$ has a tendency to grow due to positive feedback.

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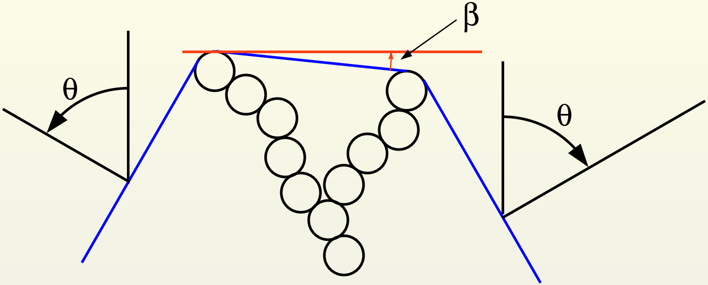

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$|\beta|$ may stay close to 0 due to random fluctuations.

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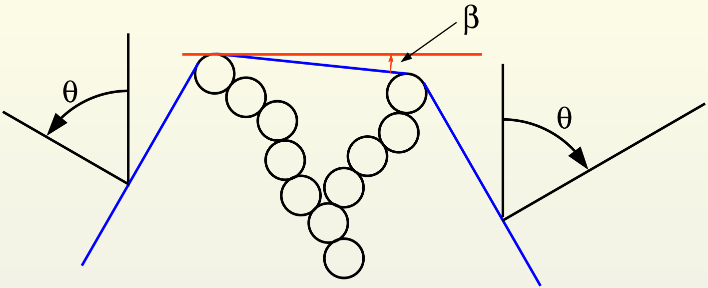

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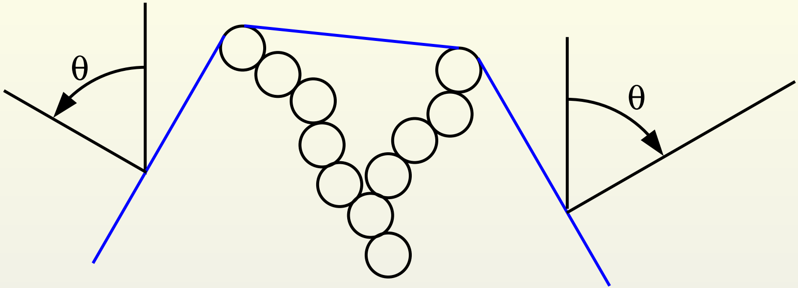
T - lifetime of a branch, $\mu = \cos\theta/(1-\cos\theta)$

THEOREM (B (2026))

$$\mathbb{P}(T > t) \approx ct^{-\mu}$$

Branch lifetime



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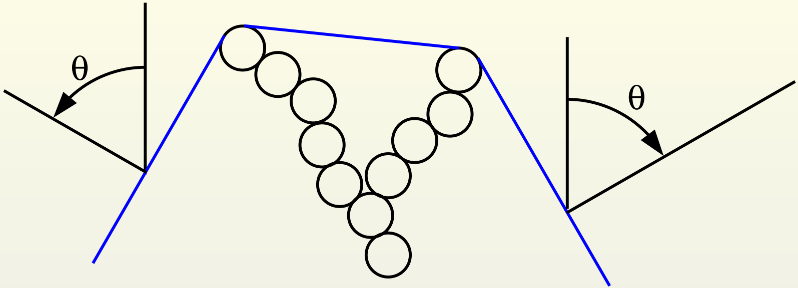

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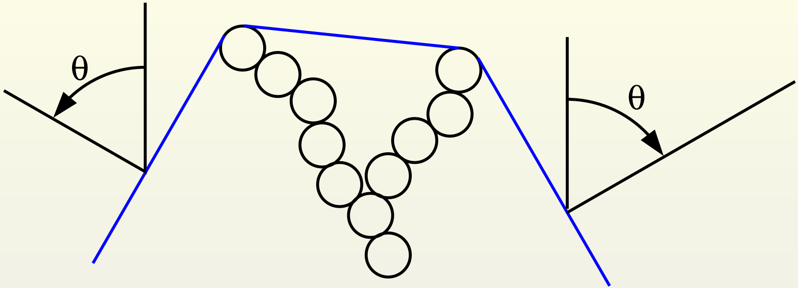
Macroscopic branches will occur if and only if $\mu < 1$, i.e., $\theta > \pi/3$, i.e., the corner angle is smaller than that of the equilateral triangle. Empirical $\mu \approx 0.95$ for triangles.

