Brownian loop soup via SLE and LQG

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Joint work with

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Random Explorations: From Random Walks to Random Geometry

In honor of Greg Lawler's 70th Birthday

IMSI, University of Chicago

10 July, 2026

Brownian loop soup

Introduced by [Lawler-Werner]

The infinite measure μ^{loop} on unrooted Brownian loops

$$\mu^{loop} = \int_{\mathbb{R}^2} \int_0^\infty \frac{1}{2\pi t^2} \mathbb{P}_{z,t} dt d\lambda(z)$$

where

- $d\lambda$ is the (infinite) Lebesgue measure on $\mathbb{R}^2$
- $\mathbb{P}_{z,t}$ is the probability measure on two-dimensional Brownian bridges rooted at z with time length t.

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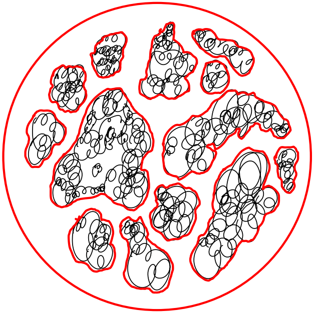
A Brownian loop soup is a Poisson point process with intensity $\frac{c}{2}\mu^{loop}$.

The bigger c is, the more loops there are in the loop-soup.

Brownian loop soup clusters

Consider a loop soup in the unit disk with intensity $c > 0$
[Sheffield-Werner]

Phase transition at $c = 1$:
$c > 1$: a.s. one single cluster.
$c \in (0, 1]$: a.s. infinitely many clusters.



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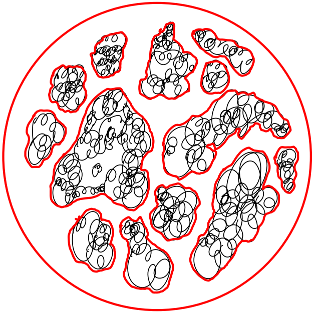
For $c \in (0, 1]$, the outer boundaries of the closures of the outermost clusters form a collection of simple loops which is distributed as a conformal loop ensemble (CLE).



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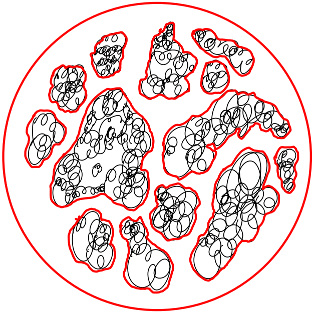


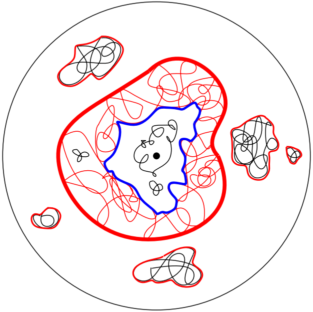
The parameter $c \in (0, 1]$ is also the central charge and

$$c(\kappa) = (6 - \kappa)(3\kappa - 8)/(2\kappa).$$

This corresponds to $\kappa \in (8/3, 4]$.

One loop-soup cluster

Consider a loop soup with intensity $c \in (0, 1]$. Let $\mathcal{C}_0$ be the outermost cluster that encircles the origin. Let η^{out} be the outer boundary of $\mathcal{C}_0$. Let η^{inn} be the boundary of the connected component of $\mathbb{C} \setminus \mathcal{C}_0$ containing the origin.



