A Rate of Convergence for the Discrete Green's Function

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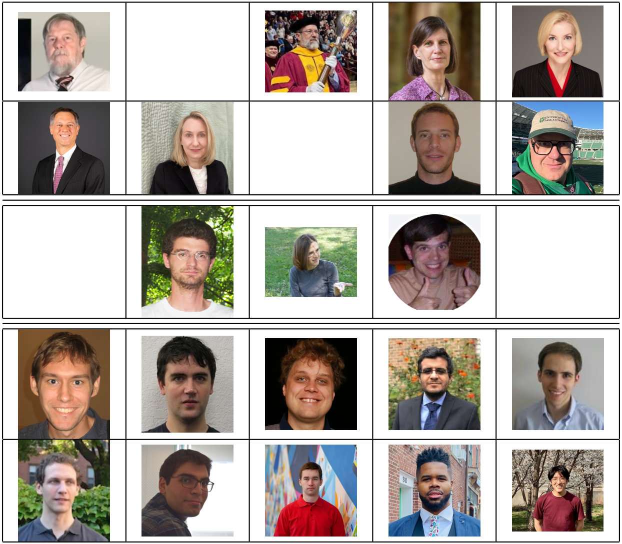
July 9, 2026

A Dedicated Adviser - Thank You, Greg!



Outline

Coupling Brownian motion and random walk

- $\tilde{B}$: standard d-dimensional Brownian motion
- S: d-dimensional simple random walk

If

$$B_t = \tilde{B}(t/d)$$

and

$$S_t = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)(S_{\lceil t \rceil} - S_{\lfloor t \rfloor}),$$

then B_t and S_t have similar distributions.

For the purpose of transferring results between B and S, we'd like the *paths* to be close.

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Skorokhod Embedding

Theorem

There exists a coupling of B and S such that $\forall\, g(n) \geq 1$ satisfying $g(n) = \mathcal{O}\left(n^{1/4}\right)$, there exist constants $b, K > 0$ such that

$$\mathbb{P}\left\{ \sup_{0\leq t\leq n} |B_t - S_t| \geq n^{1/4} g(n) \} \right\} \leq Kne^{-bg(n)}.$$

True for more general random walks (see Lawler-Limic; Random Walk: A Modern Introduction)

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Skorokhod Embedding

See it in action (e.g.) here:

- “A Lower Bound on the Growth Exponent for Loop-Erased Random Walk in Two Dimensions" (Lawler, Esaim: PS, 1999)
- “The Intersection Exponent for Simple Random Walk" (Lawler-Puckette, Combinatorics, Probability, and Computing, 2000)
- “Conformal invariance of planar loop-erased random walks and uniform spanning trees" (Lawler-Schramm-Werner, AOP (2004))

KMT ("Hungarian") Approximation

Theorem (Komlós, Major, Tusnády)

There exist a probability space containing a Brownian motion B and a simple random walk S, constants $K, \lambda, C > 0$ such that for all $x > 0$ and every n,

$$\mathbb{P}\left\{\max_{1\le k\le n} |S_k - B_k| > C\log n + x\right\} \le K\exp\{-\lambda x\}.$$

An immediate consequence is that there exists $K > 0$ such that for every $\lambda > 0$ we can find a $C = C(\lambda) > 0$ such that $\forall n \ge 2$,

$$\mathbb{P}\left\{\max_{1\le k\le n} |S_k - B_k| > C\log n\right\} \le Kn^{-\lambda}.$$

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KMT Approximation

A two-dimensional version for random times:

Theorem

There exists a probability space containing a planar Brownian motion B and a two-dimensional simple random walk S with $B_0 = S_0$ such that for every $b > 0$ there is a constant c_0 such that for any $n \in \mathbb{N}$, if D is a domain with inner radius between n and $2n$

$$P\left(\sup_{0 \le t \le T_D \vee \tau_D} |S_t - B_t| > c_0 \log n\right) = \mathcal{O}\left(n^{-b}\right),$$

where $T_D = \inf\{t \ge 0 : S_t \notin D\}$ and $\tau_D = \inf\{t \ge 0 : B_t \notin D\}$.

.