Branch lifetime



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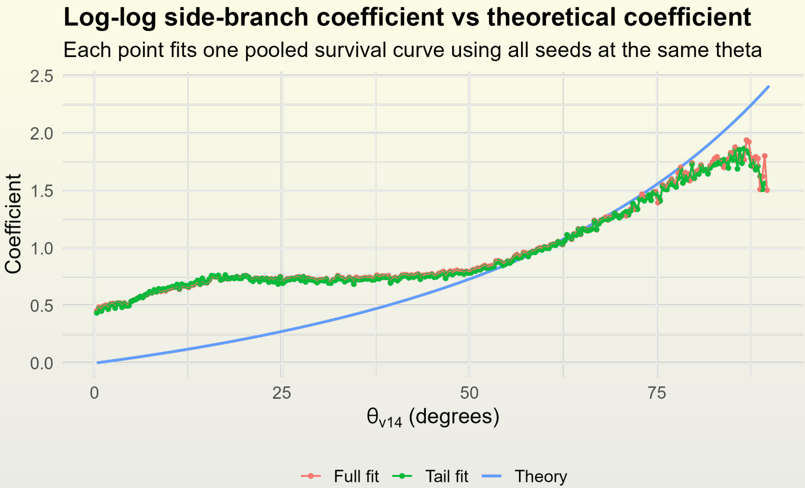
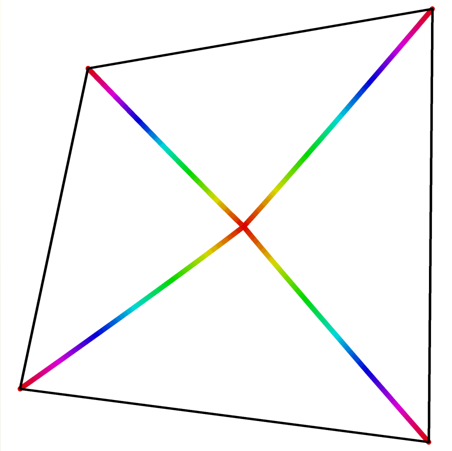
The ballistic aggregation model is "critical" in more than one sense.