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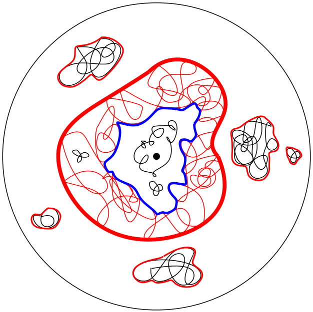

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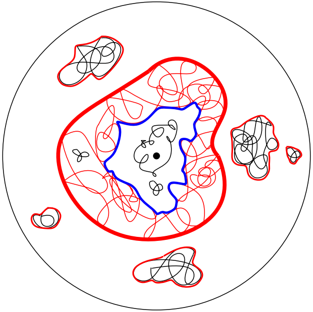

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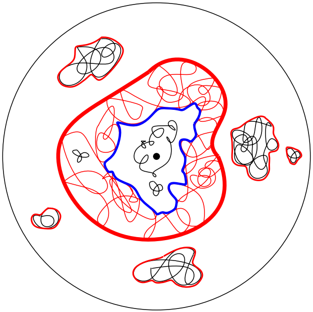

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Specific to $c = 1$ ($\kappa = 4$), thanks to the coupling with the GFF.

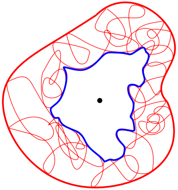
Boundaries-touching probaiblity $c \in (0,1)$

Theorem 1 (Cai, Liu, Q., Sun, Wu)

For $c \in (0,1)$*, the boundaries-touching probability is given by*

$$\mathbb{P}[\eta^{\text{out}} \cap \eta^{inn} = \emptyset] = \frac{c}{2\cos(\pi(\frac{4}{\kappa}-1))} \frac{\sin(\frac{4\pi}{\kappa}(1-\frac{\kappa}{4})^2)}{\sin(\pi(1-\frac{\kappa}{4}))} := \beta(c) \in (0,1).$$

non-touching



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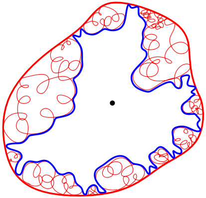

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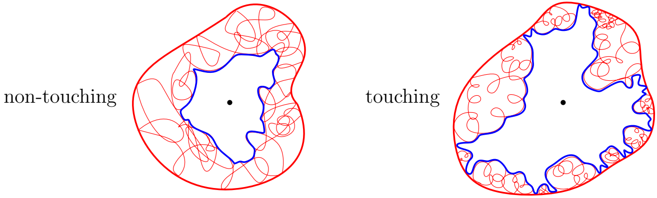

Note that

$$\lim_{c\to 1} \beta(c) = 0, \quad \lim_{c \to 0} \beta(c) = \frac{5\sqrt{3}}{9\pi}.$$

In the $c \to 0$ limit, a loop-soup cluster with intensity c "tends to" a Brownian loop.

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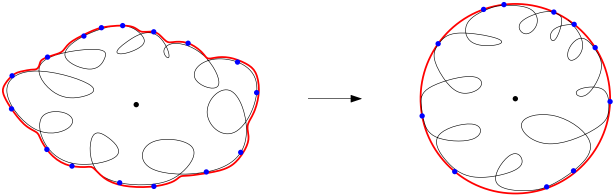

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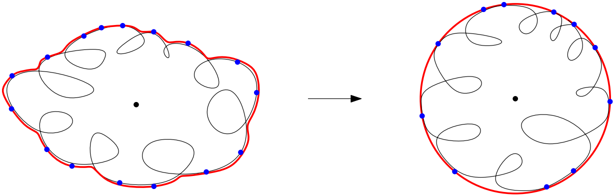

cut points seen from 0

Existence of "cut-points" for the planar Brownian motion first shown by [Burdzy,1989].

Hausdorff dimension $3/4$ computed by [Lawler-Schramm-Werner, 2000].