KMT

Some places where the KMT approximation was instrumental:

- “Cover Times for Brownian Motion and Random Walks in Two Dimensions" (Dembo-Peres-Rosen-Zeitouni, AOM, 2004)
- “Estimates of random walk exit probabilities and application to loop-erased random walk" (Kozdron-Lawler, EJP, 2005)
- “Random Walk Loop Soup" (Lawler-Trujillo, TAMS, 2007)
- “On the rate of convergence of loop-erased random walk to SLE(2)" (B-Johansson Viklund-Kozdron, CMP, 2013)

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Beurling Estimate

Theorem

If $\mathbb{D}$ is the unit disk centered at the origin and $K \subset \mathbb{D}$ is such that $\{|z| : z \in K\} = [0, 1]$, then if $\tau_A = \inf\{t \geq 0 : B_t \in A\}$, we have

$$P^{-\epsilon}(\tau_{\partial \mathbb{D}} \leq \tau_K) \lesssim \epsilon^{1/2}.$$

Beurling Estimate

Used abundantly by Greg and coauthors. E.g.

- “Almost sure multifractal spectrum for the tip of an SLE curve" (Johansson Viklund-Lawler; Acta Mathematica, 2012)
- “The Green function for the radial Schramm-Loewner evolution" (Alberts-Kozdron-Lawler; J Phys A, 2012)
- “Multi-point Green’s functions for SLE and an estimate of Beffara" (Lawler-Werness; AOP, 2013)
- “Minkowski content and natural parameterization for the Schramm-Loewner evolution" (Lawler-Rezaei; AOP, 2015)
- “Escape probability and transience for SLE" (Field-Lawler; EJP, 2015)
- “Scaling limit of the loop-erased random walk Green’s function" (B, Lawler-Viklund; PTRF, 2016)
- “Minkowski content of Brownian cut points" (Holden-Lawler-Sun; AIHP, 2022)

Discrete Beurling Estimate

Theorem

There exists $c > 0$ such that for any $n \in \mathbb{N}$, any $x \in \mathbb{Z}^2$ with $|x| \leq n$, any connected set $A \subset \mathbb{Z}^2$ with inner radius at least n,

$$P^x(T_n < T_A) \leq c\left(\frac{|x|}{n}\right)^{1/2},$$

where $T_A = \inf\{t \geq 0 : S_t \notin A\}$ and $T_n = \inf\{t \geq 0 : |S_t| \geq n\}$.

Brownian Motion Green's Function

Take a domain $D \subsetneq \mathbb{C}$ such that for any $z \in D$, if B is Brownian motion started at z,

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\} < \infty, \text{a.s.}$$

We can define

$$p_D(t, z, w) = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} P^z(|B_t - w| \leq \epsilon, t \leq \tau_D),$$

the transition density for B from z to w before exiting D. The **Green's function** in D is

$$g_D(z, w) = \pi \int_0^\infty p_D(t, z, w)\, dt.$$

We'll write $g_D(w) = g_D(0, w)$.

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Random Walk Green's Function

For a domain $D \subsetneq \mathbb{Z}^2$, if S is simple random walk started at z and $T_D = \min\{k \geq 0 : S_K \notin D\}$, the **discrete Green's function** in D is

$$G_D(z, w) = E^z \left[\sum_{k \geq 0} \mathbb{1}\{S_k = w; k < T_D\} \right],$$

the expected number of visits to w before leaving D by S started at z. We'll write $G_D(w) = G_D(0, w)$.

Green's Function in the Disk

Lawler (*Intersections of Random Walks*, 1991) showed for D a disk of radius n that for all $\alpha < 2$,

$$G_D(x) = \frac{2}{\pi}\log(n/|x|) + o(|x|^{-\alpha}) + \mathcal{O}\left(n^{-1}\right).$$

Since

$$g_D(x) = \log(n/|x|),$$

this means

$$\left|G_D(w) - \frac{2}{\pi}g_D(w)\right| = o(|w|^{-\alpha}) + \mathcal{O}\left(n^{-1}\right).$$

In particular for $|w| \gg n^{1/2}$,

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Green's Function in Simply Connected Domains

Generalization by Kozdron and Lawler (2005) for simply connected domains of inner radius about n:

$$\left| G_D(w) - \frac{2}{\pi} g_D(w) \right| = \mathcal{O}\left(|w|^{-2}\right) + \mathcal{O}\left(n^{-1/3} \log n\right).$$

And an improvement in B, Johansson Viklund, Kozdron (2013): For all $\epsilon > 0$,

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A Different Walk in Smooth Domains

Jiang and Kennedy (2017) considered a discrete-time continuous-space r.w. whose increments are uniformly distributed in the disk of radius $1/n$.

They showed for smooth domains D

$$\left| n^{-2} G_D^n(w) - 8 g_D(w) \right| = \mathcal{O}\left(n^{-1}\right)$$

for $|w| \geq 1/\sqrt{n}$.

Different Rates

The best bounds for the rate of convergence are dependent on the regularity of the domain:

- For the disk and smooth domains: order n^{-1}
- For arbitrary domains: order about $n^{-1/2}$

What about domains with boundary regularity between these two extreme cases?