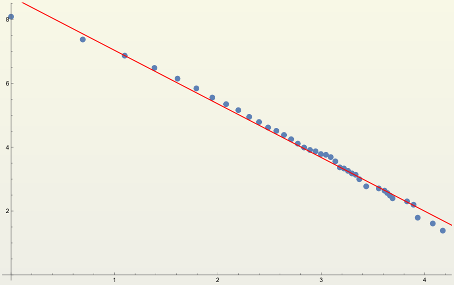
Tail decay rate



$$\mu = \cos\theta/(1-\cos\theta), \qquad \mathbb{P}(T>t)\approx ct^{-\mu}$$

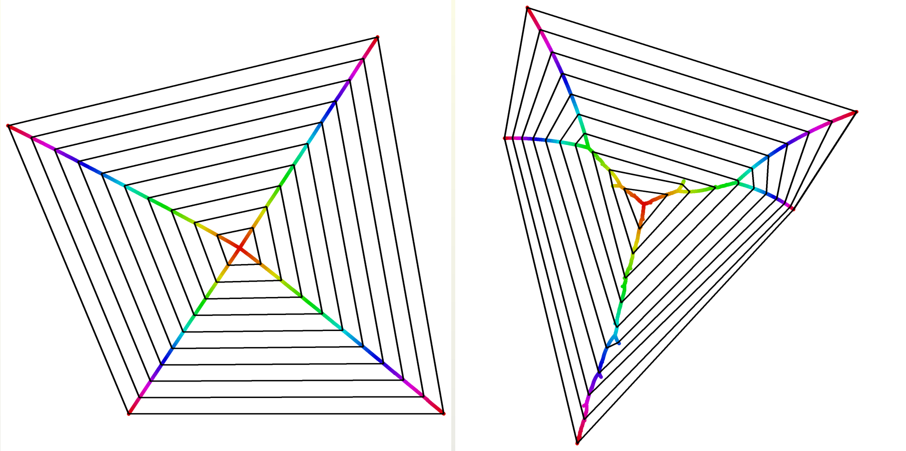
Simulations and picture: Ruiqi Zhou





Frequency of side branches longer than $n \approx \boxed{n^{-\mathbf{1.7}}}$

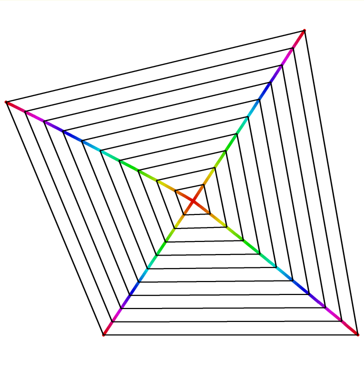
Quadrilaterals vs. Triangles



10^6 discs.

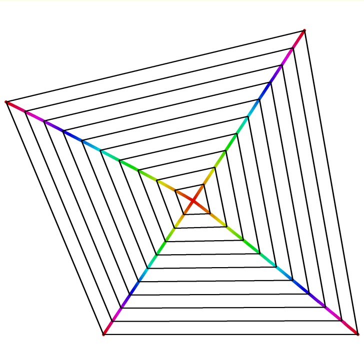
Frequency of quadrilaterals: 1% to 5%.

Deterministic evolution



1. Is any polygon a fixed shape for the shape evolution?

Deterministic evolution



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Yes, all regular polygons are fixed shapes. Are there any other examples?

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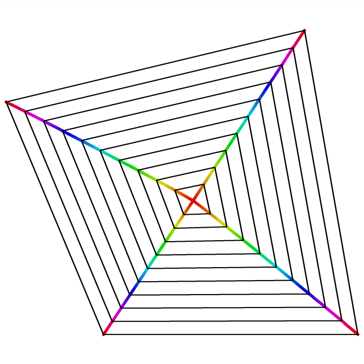

1. Is any polygon a fixed shape for the shape evolution?

Yes, all regular polygons are fixed shapes. Are there any other examples?

2. Are there any stable polygons?

Fixed shapes

Fixed shapes for deterministic growth:

1. All regular convex polygons.

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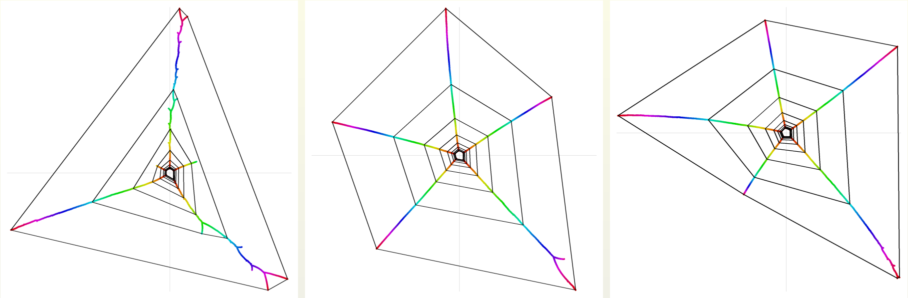
2. Pentagon with four angles $= 2.0358$ and one angle $= 1.28159$.

Fixed shapes

Fixed shapes for deterministic growth:

1. All regular convex polygons.
2. Pentagon with four angles $= 2.0358$ and one angle $= 1.28159$.
3. Possibly a pentagon with 2 small angles and 3 large angles.

Pentagon with a fixed shape



Center: pentagon with four angles $= 2.0358$ and one angle $= 1.28159$.

Sides: slightly different initial angles.

The final diameter is 100 times the initial diameter.

Stability

No polygons are stable.

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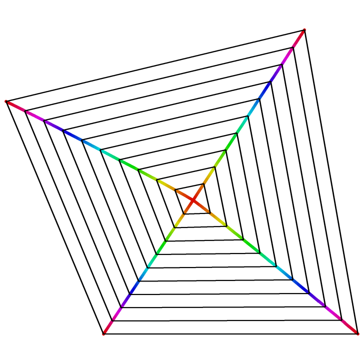
No polygons are stable.

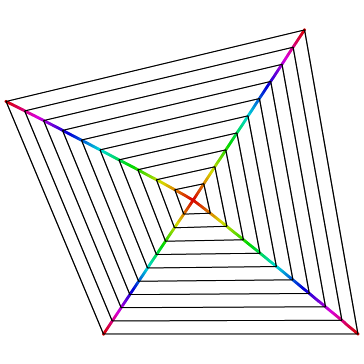
No polygons are stable for the **deterministic** evolution.