Corollary for the Brownian loop (Continuity as $c \to 0$): Let η_0 be the boundary of the connected component of $\mathbb{C} \setminus \mathcal{P}_0^{inn}$ that contains the origin. Then

$$\mathbb{P}(\eta_0 \cap \mathbb{S}^1 \neq \emptyset) = \frac{5\sqrt{3}}{9\pi}.$$

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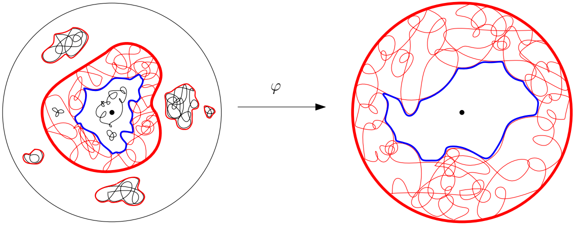
Let φ be the conformal map from the domain encircled by η^{out} onto the unit disk with $\varphi(0) = 0, \varphi'(0) > 0$.



[Q.-Werner] The law of $\varphi(\mathcal{C}_0)$ is invariant under all conformal automorphisms of the unit disk, and independent from η^{out}.
Let $\mathcal{P}_c^{inn}$ denote the law of $\varphi(\mathcal{C}_0)$.

Continuity as $c \to 0$

Theorem 2 (Cai, Liu, Q., Sun, Wu)

The law of $\mathcal{P}_c^{\mathrm{inn}}$ *converges to that of* $\mathcal{P}_0^{\mathrm{inn}}$ *as* $c \to 0$.

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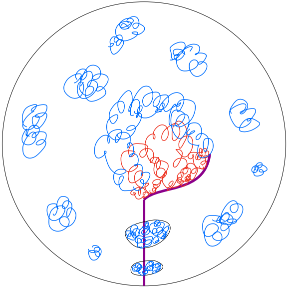

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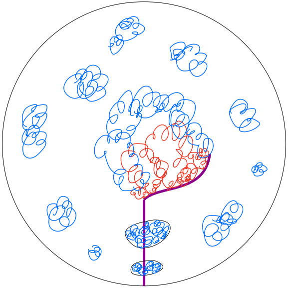
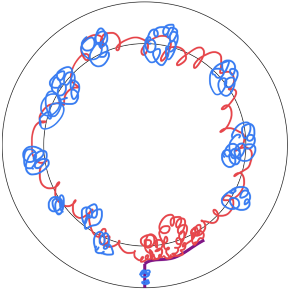


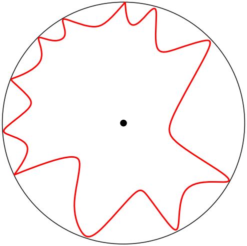


The law of the inner boundary η^{inn}: touching case

Theorem 3 (Cai, Liu, Q., Sun, Wu)

The law of η^{inn} under $\mathcal{P}_c^{\mathrm{inn}}$ conditioned on $\eta^{\mathrm{inn}} \cap \mathbb{S}^1 \neq \emptyset$ is an $SLE_\kappa(\kappa - 4)$ bubble in $\mathbb{D}$ conditioned to surround 0.

unrooted $\mathrm{SLE}_\kappa(\rho)$ bubble measure: [Zhan]



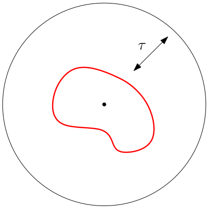
The law of the inner boundary η^{inn}: non-touching case

Theorem 4 (Cai, Liu, Q., Sun, Wu)

The law of η^{inn} under $\mathcal{P}_c^{\mathrm{inn}}$ restricted on $\eta^{\mathrm{inn}} \cap \mathbb{S}^1 = \emptyset$ is

$$\int_0^\infty \rho(\tau)\nu^0_{\mathbb{D},c,\tau}\,d\tau,$$

- *ρ is characterized by its Laplace transform (explicit but complicated)*
- *$\nu^0_{\mathbb{D},c}$ is the SLE$_\kappa$ loop measure $\nu_{\mathbb{D},c}$ restricted to loops surrounding the origin. For $\tau > 0$, let $\nu^0_{\mathbb{D},c,\tau}$ be the probability measure obtained from $\nu^0_{\mathbb{D},c}$ conditioning on the event that the modulus of the annulus bounded by the loop and $\mathbb{S}^1$ equals τ.*