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α-Beurling Estimates

Pacman set with parameters α and n:

$$D_\alpha^n := \{z \in \mathbb{C} : |z| \leq n; 0 < \arg(z) \leq 2\pi - \alpha\}.$$

Lemma

For every $a > 0$, there exists a constant $C = C_a > 0$ such that for all $n \geq 1$, all $r \leq a \log n$,

$$\sup_{z \in D_\alpha^r} P^z\left(\left|B_{\tau_{D_\alpha}}\right| = n\right) \leq C(n/r)^{-c_\alpha},$$

where

$$c_\alpha = \frac{1}{2} + \frac{\alpha}{4\pi - 2\alpha} = \frac{\pi}{2\pi - \alpha}.$$

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Use conformal invariance of harmonic measure, the fact that

$$h(z) = -(z/n)^{c_\alpha} - (n/z)^{c_\alpha}$$

maps D_α^n to the upper half-plane $\mathbb{H}$, and the fact that the Poisson kernel in the upper half-plane is given by

$$H(x, y; x') = \frac{1}{\pi} \frac{y}{(x - x')^2 + y^2}.$$

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Use KMT and

$$D'_\alpha := D_\alpha(n/4) + \left[\frac{c_0 \log n}{\sin(\alpha/2)} \right] e^{-i\alpha/2}.$$

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(α, h)-Cone-Convex Domains

A domain D is (α, h)-cone-convex (with $\alpha \in (0, \pi], h > 0$) if there exists $h > 0$ such that for every point $x \in \partial D$, there exists an open cone $C_\alpha(x)$ such that

$$C_\alpha(x) \cap D(x, h) \subseteq D^c.$$

We call the point x an (α, h)-*cone point*. We denote by

$$\mathcal{C}_{\alpha,h}$$

the set of (α, h)-cone-convex domains D that contain the origin, such that $\inf\{|x| : x \in \partial D\} \in [1, 2]$.

Green's Function Convergence in D

Theorem

Let D be a domain and $n \in \mathbb{N}$ be such that $D/n \in \mathcal{C}_{\alpha,h}$ and $w \in D \cap \mathbb{Z}^2$. Then

$$\left| G_D(w) - \frac{2}{\pi} g_D(w) \right| = \mathcal{O}\left(\left(\frac{\log^2 n}{n} \right)^{c_\alpha} + |w|^{-2} \right),$$

where

$$c_\alpha = \frac{1}{2} + \frac{\alpha}{4\pi - 2\alpha} = \frac{\pi}{2\pi - \alpha}.$$

In particular, for $|w| \geq (n/\log^2 n)^{c_\alpha/2}$,

$$\left| G_D(w) - \frac{2}{\pi} g_D(w) \right| = \mathcal{O}\left(\left(\frac{\log^2 n}{n} \right)^{c_\alpha} \right).$$

Comparing the Green's Function

With

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\},$$

$$T_D = \min\{k \geq 0 : S_k \notin D\},$$

we have

$$g_D(w) = E^w[\log|B_{\tau_D}|] - \log|w|.$$

and

$$G_D(w) = E^w[a(S_{T_D})] - a(w),$$

where

$$a(z) = \lim_{n\to\infty} \sum_{k=0}^{n} (P(S_k = 0) - P(S_k = z))$$

is the potential kernel.

Comparing the Green's Function

Recall

$$G_D(w) = E^w[a(S_{T_D})] - a(w)$$

and note (with $k = \frac{2\gamma}{\pi} + \frac{3}{\pi}\ln 2$)

$$a(z) = \frac{2}{\pi}\log|z| + k + \mathcal{O}\left(|z|^{-2}\right).$$

So (assume $\mathrm{inrad}(D) \geq n$).

$$G_D(w) = \frac{2}{\pi}\left(E^w[\log|S_{T_D}|] - \log(w)\right) + \mathcal{O}\left(|w|^{-2} \vee n^{-2}\right).$$

Since

$$g_D(w) = E^w[\log|B_{\tau_D}|] - \log|w|,$$

$$\left|G_D(w) - \frac{2}{\pi}g_D(w)\right| \lesssim E^w\left[\left|\log\frac{|S_{T_D}|}{|B_{\tau_D}|}\right|\right] + \mathcal{O}\left(|w|^{-2} \vee n^{-2}\right).$$

Using KMT

We are after

$$E^w\left[\left|\log\frac{|S_{T_D}|}{|B_{\tau_D}|}\right|\right]$$

for any $w \in D$.