Growth along intersecting angle bisectors



θ_k - the k-th internal angle of the polygon

Growth along intersecting angle bisectors



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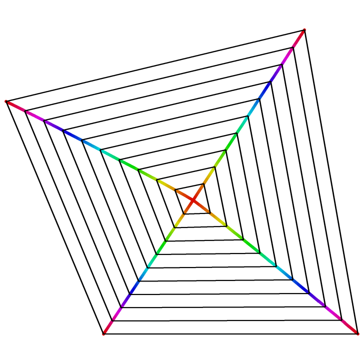
$g(\theta)$ - the rate of growth of a branch ending in a corner with angle θ

Growth along intersecting angle bisectors

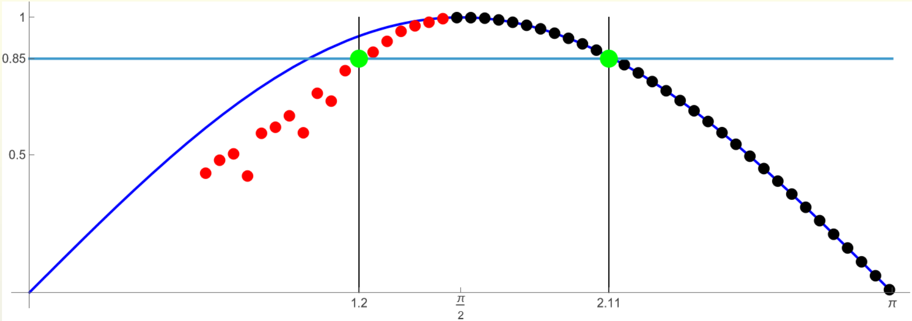


θ_k - the k-th internal angle of the polygon

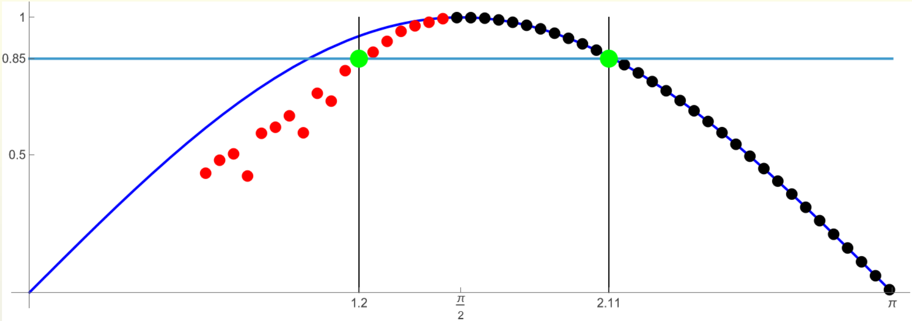
$g(\theta)$ - the rate of growth of a branch ending in a corner with angle θ

A polygonal shape is a fixed shape if $f(\theta_k) := \sin(\theta_k/2)g(\theta_k)$ has the same value for all k.

Searching for a fixed shape

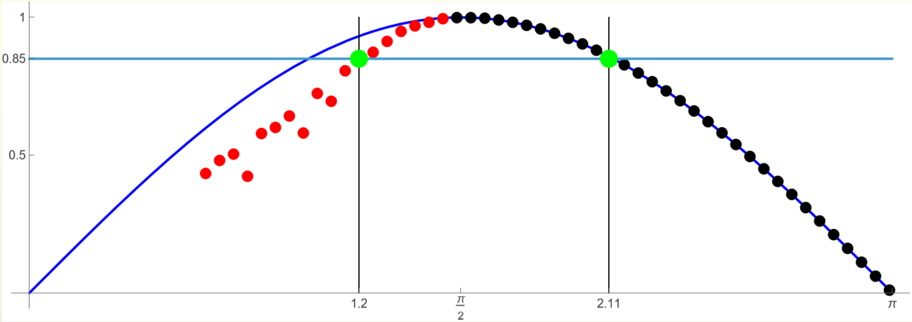


$f(\theta) = \sin(\theta/2)g(\theta)$



$f(\theta) = \sin(\theta/2)g(\theta)$

The sum of n-gon's angles is $(n-2)\pi$.