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Definition 5 (Kontsevich-Suhov)

For $c \in (-\infty, 1]$, we call a family of measures $\{\nu_{D,c}, D \subset \hat{\mathbb{C}}\}$ a Malliavin-Kontsevich-Suhov (MKS) measure of central charge c if

- Conformal invariance: f conformal map from D onto D' $\Longrightarrow \ f_* \nu_{D,c} = \nu_{D',c}$.
- Restriction: Let K_1, K_2 be two disjoint closed subsets of $\hat{\mathbb{C}}$. The renormalized Brownian loop measure [F-Lawler]

$$\Lambda^*(K_1, K_2) := \lim_{r \to 0} \Lambda_{\mathbb{C} \setminus B(z,r)}(K_1, K_2) - \log\log(r^{-1}).$$

  Then

$$d\nu_{D,c}(\cdot) = \mathbf{1}_{\cdot \subset D} \exp\left(\frac{c}{2}\Lambda^*(\cdot, \hat{\mathbb{C}} \setminus D)\right) d\nu_c(\cdot)$$

Existence and uniqueness of the MKS measure

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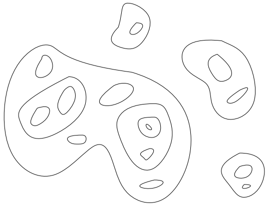

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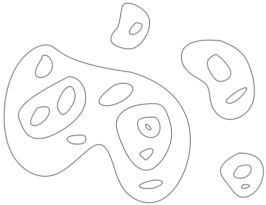

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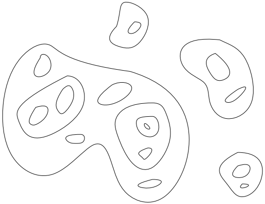

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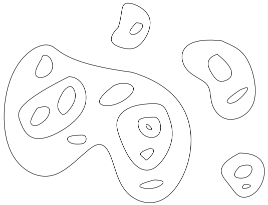

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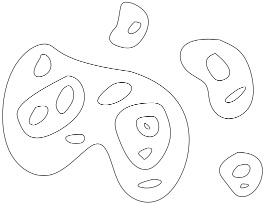

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[Cai-Gao] alternative proof for $\kappa \in (8/3, 4]$

Brownian loop soup and SLE loop measure

Let $\mathcal{L}$ be a BLS on a simply connected domain D with intensity $c \in (0, 1]$. Let $\mathcal{O} = \mathcal{O}(\mathcal{L})$ be the collection of outer boundaries of clusters in $\mathcal{L}$.

- Let $\widetilde{\nu}_{D,c}$ be counting measure in $\mathcal{O}$: $\widetilde{\nu}_{D,c}[L] = \mathbf{E}\left[\sum_{\eta \in \mathcal{O}} \mathbf{1}_{\eta \in L}\right]$.
- Let $\nu_{D,c}$ be the SLE$_\kappa$ loop measure on D (uniquely defined modulo a multiplicative constant).

Theorem 6 (Cai, Liu, Q., Sun, Wu)

Suppose $c \in (0,1]$. *There is a constant* $C \in (0,\infty)$ *such that* $\widetilde{\nu}_{\mathbb{D},c} = C\nu_{\mathbb{D},c}$.

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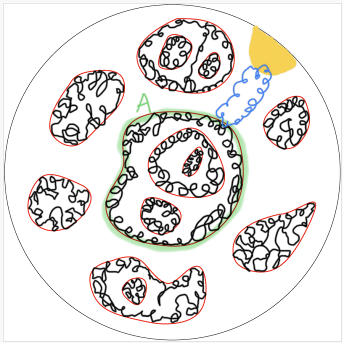
Proof relies on the conformal restriction characterization of SLE loop measure.