- Couple B and S until T, the first time B gets within $2c_0 \log n$ of ∂D. With high probability, they are within $c_0 \log n$ of each other. The low-probability case can be handled with Beurling.
- Handle the fact that B and S are not jointly Markov (one way is to sum over all possible sub-events $T \in [k-1, k]$ and start B and S independently at time $k-1$).
- The problem reduces to starting independent B and S within $2c_0 \log n$ of each other and within $3c_0 \log n$ of ∂D_α and computing

$$E^{u,v}\left[\left|\log\frac{|S_{T_D}|}{|B_{\tau_D}|}\right|\right].$$

Main term

$$E^{(u,v)}\left[\left|\log\frac{|S_{T_D}|}{|B_{\tau_D}|}\right|\right] = E^{(u,v)}\left[\left|\log\left(1+\frac{|S_{T_D}-B_{\tau_D}|}{|B_{\tau_D}|}\right)\right|\right]$$

$$\lesssim \sum_{k\le\lfloor\frac{\log n+\log h}{\log 2}\rfloor} P(\max\{|B_{\tau_D}-x|,|S_{T_D}-x|\}\in[2^{k-1},2^k))\log\left(1+\frac{2^{k+1}}{n}\right)$$

$$\lesssim \sum_{k\le\lfloor\frac{\log n+\log h}{\log 2}\rfloor} P(|B_{\tau_D}-x|\ge 2^{k-1})\log\left(1+\frac{2^{k+1}}{n}\right)$$

$$\lesssim \sum_{k\le\lfloor\frac{\log n+\log h}{\log 2}\rfloor} \left(\frac{\log n}{2^{k-1}}\right)^{c_\alpha}\frac{2^{k+1}}{n}$$

$$\lesssim \frac{\log^{c_\alpha} n}{n}\left(2^{\frac{\log n+\log h}{\log 2}}\right)^{1-c_\alpha} \lesssim \left(\frac{\log n}{n}\right)^{c_\alpha}$$

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&\lesssim \sum_{k\le\lfloor\frac{\log n+\log h}{\log 2}\rfloor} P(\max\{|B_{\tau_D}-x|,|S_{T_D}-x|\}\in[2^{k-1},2^k))\log\left(1+\frac{2^{k+1}}{n}\right) \\
&\lesssim \sum_{k\le\lfloor\frac{\log n+\log h}{\log 2}\rfloor} P(|B_{\tau_D}-x|\ge 2^{k-1})\log\left(1+\frac{2^{k+1}}{n}\right) \\
&\lesssim \sum_{k\le\lfloor\frac{\log n+\log h}{\log 2}\rfloor} \left(\frac{\log n}{2^{k-1}}\right)^{c_\alpha}\frac{2^{k+1}}{n} \\
&\lesssim \frac{\log^{c_\alpha} n}{n}\left(2^{\frac{\log n+\log h}{\log 2}}\right)^{1-c_\alpha} \lesssim \left(\frac{\log n}{n}\right)^{c_\alpha}
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&\lesssim \sum_{k\le\lfloor\frac{\log n+\log h}{\log 2}\rfloor} P(\max\{|B_{\tau_D}-x|,|S_{T_D}-x|\}\in[2^{k-1},2^k))\log\left(1+\frac{2^{k+1}}{n}\right) \\
&\lesssim \sum_{k\le\lfloor\frac{\log n+\log h}{\log 2}\rfloor} P(|B_{\tau_D}-x|\ge 2^{k-1})\log\left(1+\frac{2^{k+1}}{n}\right) \\
&\lesssim \sum_{k\le\lfloor\frac{\log n+\log h}{\log 2}\rfloor} \left(\frac{\log n}{2^{k-1}}\right)^{c_\alpha}\frac{2^{k+1}}{n} \\
&\lesssim \frac{\log^{c_\alpha} n}{n}\left(2^{\frac{\log n+\log h}{\log 2}}\right)^{1-c_\alpha} \lesssim \left(\frac{\log n}{n}\right)^{c_\alpha}
\end{aligned}
$$

Main term

$$
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\end{aligned}
$$

Happy Birthday, Greg!

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PRINCETON UNIVERSITY
Mathematics Graduate Students
1976

1) Jing Woan Wu
2) Sheldon H. Katz
3) Alan S. Grenadir
4) John F. X. Ries
5) G. Robert Meyerhoff
6) John T. Moy
7) Roy E. DeMeo
8) Gregory F. Lawler
9) David S. Jerison
10) Jonathan Rogawski
11) Mario J. D. Carneiro
12) Mark W. Saaltink
13) David Gabai
14) Douglas E. Critchlow
15) Richard M. Crew
16) Jonathan S. Carr
17) Patrick W. Keef
18) William D. Dunbar
19) C. Robin Graham
20) R. Michael Beals
21) Andrew J. Nicas