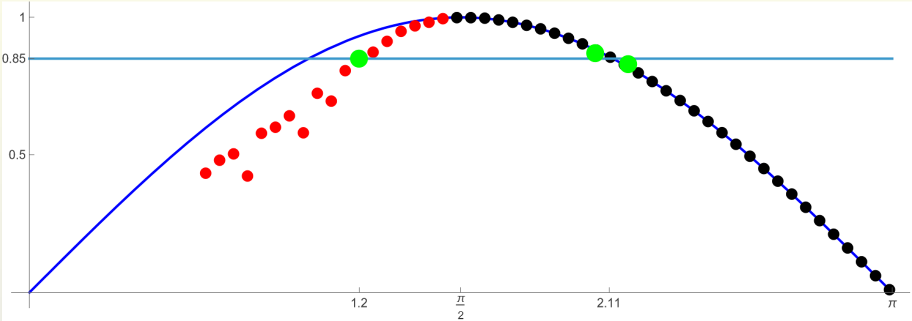
$f(\theta) = \sin(\theta/2)g(\theta)$

The sum of n-gon's angles is $(n-2)\pi$.

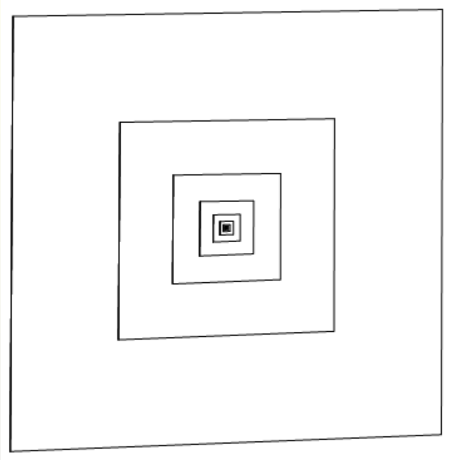
Pentagon with four angles $= 2.0358$ and one angle $= 1.28159$ is a fixed shape:

$$4 \times 2.0358 + 1 \times 1.28159 = 3\pi$$

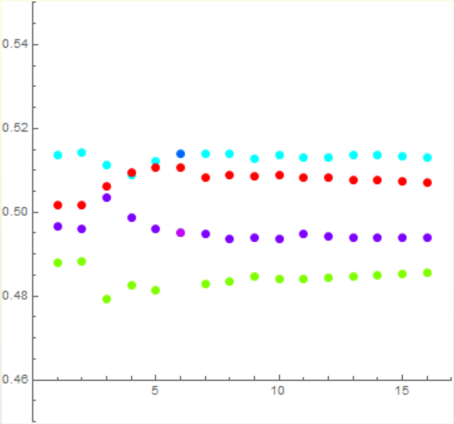
Stability



No polygons are stable for the deterministic evolution.

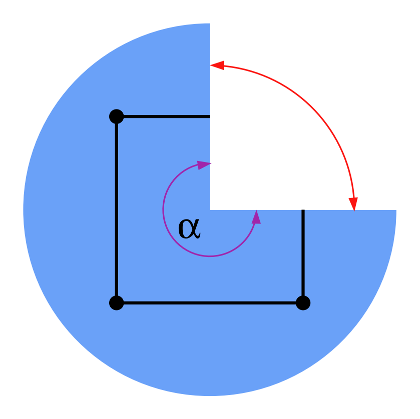


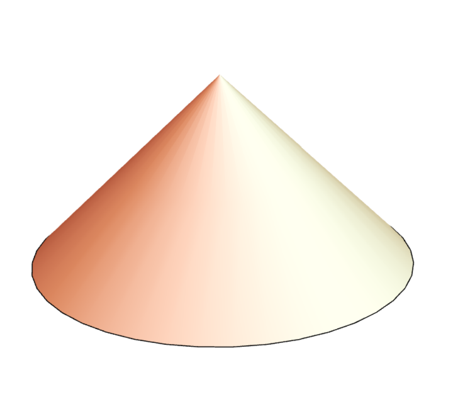
1.3×10^8 discs.
Initial square size: $2,000 \times 2,000$.