Key property: Let $D_1 \subset D_2 \subset \mathbb{C}$ be two simply connected domains. Then

$$d\widetilde{\nu}_{D_1,c}(\cdot) = \mathbf{1}_{\{\cdot \subset D_1\}} \exp\left(\frac{c}{2}\Lambda_{D_2}(\cdot, D_2 \setminus D_1)\right) d\widetilde{\nu}_{D_2,c}(\cdot).$$



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The density of $\mathrm{Mod}(A_{\eta_0})$ under CLE_κ^D (η_0 is the CLE_κ loop in D that surrounds 0) is [Ang-Remy-Sun, 25]

$$\left(1-\frac{\gamma^2}{4}\right)\frac{\cos(\pi(\frac{4}{\gamma^2}-1))}{\sin(\pi(1-\frac{\gamma^2}{4}))}\theta_1(\frac{\gamma^2}{8},\frac{\gamma^2}{4}i\tau)e^{\pi\left(\frac{2}{\gamma}-\frac{\gamma}{2}\right)^2\tau}d\tau =: \rho_0(\tau)d\tau, \quad (1)$$

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For a simple loop η in a simply connected domain D, let A_η be the annulus bounded by η and ∂D. Let $\mathrm{Mod}(A_\eta)$ be the modulus of A_η.

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Let $P(\tau)$ be the probability that there does not exist a non-contractible loop-soup cluster in an annulus of modulus τ. Let $\widetilde{\nu}_{D,c}^0$ be $\widetilde{\nu}_{D,c}$ restricted to loops surrounding 0.

Lemma 7 (Cai, Liu, Q., Sun, Wu)

For $c\in(0,1]$, $\widetilde{\nu}_{D,c}^0$ and CLE_κ^D are mutually absolutely continuous, and their Radon-Nikodym derivative equals

$$\frac{d\mathrm{CLE}_\kappa^D}{d\widetilde{\nu}_{D,c}^0}(\eta)=P(\mathrm{Mod}(A_\eta)).$$

In particular, the density of $\mathrm{Mod}(A_\eta)$ under $\widetilde{\nu}_{D,c}^0$ is $\rho_0(\tau)P(\tau)^{-1}d\tau$.

Theorem 8 (Cai, Liu, Q., Sun, Wu)

Intermediate result: For $c \in (0,1)$, as $\tau \to 0$, we have

$$1 - P(\tau) = ce^{-\frac{\pi}{\tau}}(1 + o(1)) \qquad \text{as } \tau \to 0. \tag{2}$$

Stronger result: For $c \in (0,1]$ and $\tau > 0$, we have

$$P(\tau) = \sqrt{\frac{\kappa}{8}}\,(2\tau)^{c/2}\,\eta(2i\tau)^{\,c-1}\,\theta_1\left(\frac{\kappa}{8},\,\frac{\kappa}{4}i\tau\right). \tag{3}$$

where $\eta(i\tau) = e^{-\frac{\pi\tau}{12}}\prod_{k=1}^{\infty}\left(1 - e^{-2\pi k\tau}\right)$ is the Dedekind eta function.

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Simple proof of (2): The Brownian loop mass on A with winding number n is $|n|^{-1}e^{-\frac{\pi}{\tau}}(1 + o(1))$ as $\tau \to 0$ [Wang-Xue, 25]
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Proof of (3) relies on [Baverez-Jego-Sun-Wu, 26+]: For $\kappa \in (0,4]$ the density of $\mathrm{Mod}(\eta)$ under $\nu^0_{\mathbb{D},c}$ is

$$m^{\mathrm{SLE}}(\tau)d\tau := C\frac{1}{2\sqrt{\tau}}\Big(\frac{1}{\sqrt{2\tau}\eta(2i\tau)}\Big)^{c-1}e^{\frac{1-c}{6}\pi\tau}d\tau \tag{4}$$

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Corollary of (3)

- The $c \to 0$ limit: Let A be an annulus with modulus τ, and let F be the event that the Brownian loop is non-contractible in A. We get the explicit formula:

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Corollary of Theorem 4

- One-arm probability in the BLS: [Lupu-Jego-Q., 25]
$\mathbb{P}\left(r\,\partial\mathbb{D} \xleftrightarrow{\mathcal{L}} R\,\partial\mathbb{D}\right) = C|\log r|^{-1+\frac{c}{2}+o(1)}$.
We get the sharp asymptotics:
$\mathbb{P}\left(r\,\partial\mathbb{D} \xleftrightarrow{\mathcal{L}} R\,\partial\mathbb{D}\right) = C|\log r|^{-1+\frac{c}{2}}(1+o(1))$.

Proof of touching probability based on (2)

(2), (1) and Lemma 7 imply that as $\lambda \to \infty$,

$$\widetilde{\nu}^0_{D,c}[e^{-2\pi\lambda \mathrm{Mod}(A_\eta)}] = \mathrm{CLE}^D_\kappa[e^{-2\pi\lambda \mathrm{Mod}(A_{\eta_0})}] +$$

$$\left(1 - \frac{\gamma^2}{4}\right) \frac{\cos(\pi(\frac{4}{\gamma^2} - 1))}{\sin(\pi(1 - \frac{\gamma^2}{4}))} \frac{2c}{\gamma\sqrt{2\lambda}} e^{-\pi\sqrt{2\lambda}(\frac{2}{\gamma} + \frac{\gamma}{2})}(1 + o(1)).$$

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Relating conformal radius to modulus [Ang, Remy, Sun, 25]: For $c \in (0,1)$, suppose Λ is mutually absolutely continuous with respect to $\nu^0_{\mathbb{D},c}$ such that $\frac{d\Lambda}{d\nu^0_{\mathbb{D},c}}(\eta) = F(\mathrm{Mod}(A_\eta))$ for some continuous function $F : \mathbb{R}_+ \to \mathbb{R}_+$. Then for any $\lambda \geq 0$, we have

$$\Lambda\left[\mathrm{CR}(\eta, 0)^\lambda\right] = \frac{2\sin\pi(\frac{\gamma^2}{4} - 1)}{\gamma(1 - \frac{4}{\gamma^2})} \frac{\sqrt{2\lambda - \frac{1-c}{6}}}{\sinh\left(\frac{\gamma\pi}{2}\sqrt{2\lambda - \frac{1-c}{6}}\right)} \Lambda\left[e^{-2\pi\lambda \mathrm{Mod}(\eta)}\right]. \quad (5)$$

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Applying (5) to $\widetilde{\nu}^0_{\mathbb{D},c}$ and $\mathrm{CLE}^{\mathbb{D}}_\kappa$, we get as $\lambda \to \infty$

$$\widetilde{\nu}^0_{\mathbb{D},c}[\mathrm{CR}(\eta; 0)^\lambda] = \mathrm{CLE}^{\mathbb{D}}_\kappa[\mathrm{CR}(\eta, 0)^\lambda] + 2c\cos(\pi(\frac{4}{\gamma^2} - 1))e^{-\pi\sqrt{2\lambda}(\gamma + \frac{2}{\gamma})}(1 + o(1))$$

Conformal radius of CLE_κ loop [Schramm-Sheffield-Wilson, 09]

$$R(\lambda) := \mathrm{CLE}_\kappa^{\mathbb{D}}[\mathrm{CR}(\eta,0)^\lambda] = \frac{\cos\left(\frac{\pi(4-\kappa)}{\kappa}\right)}{\cosh\left(\frac{2\pi}{\gamma}\sqrt{2\lambda-\frac{1-c}{6}}\right)}. \tag{6}$$

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Lemma 9 (Cai, Liu, Q., Sun, Wu)

For $c \in (0,1]$ and $\lambda > 0$, we have

$$\mathbb{E}[\mathrm{CR}(\eta^{\mathrm{inn}},0)^\lambda] = \frac{1}{R(\lambda)} - \frac{1}{\widetilde{\nu}^0_{\mathbb{D},c}[\mathrm{CR}(\eta,0)^\lambda]}. \tag{7}$$

Proof. Iteratively multiply the independent conformal radii

$$\begin{aligned}\widetilde{\nu}^0_{\mathbb{D},c}[\mathrm{CR}(\eta,0)^\lambda] &= \sum_{n\geq 0} \mathrm{CLE}_\kappa^{\mathbb{D}}[\mathrm{CR}(\eta,0)^\lambda]^{n+1}\mathbb{E}[\mathrm{CR}(\eta^{\mathrm{inn}},0)^\lambda]^n \\ &= \frac{R(\lambda)}{1 - R(\lambda)\cdot \mathbb{E}[\mathrm{CR}(\eta^{\mathrm{inn}},0)^\lambda]}.\end{aligned}$$

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$$\mathbb{E}[\mathrm{CR}(\eta^{\mathrm{inn}},0)^\lambda] = \frac{c}{2\cos(\pi(\frac{4}{\gamma^2}-1))}e^{-\pi\sqrt{2\lambda}(\gamma-\frac{2}{\gamma})}(1+o(1))$$

The touching case

Let T be the event that $\eta^{\mathrm{inn}} \cap \mathbb{S}^1 \neq \emptyset$.

η^{inn} on T is the unrooted $\mathrm{SLE}_\kappa(\kappa-4)$ bubble measure $\mu^0_{\mathbb{D},c}$ on $\mathbb{D}$ conditioned on surrounding the origin.

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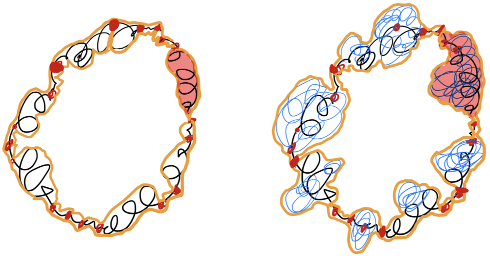
Loop case: we reduce it to the chordal case by proving a Markov property w.r.t. the beads, following [Virag]



Proposition 10

For $\kappa \in (2,4)$ and $\lambda \geq 0$, we have

$$\mu^0_{\mathbb{D},c}[\mathrm{CR}(\eta;0)^\lambda] = \frac{\sin(\pi(1-\frac{\gamma^2}{4}))}{\sin(\frac{\pi\gamma^2}{4}(\frac{4}{\gamma^2}-1)^2)} \frac{\sinh(\pi(\frac{2}{\gamma}-\frac{\gamma}{2})\sqrt{2\lambda-\frac{1-c}{6}})}{\sinh(\frac{\pi\gamma}{2}\sqrt{2\lambda-\frac{1-c}{6}})}. \quad (8)$$

Moment formula derivation: obtain the $\mathrm{SLE}_\kappa(\kappa-4)$ bubble by conformally welding a quantum disk and a thin quantum annulus.

For $\kappa \in (2,4)$ and $\rho \in (-2,\kappa-4)$, the corresponding moment for $\mathrm{SLE}^{\mathrm{bub}}_\kappa(\kappa-6-\rho)$ was computed in [Liu-Sun-Yu-Zhuang, 24]. We supply the endpoint $\rho=-2$ case.