One unit on the horizontal axis represents doubling of the disc number.
The vertical scale uses π as the unit.
$0.50 =$ the right angle.

Ballistic aggregation on a flat manifold





Equilateral triangle with right angles

$\alpha = 3\pi/2$

A regular polygon can be stable on a manifold

The sum of angles of n-gon containing the manifold vertex is equal to $\boxed{n\pi - \alpha}$, where α is the manifold angle.

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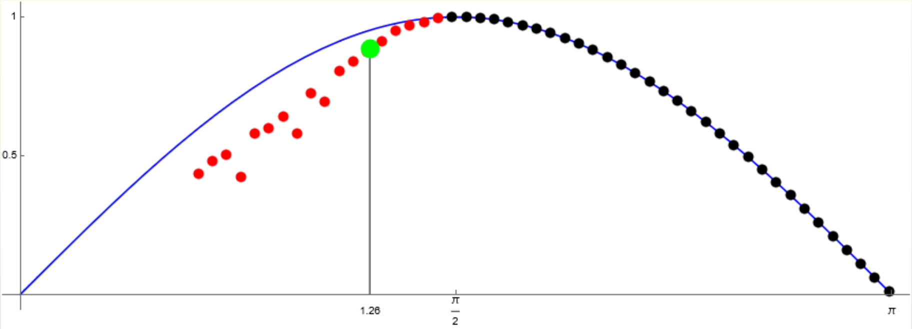

If $\alpha = 0.9(2\pi)$ then the equilateral triangle angle is $0.4\pi = \frac{1}{3}(3\pi - 0.9(2\pi)) \approx 1.26$.

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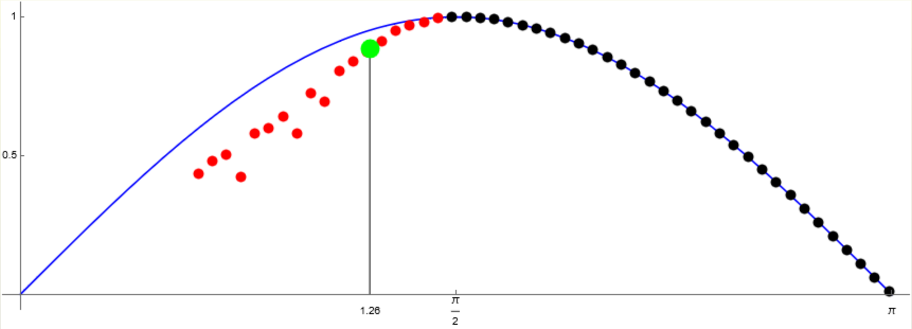
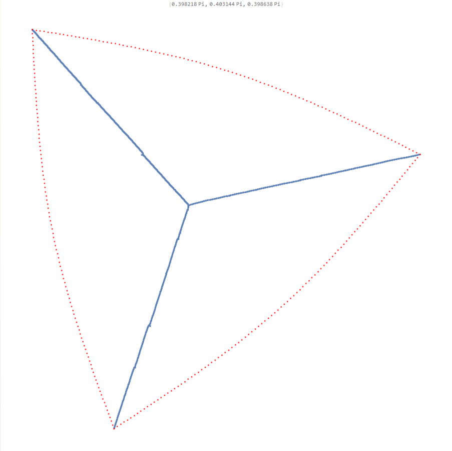
The sum of angles of n-gon containing the manifold vertex is equal to $\boxed{n\pi - \alpha}$, where α is the manifold angle.



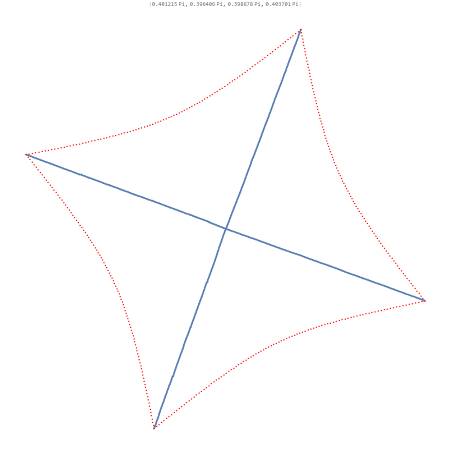
If $\alpha = 0.9(2\pi)$ then the equilateral triangle angle is $0.4\pi = \frac{1}{3}(3\pi - 0.9(2\pi)) \approx 1.26$.