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There is some $C_1 > 0$

$$\mathbb{E}\left[\mathrm{CR}(\eta^{\mathrm{inn}};0)^\lambda \mathbf{1}_T\right] = C_1 \frac{\sinh(\pi(\frac{2}{\gamma}-\frac{\gamma}{2})\sqrt{2\lambda-\frac{1-c}{6}})}{\sinh(\frac{\pi\gamma}{2}\sqrt{2\lambda-\frac{1-c}{6}})}. \quad (9)$$

We need to know the value of C_1: taking $\lambda = 0$ in (9) gives the touching probability

- $\mathbb{E}\big[\mathrm{CR}(\eta^{\mathrm{inn}};0)^{\lambda}\mathbf{1}_{T^c}\big] = \mathbb{E}\big[\mathrm{CR}(\eta^{\mathrm{inn}};0)^{\lambda}\big] - \mathbb{E}\big[\mathrm{CR}(\eta^{\mathrm{inn}};0)^{\lambda}\mathbf{1}_{T}\big]$
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- Apply (5) by [Ang, Remy, Sun, 25]: for $C_2 = \frac{\pi\gamma(1-\frac{4}{\gamma^2})}{2\sin\pi(\frac{\gamma^2}{4}-1)}$,

$$\mathbb{E}\left[e^{-2\pi\lambda\mathrm{Mod}(\eta^{\mathrm{inn}})}\mathbf{1}_{T^c}\right] = C_2\frac{\sinh\left(\frac{\gamma\pi}{2}\sqrt{2\lambda-\frac{1-c}{6}}\right)}{\pi\sqrt{2\lambda-\frac{1-c}{6}}}\mathbb{E}[\mathrm{CR}(\eta^{\mathrm{inn}},0)^{\lambda}\mathbf{1}_{T^c}] \tag{10}$$

Need to prove that the law of η^{inn} restricted to T^c is mutually absolutely continuous w.r.t. the SLE loop measure

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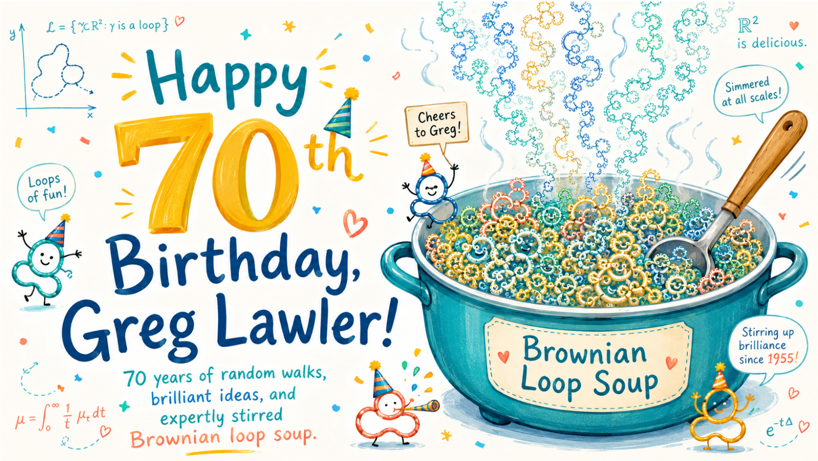
The right side of (10) has asymptotic

$$\frac{C_2}{2\pi\sqrt{2\lambda}}\left[\frac{c}{2\cos\left(\frac{\pi(4-\kappa)}{\kappa}\right)}e^{\pi(\frac{2}{\gamma}-\frac{\gamma}{2})\sqrt{2\lambda}}(1+o(1)) - C_1 e^{\pi(\frac{2}{\gamma}-\frac{\gamma}{2})\sqrt{2\lambda}}\right]$$

It must tend to 0 as $\lambda \to \infty$, hence $C_1 = \frac{c}{2\cos(\frac{\pi(4-\kappa)}{\kappa})}$.


$\mathcal{L} = \{\gamma \subset \mathbb{R}^2 : \gamma \text{ is a loop}\}$
Happy 70th Birthday, Greg Lawler!
Cheers to Greg!
$\mathbb{R}^2$ is delicious.
Simmered at all scales!
Loops of fun!
Brownian Loop Soup
Stirring up brilliance since 1955!
70 years of random walks, brilliant ideas, and expertly stirred Brownian loop soup.
$\mu = \int_0^\infty \frac{1}{t} \mu_t \, dt$
$e^{-t\Delta}$