If $\alpha = 1.2(2\pi)$ then the square angle is $0.4\pi = \frac{1}{4}(4\pi - 1.2(2\pi)) \approx 1.26$.

Stable triangles and squares

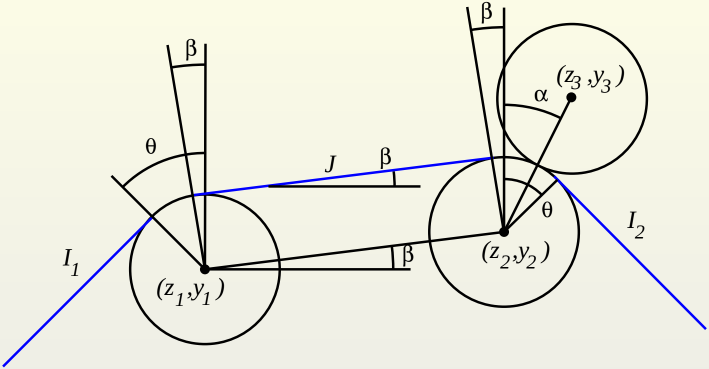


$\alpha = 0.9(2\pi)$, 10^6 discs



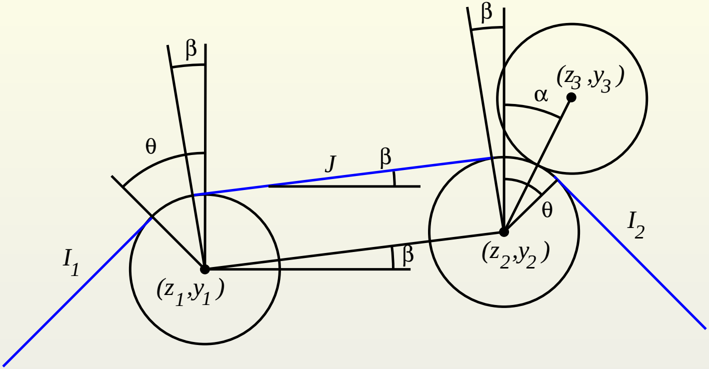
$\alpha = 1.2(2\pi)$, 10^6 discs

Competition between branches



$|\beta|$ has a tendency to grow due to positive feedback.

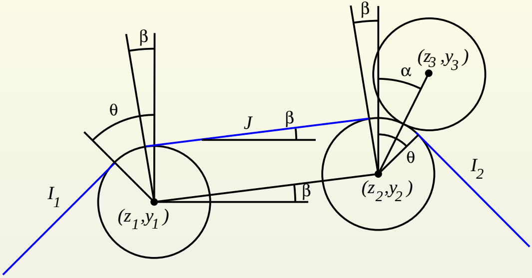
Competition between branches



$|\beta|$ has a tendency to grow due to positive feedback.

$|\beta|$ may stay close to 0 due to random fluctuations.

Competition between branches



(Z_n, Y_n) is the vector from the center of the left disc to the center of the right disc after n steps.

$\tan \beta = X_n = Y_n / Z_n$ is the tangent of the angle between the line through the centers of the discs and the horizontal.

One step transition moments

THEOREM

$$\lim_{z\to\infty} z\mathbb{E}(X_{n+1} - X_n \mid X_n = x \geq 0, Z_n = z) = \frac{x\cos\theta}{\theta},$$

$$\lim_{z\to\infty} \frac{z}{x}\mathbb{E}(X_{n+1} - X_n \mid X_n = x \geq 0, Z_n = z) = \frac{\cos\theta}{\theta},$$

$$\lim_{z\to\infty} z^2 \operatorname{Var}(X_{n+1} - X_n \mid X_n = x \geq 0, Z_n = z)$$

$$= \frac{1}{2\theta^2}\left(\left(x^2+1\right)\theta^2 - 2x^2\cos^2(\theta) - \left(x^2-1\right)\theta\sin(\theta)\cos(\theta)\right),$$

$$\lim_{x\to 0}\lim_{z\to\infty} z^2 \operatorname{Var}(X_{n+1} - X_n \mid X_n = x \geq 0, Z_n = z)$$

$$= \frac{1}{2\theta^2}\left(\theta^2 + \theta\sin(\theta)\cos(\theta)\right).$$

Fully explicit non-asymptotic formulas are given in my paper.

Informal invariance principle

CONJECTURE

The process X_n behaves like a continuous-time process X_t satisfying

$$dX_t = \frac{\sigma(\theta)}{t} dW_t + \frac{\mu(\theta) X_t}{t} dt,$$

where W_t is Brownian motion and

$$\mu(\theta) = \frac{\cos\theta}{1 - \cos\theta},$$
$$\sigma(\theta) = \frac{\sqrt{(\theta^2 + \theta \sin(\theta)\cos(\theta))/2}}{1 - \cos\theta}.$$

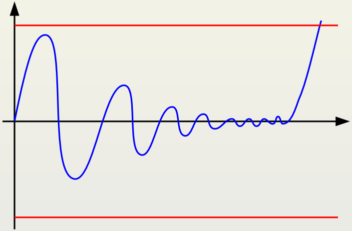
X_t — "anti-Ornstein-Uhlenbeck process"

Escape time estimate

THEOREM

$dX_t = \frac{\sigma}{t} dW_t + \frac{\mu X_t}{t} dt$. Suppose that the initial condition is $X_1 = 0$. Fix an arbitrary $a > 0$ and let $T = \inf\{t \geq 0 : |X_t| \geq a\}$. There exist $0 < c', c'' < \infty$ (depending on σ and a), and $t_1 > 0$ such that for $t > t_1$,

$$c' t^{-\mu} \leq \mathbb{P}(T > t) \leq c'' t^{-\mu}.$$



$E(X_s^2 \mid T > t) \approx 1/s$ if $t \gg s$. We need $a \approx \sigma/\mu$.

$X_t = Ct^\mu$ for $t > T$. Deterministic growth until lifetime $\approx T$.

Branch formation

We would like to determine the intensity of creation of branches of different sizes as a function of θ. If there is a fork, sub-forks are unlikely to form until the fork is very skewed, so we will assume that forks occur sequentially. Let T_k be the lifetime of the k-th fork. The limit theorems for i.i.d. random variables suggest that macroscopic branches will occur if, for large t,

$$\mathbb{P}(T_k > t) > ct^{-\mu}, \qquad \mu < 1,$$

and macroscopic branches will not occur if

$$\mathbb{P}(T_k > t) < ct^{-\mu}, \qquad \mu > 1.$$

In the first case, the rescaled partial sums of T_k's converge to a Lévy process with positive jumps, while in the second case, they converge to a straight line.

Branch formation

Macroscopic branches appear when $\mu < 1$, i.e., when $(\cos\theta)/(1-\cos\theta) < 1$, which is equivalent to $\theta > \pi/3$. Conversely, we expect not to see macroscopic branches when $\mu > 1$, i.e., when $\theta < \pi/3$. For the equilateral convex hull $\theta = \pi/3$.

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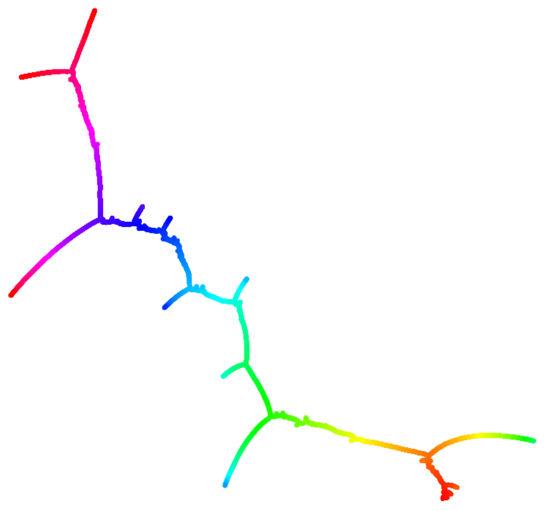
The ballistic aggregation model is "critical" in more than one sense. The side branch generation at a rate obeying the power law is the usual "criticality" in SOC.

The simulations show that the parameter $\mu \approx 0.95$. Rigorous results indicate that $\mu = 1$ corresponds to the convex hull in the shape of an equilateral triangle. The system is at the borderline of being visibly SOC.

Ballistic aggregation on a narrow manifold



$\alpha = 0.4(2\pi)$, 10^5 discs